SPECTRAL SYNTHESIS PROBLEMS ON HYPERGROUPS

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Dedicated to Professor Karl-Heinz Indlekofer on his seventieth anniversary

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Abstract. Spectral analysis and spectral synthesis deal with the description of translation invariant function spaces on topological Abelian groups. It turns out that the basic building bricks of these spaces are the exponential monomials, which store information on common eigenfunctions on the translation operators together with multiplicity. On commutative hypergroups the presence of translation operators makes it possible to formulate and study the basic problems of spectral analysis and synthesis. However, a general concept of exponential monomials is necessary to build up a theory along the lines of the group-case. Here we present some relevant problems and solutions on the subject.

1. Introduction

In this paper $\mathbb{C}$ denotes the set of complex numbers. By a *hypergroup* we always mean a commutative locally compact hypergroup. If K is a hypergroup, then $\mathcal{C}(K)$ denotes the locally convex topological vector space of all continuous

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complex valued functions defined on K, equipped with the pointwise operations and the topology of uniform convergence on compact sets. For each function f in $\mathcal{C}(K)$ we define $\check{f}$ by $\check{f}(x) = f(\check{x})$, whenever x is in K.

The dual of $\mathcal{C}(K)$ will be identified with $\mathcal{M}_c(K)$, the space of all compactly supported complex measures on K. This space is also identified with the set of all finitely supported complex valued functions on K in the obvious way that the pairing between $\mathcal{C}(K)$ and $\mathcal{M}_c(K)$ is given by the formula

$$\langle \mu, f \rangle = \int f \, d\mu \, .$$

Convolution on $\mathcal{M}_c(K)$ is defined by

$$\mu * \nu(x) = \int \mu(x * y) \, d\nu(y)$$

for any μ, ν in $\mathcal{M}_c(K)$ and x in K. Convolution converts the linear space $\mathcal{M}_c(K)$ into a commutative algebra with unit δ_e, e being the identity in K.

Convolution of measures in $\mathcal{M}_c(K)$ with arbitrary functions in $\mathcal{C}(K)$ is defined by the same formula

$$\mu * f(x) = \int f(x * y) \, d\mu(y)$$

for each μ in $\mathcal{M}_c(K)$, f in $\mathcal{C}(K)$ and x in K. The linear operators $f \mapsto \mu * f$ on $\mathcal{C}(K)$ are called *convolution operators*.

Translation with the element y in K is the operator mapping the function f in $\mathcal{C}(K)$ onto its *translate* $\tau_y f$ defined by $\tau_y f(x) = f(x * y)$ for any x in K. Clearly, τ_y is a convolution operator, namely, it is the convolution with the measure δ_y. A subset of $\mathcal{C}(K)$ is called *translation invariant*, if it contains all translates of its elements. A closed linear subspace of $\mathcal{C}(K)$ is called a *variety* on K, if it is translation invariant. For each function f the smallest variety containing f is called the *variety generated by* f and is denoted by $\tau(f)$. It is the intersection of all varieties containing f.

For basic knowledge on hypergroups the reader is referred to [2], [25].

In the case, when $K = G$ is a locally compact Abelian group in order to formulate the basic problems of spectral analysis and spectral synthesis one introduces the concept of exponential polynomials. Exponential polynomials are the elements of the function algebra generated by all continuous homomorphisms of G into the additive group of complex numbers and into the multiplicative group of nonzero complex numbers. If $G = \mathbb{R}$ is the set of real numbers with the Euclidean topology, then the exponential polynomials are precisely the functions of the form

$$f(x) = p_1(x)e^{\lambda_1 x} + p_2(x)e^{\lambda_2 x} + \cdots + p_n(x)e^{\lambda_n x}$$

with polynomials $p_1, p_2, \ldots, p_n$ and complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$. If the right hand side is a single term, then f is called an exponential monomial. In the general group situation the spectral analysis problem for a given variety is to find conditions, which imply that each nonzero subvariety of the given variety contains a nonzero exponential monomial, and the spectral synthesis problem for the variety is to study whether all exponential monomials in each subvariety of the given variety span a dense subspace in that subvariety. From this formulation it is clear that spectral synthesis for a nonzero variety implies spectral analysis for it. A pioneer result on spectral synthesis is due to L. Schwartz in [13] proving spectral synthesis for all varieties on $\mathbb{R}$. Classical results on spectral analysis and synthesis can be found in [9], [3], [11], [12]. For more recent results the reader is referred to [4], [6], [7], [8], [16], [17], [18], [19].

