Annales Univ. Sci. Budapest., Sect. Comp. 39 (2013) 415–438

ON A NUMERICAL MODEL FOR THE PIANOFORTE

Gisbert Stoyan, (Budapest, Hungary)

The work is devoted to the 70th birthday of Professor Indlekofer, remembering also an evening of four-finger joint play on his pianoforte.

Communicated by Ferenc Schipp

(Received December 22, 2012; accepted January 26, 2013)

Abstract. We investigate both theoretically and numerically a simple model for string instruments like the pianoforte emphasizing the formulation of appropriate discrete conditions in the form of difference equations at the junction point of string and frame.

1. Introduction

There exists a rather large number of papers on the mathematical modelling of string instruments, see the diploma thesis [9] and the literature cited therein: e.g., [1], [2], [5], [8]. These models are, generally, formulated as systems of partial differential equations (especially hyperbolic equations), an approach we shall follow below, too. For literature on the numerical solution of hyperbolic equations see, e.g., [12], [4], [14] and [6].

Key words and phrases: Conservation of energy, modelling the frame-string joint, weighted difference approximation, numerical experiments.
2010 Mathematics Subject Classification: 65M06, 65M12, 65F15.

2. The basic model

We shall consider the beam equation for the (cast-iron) frame (normalized to $[0,1]$):

$$
\begin{array}{lrcl}
(1) & \partial_t^2 u + a\partial_x^4 u & = & 0,\ \ 0 < x < 1,\ \ 0 < t \le T, \\
(2) & u(0,t) = u(1,t) & = & 0,\ \ \partial_x u(0,t) = \partial_x u(1,t) = 0,\ \ 0 \le t \le T, \\
(3) & u(x,0) = 0, \partial_t u(x,0) & = & 0,\ \ 0 \le x \le 1, \\
(4) & \text{joint point conditions:} & & u(x_j,t) = v(0,t), \partial_t u(x_j,t) = \partial_t v(0,t), \\
 & & & 0 \le t \le T,
\end{array}
$$

where $x_j \in (0,1)$ is the joint point of frame and string in which the solution might be less smooth so that equation (1) has to be understood in a generalized sense (to which we return later). Further, $a > 0$ is the stiffness of the beam and v the solution of the string equation:

$$
\begin{array}{lrcl}
(5) & \partial_t^2 v & = & b\partial_y^2 v,\ \ 0 < y < L,\ \ 0 < t \le T, \\
(6) & v(L,t) & = & 0, \qquad 0 < t \le T, \\
(7) & v(y,0) & = & v_0(y),\ \ \partial_t v(y,0) = v_1(y),\ \ 0 \le y \le L, \\
(8) & \text{compatibility conditions:} & & v_0(0) = 0,\ v_1(0) = 0.
\end{array}
$$

Here $b > 0$ is the square of the signal velocity of the string (depending on the density of its material, its cross section and the tension applied). The above equations do not yet determine the solution u, v of (1)-(8). E.g., one could prescribe the value of $v(0,t)$, $t > 0$. This also becomes visible if considering the conservation of energy − which should hold since there is no loss in the system.

3. Conservation of energy

In this section, we assume that all derivatives in (1), (5) are continuous in $[0,1] \times [0,T]$ (with the possible exception of the point x_j) and multiply (1) by $\partial_t u$ and twice integrate by parts on the intervals $[0,x_j]$ and $[x_j,1]$ (taking into account the boundary conditions (2)). This gives, for $t \in (0,T)$,

$$\frac{1}{2}\partial_t \int_0^1 [(\partial_t u)^2 + a(\partial_x^2 u)^2]\mathrm{d}x = a[\partial_t u \partial_x^3 u - \partial_t \partial_x u \partial_x^2 u]_{x_j-0}^{x_j+0}\,.$$

Similarly, from (5)-(6) we obtain by partial integration on $[0,L]$

$$\frac{1}{2}\partial_t \int_0^L [(\partial_t v)^2 + b(\partial_y v)^2]\mathrm{d}y = -b(\partial_t v \partial_y v)(0,t) = -b[\partial_t u(x_j,t)\partial_y v(0,t)].$$

Adding the two relations we have for the energy $E(t)$ of the frame-string system

$$(9)\ \partial_t E := \frac{1}{2}\partial_t \left\{ \int_0^1 [(\partial_t u)^2 + a(\partial_x^2 u)^2]\mathrm{d}x + \int_0^L [(\partial_t v)^2 + b(\partial_y v)^2]\mathrm{d}y \right\} =$$

$$(10)\quad = a[\partial_t u \partial_x^3 u - \partial_t \partial_x u \partial_x^2 u]_{x_j-0}^{x_j+0} - b[\partial_t u(x_j,t)\partial_y v(0,t)] =: V(t).$$

Therefore, for conservation of energy we need $V(t) = 0$, $0 < t \leq T$. If this holds true, we have from (3), (7) and (9)

$$E(t) = E(0) = \frac{1}{2}\int_0^L [v_1^2 + b(\partial_y v_0)^2]\mathrm{d}y = \text{const}\,.$$

Whether or not

$$(11)\qquad V(t) = a[\partial_t u \partial_x^3 u - \partial_t \partial_x u \partial_x^2 u]_{x_j-0}^{x_j+0} - b[\partial_t u(x_j,t)\partial_y v(0,t)] = 0$$

is satisfied obviously depends on the function spaces in which we search for a solution of (1)-(7). The appropriate spaces for generalized solutions are $u \in H^1\{0,T;H^2(0,1)\}$ and $v \in H^1\{0,T;H^1(0,1)\}$, see [15], chapter 23 (here H^k are the usual Sobolev spaces). For these spaces, $E(t)$ is well defined - but not $V(t)$. We consider the following possibilities:

1. For a solution $u \in H^2\{0,T;H^4(0,1)\}$, $v \in H^2\{0,T;H^2(0,1)\}$, $V(t)$ is well defined, moreover, the factor of a in V is zero. Namely then

$$\partial_x^{(k)} u \quad \text{is continuous at } x_j,\ \ k = 0,1,2,3, \quad 0 \leq t \leq T.$$

Mechanically: at x_j there is no hinge connection neither a support, see, e.g., [11], p. 218. Then, moreover, also

$$\partial_t \partial_x^{(k)} u \quad \text{is continuous at } x_j,\ \ k = 0,1, \quad 0 \leq t \leq T,$$

holds, and to have $V(t) \equiv 0$, $0 \leq t \leq T$, sufficient turns out to be $\partial_t u(x_j,t) = 0$, i.e. $u(x_j,t) = \text{const}$, but this would mean $u(x_j,t) = 0$ by (3). Then the beam remains in rest and the string oscillates independently (in dependence on its own initial conditions) of the beam. A second string connected to the beam and initially being in rest, would not start to cooscillate.

