

## ON THE SUM OF DIVISORS FUNCTION

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*Dedicated to Professors Zoltán Daróczy and Imre Kátai  
on their 75th anniversary*

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**Abstract.** The following assertion is proved. Let  $Q_1, Q_2$  be odd primes,  $A_{Q_1, Q_2}(x)$  be the number of those  $n \leq x$  for which  $Q_1 \nmid \sigma(n), Q_2 \nmid \sigma(n+1)$  simultaneously hold. Then  $A_{Q_1, Q_2}(x) \geq \frac{cx}{(\log x)^5}$ , if  $x > X_0$ .  $c, X_0$  are positive constants.

### 1. Introduction

#### 1.1. Notation

$\mathcal{P}$  = set of primes.  $\varphi(n)$  = Euler's totient function,  $\sigma(n)$  = sum of divisors function,  $\tau(n)$  = number of divisors.

Let  $x_1 = \log x, x_2 = \log x_1, \dots, x_{k+1} = \log x_k$ .

#### 1.2. Formulation of the theorems

In his paper [4] Kátai (Theorem 4) proved the following assertion:

Let  $\lambda > 2, I_x = \left[ \frac{\lambda x_2}{x_3}, x_2 \right], Q_1, Q_2 \in \mathcal{P}$ ,

$$E_{Q_1, Q_2}(x) := \#\{n \leq x : Q_1 \nmid \varphi(n), Q_2 \nmid \varphi(n+1)\}.$$

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Then, uniformly for  $Q_1, Q_2 \in I_x$ ,

$$\frac{1}{x} E_{Q_1, Q_2}(x) = (1 + O_x(1)) \frac{B}{2} \kappa_1 \kappa_2,$$

where

$$\kappa_j = \exp\left(-\frac{x_2}{Q_j - 1}\right), \quad (j = 1, 2),$$

and

$$B = \prod_{\substack{p \geq 3 \\ p \in \mathcal{P}}} \left(1 - \frac{2}{p(p-1)}\right).$$

Kátai notes that similar theorem can be proved for  $\sigma$  instead of  $\varphi$ , and that he is unable to count the asymptotic of those  $n$  for which  $3 \nmid \varphi(n)$ ,  $3 \nmid \varphi(n+1)$  simultaneously holds.

In this paper we shall investigate the function

$$A_{Q_1, Q_2}(x) := \#\{n \leq x : Q_1 \nmid \sigma(n), Q_2 \nmid \sigma(n+1)\},$$

where  $Q_1, Q_2$  are arbitrary odd primes,  $Q_1 = Q_2$  is included.

**Theorem 1.** *If  $Q_1, Q_2 \in \mathcal{P}$ ,  $Q_1 \neq 2$ ,  $Q_2 \neq 2$ , then there are constants  $c > 0$  and  $X_0$  such that*

$$A_{Q_1, Q_2}(x) \geq \frac{cx}{(\log x)^5},$$

if  $x > X_0$ .

**Theorem 2.** *Let  $Q \in \mathcal{P}$ ,  $Q \neq 2$ ,*

$$B_Q(x) = \#\{p \leq x : p \in \mathcal{P}, Q \nmid \sigma(p+1)\}.$$

*Then there are constants  $c > 0$  and  $X_0$  such that*

$$B_Q(x) \geq \frac{cx}{(\log x)^5}.$$

Our theorems follow from some variants of known, deep theorems.

## 2. Main auxiliary results

**2.1.** Let  $Q \in \mathcal{P}$ ,  $Q \neq 2$ ,  $\chi(n)$  be a character mod  $Q$ , such that  $\chi(-1) = -1$ . Let  $r(n) = \sum_{d|n} \chi(d)$ ,

$$T(x) = \sum_{\substack{p \leq x \\ p+1 \not\equiv 0 \pmod{Q}}} r(Qp+1) |\mu(Qp+1)|.$$

**Theorem 3.** *We have*

$$T(x) = A_0 \frac{x}{\log x} + O\left(\frac{x}{(\log x)^{1+\delta}}\right),$$

where  $\delta$  and  $A_0$  are positive constants.

**2.2.** Let  $Q_1 \neq Q_2$ ,  $Q_1, Q_2$  be odd primes,  $\chi(n)$  be a character mod  $Q_2$ , such that  $\chi(-1) = -1$ . Let  $r(n) = \sum_{d|n} \chi(d)$ . Let  $A = Q_1^a Q_2^b$ ,  $a \in \{1, 2\}$ ,  $b \in \{1, 2\}$  such that  $Q_1 \nmid \sigma(A)$ . Let

$$S(x) = \sum_{\substack{p \leq x \\ p+1 \not\equiv 0 \pmod{Q_1}}} r(Ap+1) |\mu(Ap+1)|.$$

**Theorem 4.** *We have*

$$S(x) = B_0 \frac{x}{\log x} + O\left(\frac{x}{(\log x)^{1+\delta}}\right),$$

where  $\delta$  and  $B_0$  are suitable positive constants.

## 3. Deduction of Theorem 1 and 2 from Theorem 3 and 4

In the case  $Q_1 = Q_2 = Q$  let  $A = Q$ ,  $\chi$  mod  $Q$  be a Dirichlet character such that  $\chi(-1) = -1$ , and  $r(n) = \sum_{d|n} \chi(d)$ .