Defining the substitutes of continuous homomorphisms of a hypergroup K into the additive group of complex numbers and into the multiplicative group of nonzero complex numbers is immediate using their characterizing functional equations. The continuous function $m : K \to \mathbb{C}$ is called an *exponential*, if it is not identically zero and satisfies

$$m(x * y) = m(x)m(y)$$

for each x, y in K. Exponentials are common eigenfunctions of all translation operators. Clearly, every exponential generates a one dimensional variety, and, conversely, every one dimensional variety is generated by an exponential.

The continuous function $a : K \to \mathbb{C}$ is called an *additive function*, if it satisfies

$$a(x * y) = a(x) + a(y)$$

for each x, y in K. Additive functions form a linear space.

Unfortunately, in contrast to the group-case, exponentials on hypergroups may take the value 0, and, what is even worse, the product of two exponentials is not necessarily an exponential. Also, products of additive functions, behaving polynomial-like in the group case, have very different properties on hypergroups. A consequence of these inconvenient facts is that the function algebra generated by additive functions and exponentials may contain functions, which have not much to do with translations, hence they will be inappropriate for describing varieties. This means that we need to find some different way to generalize the concept of exponential monomials for hypergroups.

In [21], [24], [22] we made attempts to study spectral analysis and synthesis problems on polynomial and Sturm–Liouville hypergroups. We used the characteristic property of exponential polynomials on Abelian groups that they span finite dimensional varieties. Although this seems to be a good idea, but its disadvantage is that it provides a rather implicit definition, which is difficult to use on particular hypergroups. Instead, we follow a different way, which depends on difference operators and modified difference operators.

2. Exponential polynomials on hypergroups

Similarly to the group case, using translation one introduces *difference operators* $\Delta_y = \tau_y - \tau_e$ and higher order difference operators $\Delta_{y_1,y_2,\dots,y_n} = \Pi_{i=1}^n \Delta_{y_i}$ for each $y, y_1, y_2 \dots, y_n$ in K. Obviously, $\Delta_{y_1,y_2,\dots,y_n}$ is a convolution operator, namely

$$\Delta_{y_1,y_2,\dots,y_n} f = \Pi_{i=1}^n (\delta_{y_i} - \delta_e) * f\,,$$

where Π denotes convolution product.

A continuous function $f : K \to \mathbb{C}$ is called a *generalized polynomial*, if there is a natural number n such that

$$\Delta_{y_1,y_2,\dots,y_{n+1}} f = 0 \tag{2.1}$$

holds for each $y_1, y_2, \dots, y_{n+1}$ in K. In this case we say that f is *of degree at most* n and the *degree* of f is the smallest natural number n for which f is of degree at most n.

We realize immediately that every nonzero additive function is a generalized polynomial of degree 1. We note that applying this definition in the group-case it follows that the product of two nonzero additive functions is a generalized polynomial of degree 2, but this is definitely not the case on hypergroups, in general.

Modified difference operators defined as follows: given an exponential m, a function f on K and an element y in K we let

$$\Delta_{m;y} f(x) = f(x * y) - m(y)\, f(x)$$

for each x in K. The iterates are defined for any positive integer n and for each $y_1, y_2, \dots, y_n$ in K by

$$\Delta_{m;y_1,y_2,\dots,y_n} = \Pi_{i=1}^n \Delta_{m;y_i} .$$

Obviously, these operators are also convolution operators, namely

$$\Delta_{m;y_1,y_2,\dots,y_n} f = \Pi_{i=1}^n (\delta_{y_i} - m(y_i)\, \delta_e) * f$$

holds. In particular, for $m = 1$ we have $\Delta_{1;y_1,y_2,\dots,y_n} = \Delta_{y_1,y_2,\dots,y_n}$.

