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MONOTONE OPERATORS AND LOCAL-GLOBAL MINIMUM PROPERTY OF NONLINEAR OPTIMIZATION PROBLEMS

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Dedicated to Professors Zoltán Daróczy and Imre Kátai on their 75*th anniversary*

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Abstract. Our main goal is to prove that every local minimizer of a certain nonlinear optimization problem is global. For this, we use some results from the theory of monotone operators and connected functions. At last, we show applications of the main results in control theory.

1. Introduction

In optimization theory sufficient conditions have a great importance. In the linear and convex cases the theory is well-developed, because the first order necessary conditions usually become sufficient.

The situation is completely different if the problem is highly nonlinear (nonconvex). In these cases sufficient conditions ensure only local optimality.

This fact motivates the development of the following question: What kind of problems possess the so-called local-global minimum property? With other

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words, when will all the local optimizers of the objective function be global as well.

The starting point is analogous to the application of monotone operators in the existence theory of PDE (see e.g. [12]). If f is a monotone real function then every local minimizer of $|f|$ is global. Our point is to generalize this fact, when the function is a nonlinear monotone operator acting on a reflexive Banach space with values from the dual space.

In the following two sections we summarize the most important definitions and facts about monotone operators (§2) and connected functions (§3). We need them to prove our main results (§4).

At last (§5), we present some possible applications in control theory.

2. Monotone operators

The theory of monotone operators is one of the main tools to prove existence theorems on nonlinear PDE (see e.g. [12] and the references therin).

This concept was introduced by Minty in [8] and its use in the theory of nonlinear PDE was developed at first by Minty and Browder (see e.g. [9] and [3]). For more historical details see the survey of Borwein [2].

Throughout this paper X always denotes a real, reflexive Banach space, X^* is its dual, namely the Banach space of all continuous linear functionals defined on X, and $\langle x^*, x\rangle$ is the canonical pairing between X^* and X.

An operator (not necessarily linear) $S\colon X \to X^*$ is said to be *monotone* if

$$\langle Sx - S\bar{x}, x - \bar{x}\rangle \geq 0, \qquad x, \bar{x} \in X.$$

It is called *strictly monotone* if there is strict inequality above, when $x \neq \bar{x}$.

A sturdier concept than the previous two is strong monotonicity. An operator $S\colon X \to X^*$ is said to be *strongly monotone with modulus c* if there exists a positive real number c such that

$$\langle Sx - S\bar{x}, x - \bar{x}\rangle \geq c\|x - \bar{x}\|^2, \qquad x, \bar{x} \in X.$$

If we do not want to emphasize the role of c, then we simply call S *strongly monotone*.

It is clear that strong monotonicity implies strict monotonicity, and the last entails monotonicity, but the reverse implications are false in general.

The operator $S\colon X \to X^*$ is called *coercive* if

$$\frac{\langle Sx, x\rangle}{\|x\|} \to \infty \quad \text{as} \quad \|x\| \to \infty.$$

If S is strongly monotone with a modulus c then it is coercive, but the reverse is not true, not even if $X = \mathbb{R}$.

The operator S is *hemi-continuous* if the real function

$$t \mapsto \langle S(\bar{x} + th), x\rangle$$

is continuous on $\mathbb{R}$ for all fixed $x, \bar{x}, h \in X$.

Theorem 2.1 (see e.g. [12]). *If X is reflexive and S is a monotone, hemi-continuous operator, then S is a continuous operator with respect to the norm topology in X, and with respect to the weak topology in X^*.*

It is trivial that continuity implies hemi-continuity, but the reverse is untrue.

In later applications it will be useful to know more about the inverse of a monotone operator.