2. For the same spaces, the second possibility for the factor of b in $V(t)$ to be zero is $\partial_y v(0,t) = 0$. With this Neumann boundary condition, the string oscillates independently from the beam, but at least excitates the latter. This, however, would mean that the spaces of **1.** are not appropriate: the beam equation has to be considered on $[0,x_j) \cup (x_j,1]$ and continuity of say $\partial_x^3 u$ at x_j is not assured, so (11) is questionable. A second string connected to the beam would, of course, lead to a second break point on the beam but to have the string excitated by the beam would ask (among the classical boundary conditions) for a Dirichlet or Robin condition at the beginning of the string. Or, if $u \in H^2\{0, T; H^3(0,1) \cap C^3([0, x_j)) \cup C^3((x_j, 1])\}$, $v \in H^2\{0, T; H^2(0,1)\}$ and $\partial_t u(x_j, t) \neq 0$, then (11) gives rise to a Neumann condition for that t.

3. Instead of looking for $V(t) = 0$, consider also the case of classical solutions: $u \in C^2\{0, T; C^4(0,1)\}$, $v \in C^2\{0, T; C^2(0,1)\}$, when, obviously, $V(t)$ is well defined and equal to the b-part $V(t) = -b[\partial_t u(x_j, t)\partial_y v(0, t)]$, and for $V(t) \equiv 0$ compare with **1.** At the same time, from (4),

$$(\partial_t^2 u)(x_j, t) = (\partial_t^2 v)(0, t), \quad t \in [0, T],$$

is plausible both mathematically and physically and yields

$$-a(\partial_x^4 u)(x_j, t) = (\partial_y^2 v)(0, t), \quad t \in [0, T]. \tag{12}$$

In case of only piecewise differentiability of u with respect to x (i.e. v being twice differentiable on $[0, L]$ as above, but u four times differentiable on $[0, x_j]$ and $[x_j, 1]$), one might require

$$\frac{1}{2}a[\partial_x^4 u(x_j + 0, t) + \partial_x^4 u(x_j - 0, t)] + b\partial_x^2 v(0, t) = 0. \tag{13}$$

Conditions (12) or (13) would connected beam and string such that each of them can react on the other.

Let us mention that in [3] the problem of biharmonic interpolation was solved numerically where, along with the biharmonic equation, scattered data are given and to be interpolated. The author reports success but the problem is different from our's since here $u(x_j)$ is not given.

4. The difference approximation

For the numerical solution of our problem, we introduce, on the frame, for $N_f \geq 4$, the equidistant grids

$$\begin{aligned} \overline{\omega}_h^f &:= \{x_i = ih_f,\, 0 \leq i \leq N_f = 1/h_f\}, \\ \omega_h^f &:= \{x_i,\, 1 \leq i \leq N_f - 1\}, \quad \mathring{\omega}_h^f := \{x_i,\, 2 \leq i \leq N_f - 2\}, \end{aligned}$$

assuming that the (fixed) junction point belongs to ω_h^f: $x_j = j(h_f)h_f$.

Correspondingly, for the parts of the frame divided by the junction point $x_j(h_f)$, we have

$$\begin{aligned}
\overline{\omega}_h^- &:= \{x_i = ih_f,\, 0 \le i \le j(h_f)\},\ \overline{\omega}_h^+ := \{x_i,\, j(h_f) \le i \le N_f\},\\
\omega_h^- &:= \{x_i,\, 1 \le i \le j(h_f)-1\}, \quad \omega_h^+ := \{x_i,\, j(h_f)+1 \le i \le N_f - 1\},\\
\mathring{\omega}_h^- &:= \{x_i,\, 2 \le i \le j(h_f)-2\}, \quad \mathring{\omega}_h^+ := \{x_i,\, j(h_f)+2 \le i \le N_f - 2\}.
\end{aligned}$$

Likewise, on the string, for $N_s \ge 2$, we use the equidistant grids

$$\begin{aligned}
\overline{\omega}_h^s &:= \{y_i = ih_s,\, 0 \le i \le N_s = L/h_s\},\\
\omega_h^s &:= \{y_i,\, 1 \le i \le N_s\}, \quad \mathring{\omega}_h^s := \{y_i,\, 1 \le i \le N_s - 1\}.
\end{aligned}$$

Moreover, we shall need the time grids

$$\overline{\omega}_\tau := \{t_k = k\tau,\, 0 \le k \le M = T/\tau\}, \quad \omega_\tau := \{t_k,\, 1 \le k \le M\}.$$

For any of these grids, say $\mathring{\omega}_h \subset \omega_h$ with equidistant stepsize h, we also introduce the following scalar products:

$$(w,z)_{\mathring{\omega}_h} := \sum_{x_i \in \mathring{\omega}_h} w(x_i)z(x_i)h, \tag{14}$$

$$(w,z)_{\omega_h} := \sum_{x_i \in \omega_h} w(x_i)z(x_i)h. \tag{15}$$

Since we will not investigate theoretically (but only numerically) the accuracy of our approximations, we use the same symbols u and v for the solution of (1)-(8) and for their difference approximations, namely

$$u_i^k \approx u(x_i,t_k),\ x_i \in \overline{\omega}_h^f,\ \ v_i^k \approx v(y_i,t_k),\ y_i \in \overline{\omega}_h^s, \quad t_k \in \overline{\omega}_\tau\,.$$

