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ON SOME ORTHOGONALLY ADDITIVE FUNCTIONS ON INNER PRODUCT SPACES

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Dedicated to Professors Zoltán Daróczy and Imre Kátai on their 75th birthday

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Abstract. Let E be a real inner product space of dimension at least 2. If $f : E \to E$ satisfies

$$f(x+y) = f(x) + f(y) \quad \text{for all orthogonal } x, y \in E$$

and

$$f(f(x)) = x \quad \text{for } x \in E,$$

then f is additive.

Let E be a real inner product space of dimension at least 2.

A function f mapping E into on abelian group is called orthogonally additive, if

$$f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in E \text{ with } x \perp y.$$

It is well known, see [3, Corollary 10] and [1, Theorem 1], that every orthogonally additive function f defined on E has the form

$$f(x) = a(\|x\|^2) + b(x) \quad \text{for } x \in E, \tag{1}$$

where a and b are additive functions uniquely determined by f.

Our main result says that every involutory orthogonally additive function is additive.

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Theorem 1. *If $f : E \to E$ is orthogonally additive and*

$$f(f(x)) = x \quad \textit{for } x \in E, \tag{2}$$

then f is additive.

Proof. Let $(u|v)$ denote the inner product of $u, v \in E$.

As mentioned above f has form (1) with additive functions $a : \mathbb{R} \to E$ and $b : E \to E$. It follows from (1) that

$$\|f(x)\|^2 = \|a(\|x\|^2)\|^2 + 2(a(\|x\|^2)|b(x)) + \|b(x)\|^2 \quad \text{for } x \in E,$$

which jointly with (2) and (1) gives

$$\begin{aligned} x &= a(\|f(x)\|^2) + b(f(x)) = \\ &= a(\|a(\|x\|^2)\|^2 + 2(a(\|x\|^2)|b(x)) + \|b(x)\|^2) + b(a(\|x\|^2) + b(x)) \end{aligned}$$

for $x \in E$. Hence, if $x \in E$ and $r \in \mathbb{Q}$, then

$$\begin{aligned} rx = r^4 a(\|a(\|x\|^2)\|^2) + 2r^3 a((a(\|x\|^2)|b(x))) + r^2 a(\|b(x)\|^2) + \\ + r^2 b(a(\|x\|^2)) + rb(b(x)). \end{aligned}$$

Consequently,

$$b(b(x)) = x \quad \text{and} \quad a((a(\|x\|^2)|b(x))) = 0 \quad \text{for } x \in E. \tag{3}$$

In particular for all $x, y \in E$ we have

$$\begin{aligned} 0 &= a((a(\|x+y\|^2)|b(x+y))) = \\ &= a((a(\|x\|^2)|b(y)) + 2(a((x|y))|b(x+y)) + (a(\|y\|^2)|b(x))), \end{aligned}$$

i.e.,

$$a((a(\|x\|^2)|b(y)) + 2(a((x|y))|b(x))) = -a((a(\|y\|^2)|b(x)) + 2(a((x|y))|b(y))).$$

As the function of $x \in E$, the left–hand side is even, whereas the right–hand side is odd, and so on each side we have zero for every $x, y \in E$. Hence

$$a((a(\|x\|^2)|b(y))) = 0 \quad \text{for all orthogonal } x, y \in E. \tag{4}$$

Now, if $z \in E$ and $\alpha \in (0, \infty)$, then finding an $x \in E$ such that $x \perp b(z)$ and $\|x\|^2 = \alpha$ and applying (3) and (4) we see that

$$a((a(\alpha)|z)) = a((a(\|x\|^2)|b(b(z)))) = 0.$$

This shows that

$$a((a(\alpha)|x)) = 0 \quad \text{for } \alpha \in \mathbb{R} \text{ and } x \in E. \tag{5}$$

Suppose $a(\alpha) \neq 0$ for some $\alpha \in \mathbb{R}$. Then

$$\left(a(\alpha)|\alpha\frac{a(\alpha)}{\|a(\alpha)\|^2}\right) = \alpha$$

and by (5) we have

$$a(\alpha) = a\left(\left(a(\alpha)|\alpha\frac{a(\alpha)}{\|a(\alpha)\|^2}\right)\right) = 0.$$