Theorem 2.2 (see e.g. [5]). *Let X be a reflexive Banach space, and $S\colon X \to \to X^*$ be a hemi-continuous strongly monotone operator with some modulus c. Then S is bijective (that is, one-to-one and onto), and its inverse is continuous from the strong topology of X^* to the strong topology of X.*

It is very important to note that S itself is not necessarily continuous with respect to the strong topologies besides of the assumptions of the above theorem. However, it is true in this case that S is a homeomorphism between the Banach space X and the locally convex topological vector space (the topology induced by a family of semi-norms) X^*.

Another important observation here is, that the space is reflexive which simplifies the situation, because weak* topology does not come to play.

Theorem 2.3 (see e.g. [5]). *Let S be strongly monotone and continuous with respect to the strong topologies both in X and X^*, then S is a homeomorphism between X and X^* with the strong (norm) topologies.*

In simply speaking, this theorem says that S is a homeomorphism between the Banach spaces X and X^*.

3. Connected functions

The concepts of quasi-connected and connected functions were introduced in [10] by the authors. These are far and useful generalizations of quasi-convexity and convexity.

Let $D \subset X$ be a nonempty set. A function $f\colon D \to \mathbb{R}$ is called *quasi-connected (connected) on* D if for all $\bar{x}, x \in D$ there exists a continuous function (*a path joining* $\bar{x}$ *and* x) $\gamma\colon [0,1] \to D$ such that γ and f fulfill the following conditions:

(i) $\gamma(t) \in D$ for all $t \in [0,1]$;

(ii) $\gamma(0) = \bar{x}$ and $\gamma(1) = x$;

(iii) $f(\gamma(t)) \le \max\{f(\bar{x}), f(x)\}$ $\big(f(\gamma(t)) \le (1-t)f(\bar{x}) + tf(x)\big)$ for all $t \in [0,1]$.

If there is strict inequality in (iii), when $\bar{x} \neq x$, then f is called *strictly quasi-connected (strictly connected)*.

The order of the base points $\bar{x}, x$ is very important. A more correct notation would be $\gamma_{\bar{x},x}$, but this is too troublesome, therefore we will use the simpler notation, when there is no ambiguity.

It is quite straightforward that every connected function is also quasi-connected.

The concepts above are a generalization of quasi-convexity (convexity) – just take $\gamma(t) = (1-t)\bar{x} + tx$ – which have a very important role in optimization theory (see e.g. [7] and the references therein).

In [1] the authors proved that connectedness, strict connectedness or strict quasi-connectedness of a function implies that every local minimizer of this function is also a global one. However, it was proved only in the case, when $X = \mathbb{R}^n$, the proof runs in a pretty similar way if X is a topological space.

Theorem 3.1 (Avrilel-Zhang)**.** *Let X be a topological space, $D \subset X$ and $f\colon D \to \mathbb{R}$ be a function. If f is connected (strictly connected, strictly quasi-connected) on D, then every local minimizer of f is also a global one.*

The last theorem says, that connectedness remains untouched if the domain is perturbed by a homeomorphism.

Theorem 3.2 ([4])**.** *Let X and Y be metric spaces, or topological spaces, $S\colon X \to Y$ be a homeomorphism, and $f\colon Y \to \mathbb{R}$ be a quasi-connected* (*strictly quasi-connected, connected, strictly connected*) *function, then $f \circ S$ is also quasi-connected* (*strictly quasi-connected, connected, strictly connected*)*.*

4. Main results

Formulation of the problem

Let X be a reflexive Banach space, $S\colon X^* \to X$ be a continuous (with respect to the norm topology both in X and in X^*), strongly monotone operator, and $g\colon X \to \mathbb{R}$ be a convex function. We investigate the local-global minimum property of the following optimization problem besides the previous assumptions.

$$\text{(P)} \qquad \min_{x^* \in X^*} f(x^*) := g(Sx^*)$$

Theorem 4.1. *Every local minimizer of* (P) *is global.*

Proof. According to Theorem 2.3 the operator S is a homeomorphism between X and X^*. All the assumptions of Theorem 3.2 are satisfied. Using this, we immediately get the statement of our theorem from Theorem 3.1. ■

If the underlying space is of finite dimension, then milder assumptions are enough.