If  $p+1 \not\equiv 0 \pmod{Q}$ , and  $|\mu(Qp+1)|r(Qp+1) \neq 0$ , then  $Qp+1$  is a squarefree number, and  $\pi|Qp+1$ ,  $\pi \in \mathcal{P}$  implies that  $1+\chi(\pi) = 2$ , consequently  $Q \nmid \pi+1$ , thus  $Q \nmid \sigma(Qp+1)$ . Furthermore  $Q \nmid \sigma(Qp) = (Q+1)(p+1)$ .

Assume that  $Q_1 \neq Q_2$ . Let  $A = Q_1^a Q_2^b$ , where  $a$  and  $b$  are such positive integers that  $Q_1 \nmid (1+Q_2+\dots+Q_2^b)$ ,  $Q_2 \nmid (1+Q_1+\dots+Q_1^a)$ . Observe that

$a, b \in \{1, 2\}$  is a suitable choice. Let  $\chi$  be a character mod  $Q_2$ , such that  $\chi(-1) = -1$ , and  $r(n) = \sum_{d|n} \chi(d)$ .

Let  $p \not\equiv -1 \pmod{Q_1}$ . Then  $Q_1 \nmid \sigma(Ap)$ , and if  $|\mu(Ap+1)| r(Ap+1) \neq 0$ , then  $Q_2 \nmid \sigma(Ap+1)$ .

We have

$$T^2(x) \leq A_{Q_1, Q_2}(x) \sum_{\substack{p \leq x \\ p \not\equiv -1 \pmod{Q}}} r^2(Qp+1) |\mu(Qp+1)|.$$

Since  $r^2(Qp+1) \leq \tau^2(Qp+1)$ , and  $\sum_{n \leq x} \tau^2(n) \leq cx \cdot x_1^3$ , therefore  $A_{Q, Q}(x) \gg \frac{x}{x_1^5}$ .

We can obtain similarly that

$$S^2(x) \leq B_Q(x) \sum_{\substack{p \leq x \\ p+1 \not\equiv 0 \pmod{Q}}} r^2(Ap+1) |\mu(Ap+1)|,$$

and hence that  $B_Q(x) \gg \frac{x}{x_1^5}$ .

**Remark.** We could improve these inequalities by using some sieve results.

#### 4. Sketch of the proof of Theorem 3 and 4

The main ingredient of the proof is the inequality due to E. Bombieri and A.I. Vinogradov which is quoted now as Lemma 1.

Let  $\pi(z, D, l) = \#\{p \leq z : p \equiv l \pmod{D}\}$ ,

$$\text{li } z = \int_2^z \frac{du}{\log u}.$$

**Lemma 1.** (See Elliott [1], Chapter 7.)

$$\sum_{D \leq \frac{\sqrt{x}}{x_1^B}} \max_{l \pmod{D}} \max_{z \leq x} \left| \pi(z, D, l) - \frac{\text{li } z}{\varphi(D)} \right| \ll \frac{x}{x_1^A},$$

where  $B \geq 2A + 23$ .

I. Kátai considered in [4] the sum

$$T(x) = \sum_{p \leq x} r_4(p-1) |\mu(p-1)|,$$

where  $r_4(n) = \sum_{d|n} \chi_4(d)$ ,  $\chi \pmod{4}$  is the character satisfying  $\chi(-1) = -1$ , and proved that

$$T(x) = A_0 \frac{x}{\log x} + O\left(\frac{x}{(\log x)^{1+\delta}}\right).$$

To prove Theorem 3 we can follow his argument. Let

$$T(x, k) = \sum_{\substack{p \leq x \\ p+1 \not\equiv 0 \pmod{Q} \\ Qp+1 \equiv 0 \pmod{k}}} r(Qp+1).$$

Arguing as in [4], we have

$$T(x) = \sum_{d \leq x_1^3} \mu(d) T(x, d^2) + O\left(\frac{x}{x_1^{1,5}}\right).$$

Let  $k \leq x_1^6$ ,  $(k, Q) = 1$ . Then

$$T(x, k) = \sum_{\substack{p \leq x \\ Qp+1 \equiv 0(k) \\ p+1 \not\equiv 0(Q)}} \sum_{Qp+1=uv} \chi(u).$$

Thus

$$T(x, k) = \sum_{\substack{u \leq Qx+1 \\ (u, Q)=1}} \chi(u) \sum_{\substack{p \leq x \\ p \not\equiv -1(Q) \\ Qp+1 \not\equiv 0([k, u])}} 1.$$

Since  $(Q, u) = 1$ , therefore the right most sum equals

$$\sum_{j=1}^{Q-2} \pi(x, Q[k, u], l_j),$$

where  $l_j$  are those residues for which

$$Ql_j + 1 \equiv 0 \pmod{[k, u]}, \quad l_j + 1 \not\equiv 0 \pmod{Q}$$

holds. With this modification we can proceed further. The "enveloping sieve" of C. Hooley can be applied.

The proof of Theorem 4 is similar.

### References

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