We shall use the standard second and fourth order difference quotions on $\mathring{\omega}_h^s$, resp. $\mathring{\omega}_h^f$, see, e.g., [13] or [12],

$$v_{\overline{y}y} = v_{\overline{y}y,i}^k := \frac{1}{h_s^2}(v_{i+1}^k - 2v_i^k + v_{i-1}^k), \tag{16}$$

$$u_{\overline{x}x\overline{x}x} = u_{\overline{x}x\overline{x}x,i}^k := \frac{1}{h_f^4}\left(u_{i+2}^k - 4u_{i+1}^k + 6u_i^k - 4u_{i-1}^k + u_{i-2}^k\right). \tag{17}$$

For our time dependent equations, we need also corresponding approximations of the first and second order time derivative and of a symmetric average operator, see, e.g., [14]:

$$w_t = w^k_{t,i} := \frac{w^{k+1}_{t,i} - w^k_{t,i}}{\tau}, \tag{18}$$
$$w_{\bar t} = \frac{w^k_{t,i} - w^{k-1}_{t,i}}{\tau}, w_{\overset{\circ}{t}} = \frac{w^{k+1}_{t,i} - w^{k-1}_{t,i}}{2\tau},$$

$$w_{\bar t t} = w^k_{\bar t t,i} := \frac{1}{\tau^2}(\hat w_i - 2w_i + \check w_i), \tag{19}$$
$$\hat w_i := w_i^{k+1},\ w_i := w_i^k,\ \check w_i := w_i^{k-1},$$

$$w^{(\sigma)} = w_i^{(k,\sigma)} := \sigma \hat w_i + (1-2\sigma) w_i + \sigma \check w_i = w_i^k + \sigma\tau^2 w^k_{\bar t t}. \tag{20}$$

Using these notations (where $w = u, v$), we have the following approximations of (1) and (5):

$$u_{\bar t t} = -a u^{(\sigma)}_{\bar x x \bar x x}, \quad (x,t) \in [\omega_h^- \cup \omega_h^+] \times \omega_\tau, \tag{21}$$
$$u_{\bar t t} = -a\left(u^{(\sigma)}_{\bar x x \bar x x} + V_{fs}\right), \quad (x,t) \in \{x_j - h_f, x_j, x_j + h_f\} \times \omega_\tau, \tag{22}$$
$$v_{\bar t t} = b v^{(\sigma)}_{\bar y y}, \quad (y,t) \in \omega_h^s \times \omega_\tau, \tag{23}$$
$$v_0 = u_j, \quad t \in \omega_\tau. \tag{24}$$

To the terms V_{fs} in (22), which may depend on a, b and on our discrete solutions u, v, we return in Subsection 5.2 below.

The corresponding initial and boundary values (at the beam's two and the string's right end point) follow immediately below, but already here we see, compared to the continuous case, that the grid functions v in $y = 0$ and u in the three points x_{j-1}, $x_j = y_0$, x_{j+1} are determined insufficiently. These points shall be treated below in a separate section in analogy to Section 3.

The initial conditions will be replaced by

$$u^0 = 0, \qquad u^1 = 0, \quad x \in \overline{\omega}_h^f, \tag{25}$$
$$v^0 = v_0(y), \quad v^1 = v_1(y), \quad y \in \overline{\omega}_h^s. \tag{26}$$

Here, the values (25)-(26) on the first time level $t = t_1 = \tau$ give, at best, only a first order approximation and, for test aims, will be replaced by exact values of u and v on that time level.

From (4) and (6), the boundary conditions for the string are clearly, for any $t \in \omega_\tau$,

$$v_0 = u_j, \quad v_{N_s} = 0. \tag{27}$$

The question of the difference form of the boundary conditions (2) is decisive and solved by the following result.

Theorem 1. *The matrix corresponding to $u_{\overline{x}x\overline{x}x,i}$ for $x_i \in \overset{\circ}{\omega}{}_h^f$ and to the boundary conditions*

$$u_0 = 0, \qquad u_{N_f} = 0, \tag{28}$$

$$u_1 = (4u_0 + 4u_2 - u_3)/7, \quad u_{N_f-1} = (4u_{N_f} + 4u_{N_f-2} - u_{N_f-3})/7, \tag{29}$$

is, after elimination of the boundary conditions, symmetric and positive definite. Moreover, (28)-(29) *give second order approximations to* (2).

Proof. a) Second order: It is sufficient to consider the left-hand condition (29) and to omit the dependence of u on t, moreover, we may assume that $u(x)$ is six times differentiable with respect to x since then $y_{\overline{x}x\overline{x}x,i} = u^{(4)}(x_i) + O(h_f^2)$.

We start from a 4-parameter formula

$$\begin{aligned} \frac{1}{h_f}(\alpha u(x_0) + \beta u(x_1) + \gamma u(x_2) + \delta u(x_3)) &= \frac{1}{h_f}u(0)(\alpha + \beta + \gamma + \delta) + \\ + u'(0)(\beta + 2\gamma + 3\delta) &+ \frac{h_f}{2}u''(0)(\beta + 4\gamma + 9\delta) + \\ &+ \frac{h_f^2}{6}u'''(0)(\beta + 8\gamma + 27\delta) + O(h_f^3), \end{aligned} \tag{30}$$

because then the first three multiplyers on the right-hand side can be selected to give

$$\begin{aligned} \alpha + \beta + \gamma + \delta &= 0, \\ \beta + 2\gamma + 3\delta &= 1, \\ \beta + 4\gamma + 9\delta &= 0. \end{aligned} \tag{31}$$

A fourth condition will arise from the condition of symmetry of the matrix:

$$\gamma = -4\delta, \tag{32}$$

after which we obtain the (unique) solution of (31)-(32):

$$\alpha = -2, \quad \beta = \frac{7}{2}, \quad \gamma = -2, \quad \delta = \frac{1}{2}. \tag{33}$$

From here and (30) we have

$$\frac{1}{2h_f}(-4u(x_0) + 7u(x_1) - 4u(x_2) + u(x_3)) = u'(0) + \frac{h_f^2}{6}u'''(0) + O(h_f^3),$$

and remembering that $u'(0) = 0$, the discrete boundary condition becomes (29).

b) Symmetry: Let $\tilde{A}_{h_f}$ denote the matrix corresponding to $u_{\overline{x}x\overline{x}x,i}$ for $x_i \in \overset{\circ}{\omega}{}_h^f$ and to the boundary conditions (28), and, instead of (29), to the boundary condition $\alpha u_0 + \beta u_1 + \gamma u_2 + \delta u_3 = 0$, and the corresponding mirrored condition at $x = 1$. Since these boundary conditions are homogeneous, a common factor is not important. Also, only the first 5 lines of the matrix are of interest. Therefore we may write