The continuous function $f : K \to \mathbb{C}$ is called a *generalized exponential monomial*, if there exists an exponential m and a natural number n such that

$$\Delta_{m;y_1,y_2,\dots,y_{n+1}} f(x) = 0 \tag{2.2}$$

holds for each $y_1, y_2, \dots, y_{n+1}$ in K. We say that f is *related* to the exponential m. At this moment we cannot claim that a generalized exponential monomial is related to a unique exponential. Clearly, every additive function and every exponential is a generalized exponential monomial.

A generalized polynomial is called a *polynomial*, if it generates a finite dimensional variety. Similarly, a generalized exponential polynomial is called an *exponential polynomial*, if it generates a finite dimensional variety. Accordingly, additive functions are polynomials and exponentials are exponential monomials.

Now we show that these definitions are compatible with the ones in the group-case. Formally, the concepts of exponential, additive function, generalized polynomial and polynomial are the same as in the group-case. The following theorem proves the same for generalized exponential monomials and exponential monomials.

Theorem 2.1. *Let K be a hypergroup, which is a locally compact Abelian group. Then a continuous function $f : K \to \mathbb{C}$ is a generalized exponential monomial if and only if it can be written in the form $f = p\,m$, where p is a generalized polynomial and m is an exponential. In particular, f is an exponential monomial if and only if in this form p is a polynomial.*

Proof. It is easy to check that if K is a group, then

$$\Delta_{m;y_1,y_2,\dots,y_{n+1}} f(x) = m(x + y_1 + y_2 + \dots + y_{n+1})\, \Delta_{y_1,y_2,\dots,y_{n+1}} (f \cdot \check{m})(x)$$

holds for each function $f : K \to \mathbb{C}$, exponential m and elements $x, y_1, \dots, y_{n+1}$ in K. As m is never zero, the condition

$$\Delta_{m;y_1,y_2,\dots,y_{n+1}} f(x) = 0$$

is equivalent to

$$\Delta_{y_1,y_2,\ldots,y_{n+1}}(f\cdot\check{m})(x)=0\,,$$

which is equivalent to the fact that $f\cdot\check{m}$ is a generalized polynomial, implying our statement. The last statement follows from the well-known characterization of exponential monomials on Abelian groups: they are exactly those generalized exponential monomials, which generate a finite dimensional variety (see e.g. [1], [5], [10], [14], [15], [26]). ■

3. Spectral analysis and spectral synthesis on commutative hypergroups

Having defined exponential monomials we are ready to define spectral analysis and spectral synthesis on commutative hypergroups exactly in the same way as in the group-case. Let K be a commutative hypergroup. We say that *spectral analysis* holds for a variety on K, if every nonzero subvariety of it contains a nonzero exponential monomial. We say that *spectral synthesis* holds for a variety on K, if in every subvariety of it the exponential monomials span a dense subspace. We say that *spectral analysis*, resp. *spectral synthesis holds on* K, if spectral analysis, resp. spectral synthesis holds for every nonzero variety on K.

In [21] and in [24] we proved that spectral synthesis holds on polynomial hypergroups. However, in those papers we used a formally different concept of exponential monomials. Now we show that the terminology we used there is compatible with our new definition.

Theorem 3.1. *Let K be a polynomial hypergroup in d variables generated by the family of polynomials $(Q_x)_{x\in K}$. For each polynomial $P:\mathbb{C}^d\to\mathbb{C}$ the function $x\mapsto P(\partial)Q_x$ is an exponential monomial.*

Proof. Although we have not defined the degree of exponential monomials, but in this proof we shall use this terminology in the following sense: we say that the exponential monomial f is of degree at most n, if it satisfies

$$\Delta_{m,y_1,y_2,\ldots,y_{n+1}}f(x)=0$$

for each $y_1,y_2,\ldots,y_{n+1}$ in K. Using this terminology we will show that if the polynomial P is of degree N, then the function $x\mapsto P(\partial)Q_x$ is an exponential monomial of degree at most N.