The contradiction obtained proves that $a = 0$ and (1) gives $f = b$. ■

Remark 1. Let H_0 be a basis of the vector space $\mathbb{R}$ over $\mathbb{Q}$ and let H be a basis of the vector space E over $\mathbb{Q}$. Then (cf. [2, Theorem 4.2.3])

$$\mathfrak{c} = \text{card } H_0 \leq \text{ card } H.$$

If H_1 and H_2 are disjoint subsets of H such that

$$1 \leq \text{card } H_1 \leq \mathfrak{c} \quad \text{and} \quad \text{card } H_2 = \text{ card } H,$$

and $a : \mathbb{R} \to E$ and $b : E \to E$ are additive functions such that

$$a(H_0) = H_1, \quad b(H) = H_2$$

and b is injective, then the function $f : E \to E$ given by (1) is orthogonally additive, injective and it is not additive.

To see that f is injective it is enough to observe that if $x, y \in E$ and $f(x) = f(y)$, then

$$a(\|x\|^2) - a(\|y\|^2) = b(y) - b(x),$$

the left–hand side belongs to $\text{Lin}_{\mathbb{Q}} H_1$ and the right–hand side is in $\text{Lin}_{\mathbb{Q}} H_2$, whence $b(x) = b(y)$ and, consequently, $x = y$.

Remark 2. Assume

$$E = E_1 \oplus E_2, \quad E_1 \perp E_2 \quad \text{and} \quad \dim E_1 = 1.$$

Fix an $e \in E_1$ with $\|e\| = 1$, let $a_0 : \mathbb{R} \to \mathbb{R}$ and $b_0 : E_2 \to E_2$ be additive functions such that

$$a_0([0, \infty)) = \mathbb{R}, \quad b_0(E_2) = E_2$$

and define $a : \mathbb{R} \to E$ and $b : E \to E$ by

$$a(\alpha) = a_0(\alpha)e, \quad b(\alpha e + x_2) = b_0(x_2) \quad \text{for } \alpha \in \mathbb{R} \text{ and } x_2 \in E_2.$$

Then the function $f : E \to E$ given by (1) is orthogonally additive, $f(E) = E$ and f is not additive.

To see that $f(E) = E$ fix arbitrarily $y \in E$. Then $y = \beta e + y_2$ where $\beta \in \mathbb{R}$, $y_2 \in E_2$ and $y_2 = b_0(x_2)$ for some $x_2 \in E_2$, $\beta - a_0(\|x_2\|^2) = a_0(\alpha)$ for some $\alpha \in [0, \infty)$. Consequently $\|\sqrt{\alpha}e + x_2\|^2 = \alpha + \|x_2\|^2$ and

$$f(\sqrt{\alpha}e + x_2) = a_0(\alpha + \|x_2\|^2)e + b_0(x_2) = \beta e + y_2 = y.$$

We have been unable to find an example of a bijective orthogonally additive function $f : E \to E$ which is not additive.

Remark 3. If $a : \mathbb{R} \to E$ and $b : E \to E$ are linear and the function $f : E \to E$ given by (1) is bijective, then it is linear.

Proof. As for some $x_0 \in E$ we have

$$-a(1) = f(x_0) = \|x_0\|^2 a(1) + b(x_0),$$

it follows that

$$(\|x_0\|^2 + 1)a(1) = -b(x_0)$$

and so

$$a(1) = b(y_0),$$

where $y_0 = -\frac{1}{1+\|x_0\|^2} x_0$. Consequently

$$f(x) = \|x\|^2 a(1) + b(x) = b(\|x\|^2 y_0 + x)$$

for $x \in E$. Suppose $y_0 \neq 0$. Then

$$f\left(-\frac{1}{\|y_0\|^2} y_0\right) = b\left(\frac{1}{\|y_0\|^2} y_0 - \frac{1}{\|y_0\|^2} y_0\right) = 0 = f(0)$$

which contradicts the injectivity of f. Hence $y_0 = 0$ and $f = b$. ∎

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