(34)
$$\tilde{A}_{h_f} = \frac{1}{h_f^4} \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & \dots & \dots & 0 \\ \alpha & \beta & \gamma & \delta & 0 & \dots & \dots & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & \dots & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 & 0 & \dots \\ 0 & 0 & 1 & -4 & 6 & -4 & 1 & \dots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \in \mathbb{R}^{(N_f+1)\times(N_f+1)}.$$

Due to the second line, only the first four columns are important, and after eliminating the first line and column (and in the same way eliminating the last line and column) we get

$$\tilde{\tilde{A}}_{h_f} = \frac{1}{h_f^4} \begin{pmatrix} \beta & \gamma & \delta & 0 & \dots \\ -4 & 6 & -4 & 1 & \dots \\ 1 & -4 & 6 & -4 & \dots \\ 0 & 1 & -4 & 6 & \dots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \in \mathbb{R}^{(N_f-1)\times(N_f-1)}.$$

Assuming now $\beta \neq 0$ (remember that above $\beta = \frac{7}{2}$), we eliminate here the first and last line and column, denoting the result by A_{h_f}:

$$A_{h_f} = \frac{1}{h_f^4} \begin{pmatrix} 6 + \frac{4\gamma}{\beta} & -4 + \frac{4\delta}{\beta} & 1 & \dots \\ -4 - \frac{\gamma}{\beta} & 6 - \frac{\delta}{\beta} & -4 & \dots \\ 1 & -4 & 6 & \dots \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix} \in \mathbb{R}^{(N_f-3)\times(N_f-3)}. \tag{35}$$

Therefore, to get a symmetric matrix, we need (32).

c) Positive definiteness: For this investigation, it is more convenient to start from $\tilde{A}_{h_f}$ (see (34)) and $y = (y_0, \dots, y_N)^T$ and then taking into account (28)-(29).

To obtain our result, basic is the following identity of partial summation, see, e.g., [11], p. 257, or also [13] (proof of Theorem 11.14.), which we shall use for grid functions w and z defined on $\overline{\omega}_h^f$ of step size h_f, and referring to the corresponding scalar products (14)-(15) of Section 4:

$$(w_{\overline{x}x\overline{x}x}, z)_{\overset{\circ}{\omega}{}_h^f} = (w_{\overline{x}x}, z_{\overline{x}x})_{\omega_h^f} + [w_{\overline{x}x} z_{\overline{x}} - w_{x\overline{x}x} z]_1 + [w_{\overline{x}x\overline{x}} z - w_{\overline{x}x} z_x]_{N_f-1}\ . \tag{36}$$

Now consider the boundary terms and assume that w and z satisfy the boundary conditions (28)-(29), see also (40) below. Then

$$\begin{aligned}
&[w_{\overline{x}x}z_{\overline{x}} - w_{x\overline{x}x}z]_1 + [w_{\overline{x}x\overline{x}}z - w_{\overline{x}x}z_x]_{N_f-1} = \\
&= \Big[(w_{\overline{x}x,1} - \frac{1}{h_f}(w_{\overline{x}x,2} - w_{\overline{x}x,1})h_f)z_{\overline{x},1} + \\
&+ (\frac{1}{h_f}(w_{\overline{x}x,N_f-1} - w_{\overline{x}x,N_f-2})(-h_f) - w_{\overline{x}x,N_f-1})z_{x,N_f-1}\Big] = \\
&= [(2w_{\overline{x}x,1} - w_{\overline{x}x,2})z_{\overline{x},1} + (w_{\overline{x}x,N_f-2} - 2w_{\overline{x}x,N_f-1})z_{x,N_f-1}] = \\
&= \frac{h_f}{2}[(2w_{\overline{x}x,1} - w_{\overline{x}x,2})(2z_{\overline{x}x,1} - z_{\overline{x}x,2}) + \\
&+ (2w_{\overline{x}x,N_f-1} - w_{\overline{x}x,N_f-2})(2z_{\overline{x}x,N_f-1} - z_{\overline{x}x,N_f-2})].
\end{aligned}$$

This shows (together with (36)) once more the symmetry, and for $z = w = y$ gives

$$(y_{\overline{x}x\overline{x}x}, y)_{\overset{\circ}{\omega}{}_h^f} = (y_{\overline{x}x}, y_{\overline{x}x})_{\omega_h^f} + \frac{h_f}{2}\left[(2y_{\overline{x}x,1} - y_{\overline{x}x,2})^2 + (2y_{\overline{x}x,N_f-1} - y_{\overline{x}x,N_f-2})^2\right],$$

or, in other words,

$$(A_{h_f}y, y)_{\overset{\circ}{\omega}{}_h^f} = (y_{\overline{x}x\overline{x}x}, y)_{\overset{\circ}{\omega}{}_h^f} \geq (y_{\overline{x}x}, y_{\overline{x}x})_{\omega_h^f} = \|y_{\overline{x}x}\|^2_{\omega_h^f} =: \|y\|^2_{2,\omega_h^f}\,. \tag{37}$$

Now we can refer to the discrete embedding theorem from [11], p. 292, according to which, for grid functions w on $\overline{\omega}_h$ being 0 in the end points, i.e. $w_0 = w_N = 0$, there holds for any $\varepsilon > 0$

$$\sum_{i=1}^{N}(w_{x,i})^2 h =: \|w\|^2_{1,\omega_h} \leq \varepsilon\|w\|^2_{2,\omega_h} + \frac{1}{4\varepsilon}\|w\|^2_{\omega_h}. \tag{38}$$

Moreover, well known is the discrete embedding (see, e.g., [13], Lemma 11.5.)