Corollary 4.1. *If X is a finite dimensional Banach space, and S is a hemicontinuous and strongly monotone operator, then every local minimizer of* (P) *is also global.*

Proof. Besides the required assumptions X is automatically reflexive, and S is a continuous, strongly monotone operator. We get now the statement of the corollary from the previous theorem. ■

Let us consider a constrained version of (P).

$$\text{(CP)} \qquad \begin{aligned} &\min g(x) \\ \text{s.t.} \quad &Ax = x^*, \quad x \in X, \end{aligned}$$

where g is assumed to be the same as in (P), and $A\colon X \to X^*$ is a strongly monotone, continuous operator (with respect to the norm topology both in X and in X^*).

Theorem 4.2. *Every local minimizer of* (CP) *is global.*

Proof. According to Theorem 2.3 the operator A is a homeomorphism between X and X^*. With $S := A^{-1}$ we get from Theorem 3.2 that $g \circ S$ is connected. At last, our result follows from Theorem 3.1. ■

In a very similar way, we get the finite dimensional version of (CP).

Corollary 4.2. *If X is a finite dimensional Banach space, and A is a hemicontinuous and strongly monotone operator, then every local minimizer of* (CP) *is also global.*

5. Applications

5.1. An optimal control problem governed by a nonlinear ODE

Let us minimize the following objective function subject to an ODE.

$$\min \|x\|^2_{L^2(0,1)} := \int_0^1 x^2(t)dt$$
$$\text{s.t.} \quad -x''(t) + h(x(t)) = u(t), \quad t \in]0,1[\tag{5.1}$$
$$x(0) = x(1) = 0,$$

where the nonlinearity term $h\colon \mathbb{R} \to \mathbb{R}$ is a bounded, continuous, monotone function, and $u \in L^2(0,1)$ are given. We would like to reformulate this problem such that it fits better to our earlier scheme. For this, we need the weak formulation of the above boundary value problem.

We denote with $V = H_0^1(0,1)$ the Hilbert space of such $L^2(0,1)$ functions, whose first weak derivative is also in $L^2(0,1)$, and fulfill the previously given boundary condition. This makes sense in this case, because the space is one dimensional so, all the elements of V are continuous.

We called the function $x \in V$ the *weak solution* of the above boundary value problem if

$$\int_0^1 x'\varphi' + \int_0^1 h(x)\varphi = \int_0^1 u\varphi \qquad \text{for all } \varphi \in V.$$

Let us define the operator $A\colon V \to V^*$, and the functional $F\colon V \to \mathbb{R}$ in the following way.

$$\langle Ax, v\rangle := \int_0^1 x'v' + \int_0^1 h(x)v, \qquad v \in V$$
$$F(v) := \int_0^1 uv, \qquad v \in V.$$

The operator A is continuous and strongly monotone moreover, $F \in V^*$ (for the details see [6]).

According to Theorem 4.2 every local minimizer of (5.1) is global.

5.2. An optimal control problem governed by a semilinear elliptic PDE

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with boundary Γ, $g\colon \mathbb{R} \to \mathbb{R}$ bounded, continuous and monotone, $y_\Omega, u \in L^2(\Omega)$. Let us consider the following optimal control problem governed by a semilinear elliptic equation (in weak sense).

$$\begin{aligned} \min \|y - y_\Omega\|^2_{L^2(\Omega)} &:= \int_\Omega |y(x) - y_\Omega(x)|^2 dx \\ \text{s.t.} \quad -\Delta y + g(y) &= u \quad \text{in } \Omega \\ y &= 0 \qquad \text{on } \Gamma. \end{aligned}$$

Similarly as above, one can derive the same conclusion.

Remark 5.1. Both in this and in the previous (ODE) case, the above presented process works with a more general boundary value problem. The interested reader can consult with e.g. [12] and [11].

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