Obviously it is enough to prove the statement for polynomials of the form $P(\xi)=\xi^\alpha$, where α is a multi-index in $\mathbb{N}^n$ and $\xi^\alpha=\xi_1^{\alpha_1}\xi_2^{\alpha_2}\cdot\dots\cdot\xi_n^{\alpha_n}$. We prove by induction on $|\alpha|$ and the statement is obviously true for $|\alpha| = 0$. Let $m_\lambda(x) = Q_x(\lambda)$, then $m : K \to \mathbb{C}$ is an exponential and we have

$$
\begin{aligned}
P(\partial_\lambda)m_\lambda(x) &= \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdot\dots\cdot\partial_n^{\alpha_n}\big(m_\lambda(x)\cdot m_\lambda(y)\big) = \\
&= \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\partial_\lambda^\beta Q_x(\lambda)\partial_\lambda^{\alpha-\beta}Q_y(\lambda) = \\
&\sum_{\beta<\alpha}\binom{\alpha}{\beta}\partial_\lambda^\beta Q_x(\lambda)\partial_\lambda^{\alpha-\beta}Q_y(\lambda) + P(\partial_\lambda)Q_y(\lambda)\,.
\end{aligned}
\tag{3.1}
$$

With the notation $l = |\alpha|$ it follows, by (3.1)

$$
\begin{aligned}
\Delta_{m,y_1,y_2,\dots,y_{l+1}}\partial_\lambda^\alpha Q_x(\lambda) &= \Delta_{m,y_1,y_2,\dots,y_l}\big(\Delta_{m;y_{l+1}}\partial_\lambda^\alpha Q_x(\lambda)\big) = \\
&= \Delta_{m,y_1,y_2,\dots,y_l}\big(\partial_\lambda^\alpha m_\lambda(x*y_{l+1}) - \partial_\lambda^\alpha m_\lambda(y_{l+1})m_\lambda(x) = \\
&= \Delta_{m,y_1,y_2,\dots,y_l}\big(P(\partial_\lambda)m_\lambda(x*y_{l+1}) - P(\partial_\lambda)m_\lambda(y_{l+1})\big) = \\
&= \Delta_{m,y_1,y_2,\dots,y_l}\big(\sum_{\beta<\alpha}\binom{\alpha}{\beta}\partial_\lambda^\beta Q_x(\lambda)\partial_\lambda^{\alpha-\beta}Q_{y_{l+1}}(\lambda)\big) = \\
&= \sum_{\beta<\alpha}\binom{\alpha}{\beta}[\Delta_{m,y_1,y_2,\dots,y_l}\partial_\lambda^\beta Q_x(\lambda)]\partial_\lambda^{\alpha-\beta}Q_{y_{l+1}}(\lambda) = 0\,,
\end{aligned}
$$

as the function $x \mapsto \partial_\lambda^\beta Q_x(\lambda)$ is an exponential monomial of degree at most $|\alpha| - 1 < l$. This means that the function $x \mapsto P(\partial_\lambda)Q_x(\lambda)$ is a generalized exponential monomial of degree at most l and equation (3.1) implies that it generates a finite dimensional variety, hence it is actually an exponential monomial. ■

References

[1] **Anselone, P.M. and J. Korevaar,** Translation invariant subspaces of finite dimension, *Proc. Amer. Math. Soc.*, **15** (1964), 747–752.

[2] **Bloom, W.R. and H. Heyer,** *Harmonic Analysis of Probability Measures on Hypergroups*, de Gruyter Studies in Mathematics, de Gruyter, Berlin, New York, 1995.

[3] **Ehrenpreis, L.,** Mean periodic functions. I. Varieties whose annihilator ideals are principal, *Amer. J. Math.*, **77** (1955), 293–328.

[4] **Elliot, R.J.,** Two notes on spectral synthesis for discrete Abelian groups, *Math. Proc. Cambridge Phil. Soc.*, **61** (1965), 617–620.