$$\|w\|^2_{\omega_h} \leq \frac{1}{6}\|w\|^2_{1,\omega_h},$$

which, however, by using discrete Fourier expansion, can be improved to

$$\|w\|^2_{\omega_h} \leq \frac{1}{8}\|w\|^2_{1,\omega_h},$$

and here the 8 cannot be enlarged further. This means, together with (38), that

$$\|w\|^2_{\omega_h} \leq \frac{\varepsilon}{8}\|w\|^2_{2,\omega_h} + \frac{1}{32\varepsilon}\|w\|^2_{\omega_h},$$

or, for $32\varepsilon > 1$,

$$\|w\|^2_{\omega_h} \leq \frac{\varepsilon}{8(1-\frac{1}{32\varepsilon})}\|w\|^2_{2,\omega_h} = \frac{1}{64}\|w\|^2_{2,\omega_h}$$

when $\varepsilon = \frac{1}{16}$. From here (taking $w = y$, $h = h_f$ and (37)) then follows, finally,

$$\|y\|^2_{\overset{\circ}{\omega}{}^f_h} \leq \|y\|^2_{\omega^f_h} \leq \frac{1}{64}\|y\|^2_{2,\omega^f_h} \leq \frac{1}{64}(y_{\overline{x}x\overline{x}x}, y)_{\overset{\circ}{\omega}{}^f_h} = \frac{1}{64}(A_{h_f}y, y)_{\overset{\circ}{\omega}{}^f_h}\,.$$

This shows the positive definiteness of A_{h_f} for vectors $y \in \mathbb{R}^{N_f-3}$ and also the estimate of its first eigenvalue

$$\lambda^h_1(A_{h_f}) \geq 64. \tag{39}$$

■

Remarks. 1. The relations (29) in the presence of (28) can be rewritten as

$$\begin{aligned} u_1 = h_f u_{\overline{x},1} &= \frac{h_f^2}{2}\big(2u_{\overline{x}x,1} - u_{\overline{x}x,2}\big)\,, \\ u_{N_f-1} = -h_f u_{x,N_f-1} &= \frac{h_f^2}{2}\big(2u_{\overline{x}x,N_f-1} - u_{\overline{x}x,N_f-2}\big)\,. \end{aligned} \tag{40}$$

2. A third-order approximation to $u'(0) = 0$ is $u_{\overline{x},1} - \frac{h_f}{2}u_{\overline{x}x,1} + \frac{h_f^2}{3}u_{x\overline{x}x,1} = 0$, but, together with the corresponding approximation at $x = 1$, would not lead to a symmetrix matrix, according to the proof of Theorem 1.

3. For a numerical investigation of the eigenvalues of A_{h_f}, see Section 6.

5. The discrete junction point conditions

Whereas for the continuous case, after multiplication of the equations by $\partial_t u$ and $\partial_t v$, respectively, the frame-string system's energy and loss term is defined uniquely (see Section 3), in the discrete case already the selection of one of the three discrete time derivatives (18) is not clear, and so, for the choice of the discrete energy there are several possibilities which then define the loss terms. Which form of discrete equations can be recommended should be decided by the stability of the resulting approach, its behaviour (physical interpretability, and accuracy) in several test cases and its ease of programming.

A first option is to replace (4) by a convenient discrete equation. A second one consists in deriving the condition of discrete energy conservation which we follow now.

For this aim we cite the following general three-level difference scheme (remember (18)-(19))

$$By_{\overset{\circ}{t}} + Ry_{\bar{t}t} + Ay = \varphi, \quad k \geq 1, \quad \text{adott } y^0, y_t^0 \in H, \tag{41}$$

where we assume that B, R, A are operators acting in a Hilbert H space endowed with a scalarproduct $(\cdot, \cdot)$ (e.g. matrices in some $\mathbb{R}^{N\times N}$ with a weighted Euclidean scalarproduct for vectors from $\mathbb{R}^N$, like (14)-(15)), satisfying

$$A = A^T > 0, \quad R = R^T > 0, \quad B \geq 0. \tag{42}$$

We shall be interested especially in the case

$$R = I + \sigma\tau^2 A, \quad B = 0, \quad \varphi \equiv 0, \tag{43}$$

compare with (20). Here and in (42), $0, I$ are the zero and the identity operator, respectively. B may depend on t but A and R not. Observe that the operator acting on the new value y^{k+1} is $\frac{1}{2\tau}B + \frac{1}{\tau^2}R$, and, due to (42), invertible.

For (41) holds the following classical result [10], see also [14], Section 18.4.5 (Sect. 18.5 in the first edition) in which there appears the norm

$$\|y\|_A = (Ay, y)^{1/2} (=: \|y\|_{(1,h)}).$$

Proposition. The solution y of (41) under the conditions (42) has the properties below:

a) scalarly multiplying (41) by $2\tau y_{\overset{\circ}{t}}$, there follows the identity

$$2\tau(By_{\overset{\circ}{t}}, y_{\overset{\circ}{t}}) + E(\hat{y}, y) - E(y, \check{y}) = 2\tau(\varphi, y_{\overset{\circ}{t}}), \tag{44}$$

(45) where
$$E(\hat{y}, y) := (Ry_t, y_t) + \frac{1}{2}\left[(A\hat{y}, y) + (Ay, \hat{y})\right].$$

b) If

$$C := R - \frac{\tau^2}{4}A > 0, \tag{46}$$

then E can be interpreted as an energy since then

$$E(\hat{y}, y) = (Cy_t, y_t) + \frac{1}{4}\|\hat{y} + y\|_A^2 \geq 0 \tag{47}$$

and is zero only for $y = \hat{y} = 0$. If, however $C \geq 0$, then $E(\hat{y}, y)^{1/2}$ is, in general, only a seminorm.

c) In the case (43), (44) expresses the conservation of energy and the stability of (41):

$$E(\hat{y}, y) = E(y^{k+1}, y^k) = E(y^k, y^{k-1}) = \ldots = E(y^1, y^0).$$

d) In general, the solution of the scheme (41) satisfies

$$(E(y^{k+1},y^k))^{1/2} \le (E(y^1,y^0))^{1/2} + \left(\frac{1}{2\varepsilon}\sum_{m=1}^{k}\tau\|\varphi^m\|^2\right)^{1/2} \tag{48}$$

if there holds, along with (42), (46), also $B \ge \varepsilon I > 0$.

e) If, instead of (46), there holds

$$R - \frac{1+\varepsilon}{4}\tau^2 A \ge 0 \quad (\varepsilon = \mathrm{const} > 0), \tag{49}$$

then

$$E(\hat{y},y) \ge \frac{\varepsilon}{1+\varepsilon}\|\hat{y}\|_A^2\,. \tag{50}$$

f) For $B = 0$, the solution of (41) satisfies

$$(E(y^{k+1},y^k))^{1/2} \le (E(y^1,y^0))^{1/2} + \left(\sum_{m=1}^{k}\tau^2\|\varphi^m\|_{R^{-1}}^2\right)^{1/2} \tag{51}$$

and for $y^0 = y^1 = 0$, from (49)-(50),

$$\|y^{k+1}\|_A \le \sqrt{\frac{1+\varepsilon}{\varepsilon}}\sum_{m=1}^{k}\tau\|\varphi^m\|_{R^{-1}}\,, \tag{52}$$

where ε is the number of (49).

g) Independently of $B = 0$ and (49), we have the upper estimate

$$E(\hat{y},y) \le (\|y\|_A + \|y\|_R)^2\,. \tag{53}$$

5.1. The string part of the discrete energy

Now we turn to the string. For its discrete equations (23) in case $u_j^k = 0$ for all k we refer to the above proposition and get

$$\begin{gathered}\text{condition for stability with respect to initial values :}\\ \sigma = 0: \ \frac{b\tau^2}{h_s^2} =: \gamma_s^2 \le 1, \quad \sigma > \frac{1}{4}: \ \text{no condition,}\end{gathered} \tag{54}$$

$$\begin{gathered}\text{condition for stability with respect to right-hand sides :}\\ \sigma \ge \frac{1+\varepsilon}{4}, \quad \varepsilon > 0.\end{gathered} \tag{55}$$

In more detail:

In our case $B = 0$, $R = R_{h_s} = I_{h_s} + \sigma\tau^2 A_{h_s}$, $(A_{h_s}v)_i = -bv_{\overline{y}y,i}$, $1 \le i \le$ $\le N_s - 1$, $v_0 = v_{N_s} = 0$, (41) is equation (23), and because of $A = A_{h_s} =$ $= A_{h_s}^T > 0$ there also holds $R = R^T \ge I_{h_s} > 0$ if $\sigma \ge 0$. Then conditions (46) and (49) read

$$I_{h_s} + \left(\sigma - \frac{1}{4}\right)\tau^2 A_{h_s} \ge 0, \quad \text{and } I_{h_s} + \left(\sigma - \frac{1+\varepsilon}{4}\right)\tau^2 A_{h_s} \ge 0.$$

But, because of the well-known estimate $I_{h_s} > \frac{h_s^2}{4b} A_{h_s}$ we have

$$\begin{aligned} I_{h_s} + \left(\sigma - \frac{1+\varepsilon}{4}\right)\tau^2 A_{h_s} \quad &> \quad \left(\frac{h_s^2}{4\tau^2 b} + \sigma - \frac{1+\varepsilon}{4}\right)\tau^2 A_{h_s} \ge 0 \\ \text{if} \qquad \sigma \quad &\ge \quad \frac{1}{4}(1 + \varepsilon - \frac{1}{\gamma_s^2}) \ge 0. \end{aligned}$$

Here $|\gamma_s| := \frac{\sqrt{b}\tau}{h_s}$ is the Courant number. According to this the explicit scheme is conditionally stable (with respect to initial values and the right-hand side) in case

$$|\gamma_s| \le \frac{1}{\sqrt{1+\varepsilon}}.$$

Unconditional stability follows for

$$\sigma \ge \frac{1+\varepsilon}{4}$$

from (49), and $\|\varphi^m\|_{R^{-1}}$ in (51) and (52) can be replaced by $\|\varphi^m\|_{\overset{\circ}{\omega}{}_h^s}$ because of $R_{h_s}^{-1} \le I_{h_s}$, resulting by (52) and (53) into

$$\|y^{k+1}\|_A \le (\|y^0\|_A + \|y_t^0\|_R) + \sqrt{\frac{1+\varepsilon}{\varepsilon}} \sum_{m=1}^{k} \tau \|\varphi^m\|_{\overset{\circ}{\omega}{}_h^s}\,, \tag{56}$$

whereas stability with respect to the initial values only we obtain from (46), that is when $|\gamma_s| \le 1$. This is the well-known Courant–Friedrichs–Lewy condition. Namely, in this case (46) takes the form

$$C = C_{h_s} = I_{h_s} - \frac{\tau^2}{4} A_{h_s} > \left(\frac{h_s^2}{4b} - \frac{\tau^2}{4}\right) A_{h_s} \ge 0.$$

Since our problem is one-dimensional, moreover the y-interval $[0, L]$, from (52) we get an estimate in the maximum norm $\|\cdot\|_{C(\omega_{h_s})}$, too, applying the discrete embedding theorem, see [11], p. 289:

$$\|v\|_{C(\omega_{h_s})}^2 \le \frac{L}{4b}\|v\|_{A_{h_s}}^2.$$

Until now, we have considered the case $u_j^k = 0$, $k \geq 0$. To have a clear account of the role of the unknown values u_j^k for $k > 0$ (remember $u_j^0 = 0$) on the string, in our equations (23)-(24), (26)-(27), it seems straightforward to put, for any fixed time index $k \geq 0$,

$$v = \overline{v} + u_j \ell(y), \quad \ell(y) := 1 - y/L, \quad y \in \overline{\omega}_h^s \,, \tag{57}$$

resulting in equations with homogeneous boundary values:

$$\overline{v}_{\overline{t}t} = b\overline{v}_{\overline{y}y}^{(\sigma)} - (u_j)_{\overline{t}t}\ell \,, \quad (y,t) \in \omega_h^s \times \omega_\tau \,, \tag{58}$$
$$\overline{v}^0 = v_0(y), \;\; \overline{v}^1 = v_1(y) - u_j^1 \ell(y), \quad y \in \overline{\omega}_h^s \,, \tag{59}$$
$$\overline{v}_0 = \overline{v}_{N_s} = 0, \quad t \in \omega_\tau \,. \tag{60}$$

In the scalar product (14) on $\overset{\circ}{\omega}{}_h^s$ and the norm belonging to it we then would have, for any $k \geq 1$ and from (44),

$$E(\hat{\overline{v}}, \overline{v}) - E(\overline{v}, \check{\overline{v}}) = -2\tau (u_j)_{\overline{t}t} (\ell, \overline{v}_{\overset{\circ}{t}}). \tag{61}$$

From here and above, various estimates could be obtained for the string energy. However, our aim is to get an expression for the joint energy of the frame-string system and there to let disappear the expressions connected to u_j by finding an additional equation corresponding to the junction point (and referring to u and v). Then, the quadratic expressions of E and the term $(\ell, \overline{v}_{\overset{\circ}{t}})$ in (61) would be difficult to handle.