[5] **Engert, M.,** Finite dimensional translation invariant subspaces, *Pacific. J. Math.*, **32** (1970), 333–343.

[6] **Gurevič, D.I.,** Counterexamples to a problem of L. Schwartz, *Funkcional. Anal. i Priložen.*, **9(2)** (1975), 29–35. (English translation: *Functional Anal. Appl.*, **9(2)** (1975), 116–120.)

[7] **Laczkovich, M. and G. Székelyhidi,** Harmonic analysis on discrete Abelian groups, *Proc. Amer. Math. Soc.*, **133(6)** (2005), 1581–1586.

[8] **Laczkovich, M. and L. Székelyhidi,** Spectral synthesis on discrete Abelian groups, *Math. Proc. Camb. Phil. Soc.*, **143(01)** (2007), 103–120.

[9] **Lefranc, M.,** L'analyse harmonique dans $\mathbb{Z}^n$, *C. R. Acad. Sci. Paris.*, **246** (1958), 1951–1953.

[10] **McKiernan, M. A.,** Equations of the form $H(x \circ y) = \sum_i f_i(x)g_i(y)$, *Aequationes Math.*, **16** (1977), 51–58.

[11] **Malgrange, B.,** Sur quelques propriétés des équations de convolution, *C. R. Acad. Sci. Paris*, **238** (1954), 2219–2221.

[12] **Malliavin, P.,** Impossibilité de la synthèse spectrale sur les groupes abéliens non compacts, *Inst. Hautes Études Sci. Publ. Math.*, **1959** (1959), 85–92.

[13] **Schwartz, L.,** Théorie génerale des fonctions moyenne-périodiques, *Ann. of Math.*, **48(4)** (1947), 857–929.

[14] **Stone, J.J.,** Exponential polynomials on commutative semigroups, *Appl. Math. and Stat. Lab. Technical Note, Stanford University*, **14** (1960)

[15] **Székelyhidi, L.,** Note on exponential polynomials, *Pacific J. Math.*, **103** (1982), 583–587.

[16] **Székelyhidi, L.,** A Wiener Tauberian theorem on discrete Abelian torsion groups, *Annales Acad. Paedag. Cracov. Studia Mathematica I.*, **4** (2001), 147–150.

[17] **Székelyhidi, L.,** The failure of spectral synthesis on some types of discrete Abelian groups, *Jour. Math. Anal. Appl.*, **291** (2004), 757–763.

[18] **Székelyhidi, L.,** Polynomial functions and spectral synthesis, *Aequationes Math.*, **70(1–2)** (2005), 122–130.

[19] **Székelyhidi, L.,** Spectral synthesis problems on locally compact groups, *Monatsh. Math.*, **161(2)** (2010), 223–232.

[20] **Székelyhidi, L.,** Functional equations on hypergroups, in: Th. M. Rassias (Ed.) *Functional Equations, Inequalities and Applications*, Kluwer Academic Publishers, 2003, 167–181.

[21] **Székelyhidi, L.,** Spectral Analysis and Spectral Synthesis on Polynomial Hypergroups, *Monatsh. Math.*, **141(1)** (2004), 33–43.

[22] **Székelyhidi, L.,** Functional equations on Sturm–Liouville hypergroups, *Math. Pannonica*, **17(2)** (2006), 169–182.

[23] **Székelyhidi, L.,** *Discrete Spectral Synthesis and Its Applications*, Springer Monographs in Mathematics, Springer, Dordrecht, The Netherlands, 2006.

[24] **Székelyhidi, L.,** Spectral Synthesis on multivariate polynomial hypergroups, *Monatsh. Math.*, **153** (2008), 145–152.

[25] **Székelyhidi, L.,** *Functional Equations on Hypergroups*, World Scientific Publishing Co. Pte. Ltd., Singapore, New Jersey, London, Hong Kong, 2012.

[26] **Székelyhidi, L.,** A characterization of exponential polynomials, submitted

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