Therefore, we do the following: We introduce $\overline{v}_i := v_i$, $x_i \in \omega_h^s$, $\overline{v}_0 = 0$, so that v and $\overline{v}$ differ only in one point (where $v_0 = u_j$), and in (23) we take u_j out of $bv_{\overline{y}y}^{(\sigma)}$ considering it a known right-hand side:

$$\overline{v}_{\overline{t}t} = \overline{v}_{\overline{t}t,i} = b\overline{v}_{\overline{y}y,i}^{(\sigma)} + \delta_{1i} \frac{b}{h_s^2} u_j^{(\sigma)} = b\overline{v}_{\overline{y}y}^{(\sigma)} + \varphi_s, \quad (\varphi_s)_i := \delta_{1i} \frac{b}{h_s^2} u_j^{(\sigma)}.$$

This means that on $\overline{v}$ there acts the matrix A_{h_s} and we can apply the above results based on Theorem 1. Then (44) gives:

$$E_s(\hat{\overline{v}}, \overline{v}) = E_s(\overline{v}, \check{\overline{v}}) + 2\tau (\varphi_s, \overline{v}_{\overset{\circ}{t}})_{\overset{\circ}{\omega}{}_h^s}.$$

5.2. The beam part of the discrete energy

For the discrete beam equations (21), after eliminating the boundary conditions (28)-(29), we can refer to the above proposition and get

condition for stability with respect to initial values :

$$\sigma = 0 : \quad \frac{a\tau^2}{h_f^4} =: \gamma_f^2 \leq 1, \quad \sigma > \frac{1}{4} : \text{ no condition,} \tag{62}$$

condition for stability with respect to right-hand sides :

$$\sigma \geq \frac{1+\varepsilon}{4}, \quad \varepsilon > 0. \tag{63}$$

We emphasize that the first condition (62), for the explicit scheme, is very restrictive.

In more detail:

Here $B = 0$, $R = R_{h_f} = I_{h_f} + \sigma\tau^2 A_{h_f}$, $(A_{h_f} u)_i = a u_{\overline{x}x\overline{x}x,i}$, $2 \leq i \leq$ $\leq N_f - 2$ (after taking into account (28)-(29)), (41) is (21)-(22), and because of $A = A_{h_f} = A_{h_f}^T > 0$ there also holds $R = R^T \geq I_{h_f} > 0$ if $\sigma \geq 0$. Then conditions (46) and (49) read

$$I_{h_f} + \left(\sigma - \frac{1}{4}\right)\tau^2 A_{h_f} \geq 0, \quad \text{and } I_{h_f} + \left(\sigma - \frac{1+\varepsilon}{4}\right)\tau^2 A_{h_f} \geq 0.$$

But, because of $I_{h_f} > \frac{h_f^4}{16a} A_{h_f}$ (see Section 6) we have

$$\begin{array}{rcl} I_{h_f} + \left(\sigma - \frac{1+\varepsilon}{4}\right)\tau^2 A_{h_f} & > & \left(\frac{h_f^4}{16\tau^2 a} + \sigma - \frac{1+\varepsilon}{4}\right)\tau^2 A_{h_f} \geq 0 \\ \text{if} \qquad \sigma & \geq & \frac{1}{4}(1+\varepsilon - \frac{1}{\gamma_s^2}) \geq 0. \end{array}$$

Here $|\gamma_f| := \frac{\sqrt{a}\tau}{h_f^2}$ is the Courant number. According to this the explicit scheme is conditionally stable (with respect to initial values and the right-hand side) in case

$$|\gamma_f| \leq \frac{1}{\sqrt{1+\varepsilon}}.$$

Unconditional stability follows for

$$\sigma \geq \frac{1+\varepsilon}{4}$$

from (49), and $\|\varphi^m\|_{R^{-1}}$ in (51) and (52) can be replaced by $\|\varphi^m\|_{\overset{\circ}{\omega}{}_h^f}$ because of $R_{h_f}^{-1} \leq I_{h_f}$, resulting from (52) and (53) into

$$\text{(64)} \qquad \|y^{k+1}\|_A \leq (\|y^0\|_A + \|y_t^0\|_R) + \sqrt{\frac{1+\varepsilon}{\varepsilon}} \sum_{m=1}^{k} \tau \|\varphi^m\|_{\overset{\circ}{\omega}{}_h^f},$$

whereas stability with respect to the initial values only we obtain from (46), that is when $|\gamma_f| \leq 1$. In this case (46) takes the form

$$C = C_{h_f} = I_{h_f} - \frac{\tau^2}{4} A_{h_f} > \left(\frac{h_f^4}{16a} - \frac{\tau^2}{4} \right) A_{h_f} \geq 0.$$

Since our problem is one-dimensional, moreover the x-interval $[0,1]$, from (52) we get an estimate in the maximum norm $\|\cdot\|_{C(\omega_{h_f})}$, too, applying a discrete embedding theorem, see in this case the estimations in the proof of Theorem 1 after (38), leading for $\varepsilon = \frac{1}{16}$ also to

$$\|w\|_{1,\omega_h^f}^2 \leq \frac{1}{8} \|w\|_{2,\omega_h^f}^2 = \frac{1}{8a} \|v\|_{A_{h_f}}^2,$$

(we remark that in the cited theorem A_{h_f} was defined on the basis of $u_{\overline{x}x\overline{x}x}$ and not on $au_{\overline{x}x\overline{x}x}$ like here) and use then, e.g., [13], Section 11.4.6., Lemma 11.5., to obtain

$$\|v\|_{C(\omega_{h_f})} \leq \frac{1}{4\sqrt{2a}} \|v\|_{A_{h_f}}.$$

In the beam case, we have therefore from (44) and (21)-(22)

$$E_b(\hat{u}, u) = E_b(u, \check{u}) + 2\tau (\varphi_f, u_{\overset{\circ}{t}})_{\overset{\circ}{\omega}{}_h^f},$$

where

$$\varphi_f = -aV_{fs}, \quad x \in \{x_j - h_f, x_j, x_j + h_f\} \text{ , and } \varphi_f = 0 \text{ else.}$$

5.3. The frame-string system's discrete energy

Summarizing the results of Sections 5.2 and 5.1 and indicating also the dependence of the right-hand sides on the solution u, we find

$$E_b(\hat{u}, u) + E_s(\hat{\overline{v}}, \overline{v}) = E_b(u, \check{u}) + E_s(\overline{v}, \check{\overline{v}}) + 2\tau \Big((\varphi_f(u), u_{\overset{\circ}{t}})_{\overset{\circ}{\omega}{}_h^f} + (\varphi_s(u_j), \overline{v}_{\overset{\circ}{t}})_{\overset{\circ}{\omega}{}_h^s} \Big)$$

and shall require

$$(\varphi_f(u), u_{\overset{\circ}{t}})_{\overset{\circ}{\omega}{}_h^f} + (\varphi_s(u_j), \overline{v}_{\overset{\circ}{t}})_{\overset{\circ}{\omega}{}_h^s} = 0$$

since the frame-string system should be lossless. Observe that the right-hand sides φ_f, φ_s are defined only locally. Especially, if $V_{fs} \neq 0$ for $x = x_j$ only then the last equation reduces to

$$-aV_{fs}u_{\overset{\circ}{t},j}h_f + \frac{b}{h_s}u_j^{(\sigma)}\overline{v}_{\overset{\circ}{t},1} = 0$$

where we know that $\overline{v}_{\overset{\circ}{t},1} = v_{\overset{\circ}{t},1}$. Setting here

$$V_{fs} := -1, \text{ if } v_{\overset{\circ}{t},1} \geq 0, \quad V_{fs} := 1, \text{ if } v_{\overset{\circ}{t},1} < 0,$$

we would obtain the following equation for $\hat{u}_j$:

$$\hat{u}_j = \check{u}_j - \frac{\frac{2b\tau}{ah_sh_f}|v_{\overset{\circ}{t},1}|}{1 + \frac{2\sigma b\tau}{ah_sh_f}|v_{\overset{\circ}{t},1}|}\left((1-2\sigma)u_j + 2\sigma\check{u}_j\right).$$

This equation would change $\hat{u}_j$ in dependence on $u_j, \check{u}_j$ and $v_{\overset{\circ}{t},1}$ but would have the drawback to give $\hat{u}_j = 0$ if $u_j = \check{u}_j = 0$.

Instead we take (looking also at the physical dimension of V_{fs} in (22))

$$V_{fs} = \frac{u_j^{(\sigma)}}{h_f^4}, \; x = x_j, \quad V_{fs} = 0 \quad \text{else,} \tag{65}$$

$$\text{resulting into} \quad \frac{a}{h_f^3}u_{\overset{\circ}{t},j} = \frac{b}{h_s}v_{\overset{\circ}{t},1}, \quad \text{i.e.} \quad \gamma_f^2 h_f u_{\overset{\circ}{t},j} = \gamma_s^2 h_s v_{\overset{\circ}{t},1}. \tag{66}$$

Definition (65) will be used in (22), whereas (66) together with (24), for $t \in \omega_\tau$.

Observe that (65) will neither disturb the symmetry of the matrix corresponding to the right-hand side in (22) nor does is endanger the stability of the scheme, but (66) constitutes, in fact, a third-kind boundary condition of the form $-bv_{\overline{x},1} + (c-d)v_0 = e$ to (23) and ensures stability in case

$$d = \frac{b}{h_s} \leq \frac{a}{h_f^3} = c, \quad \text{i.e. } h_f^3 \leq \frac{a}{b}h_s.$$

6. The eigenproblem

To analyse the matrix (35)-(33) arising from the difference approximation of the stationary beam without a junction point, we consider also the fourth-order boundary value eigenproblem

$$u^{(4)} = \lambda u, \; 0 < x < 1, \quad u(0) = u(1) = 0, \; u'(0) = u'(1) = 0. \tag{67}$$

Its difference approximation on the equidistant grid $\overline{\omega}_h := \{x_i = ih, 0 \le i \le \le N = 1/h\}$ and $\mathring{\omega}_h := \{x_i, 2 \le i \le N-2\}$ is (remember the definition (17))

(68) $$y_{\overline{x}x\overline{x}x} = \lambda^h y, \;\; x \in \mathring{\omega}_h,$$

(69) $$y_0 = y_N = 0, \;\; y_1 = (4y_0 + 4y_2 - y_3)/7, \;\; y_{N-1} = (4y_N + 4y_{N-2} - y_{N-3})/7.$$

The symmetric and positive definite matrix representing this discrete eigenvalue problem and its boundary conditions appeared already in Section 5.2 containing the parameters $\alpha, \beta, \gamma, \delta$. We insert the specific values of these parameters and write now A_h instead of A_{h_f}:

(70)

$$A_h = \frac{1}{h^4}\begin{pmatrix} \frac{26}{7} & -\frac{24}{7} & 1 & 0 & \dots & \dots & \dots & 0 \\ -\frac{24}{7} & \frac{41}{7} & -4 & 1 & 0 & \dots & \dots & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & \dots & \dots & 0 & 1 & -4 & \frac{41}{7} & -\frac{24}{7} \\ 0 & \dots & \dots & \dots & 0 & 1 & -\frac{24}{7} & \frac{26}{7} \end{pmatrix} \in \mathbb{R}^{(N-3)\times(N-3)}.$$

From here, it is immediately clear that for the spectral radius of A_h there holds

$$\rho(A_h) \le \frac{16}{h^4} = \|A_h\|_\infty.$$

When approximating the second order derivative using the classical three-point difference quotient, the corresponding eigenvalues converge from below to the exact ones for $h \to 0$. Here, for the difference approximation of (67), we experience the same behaviour. The next table shows this for the smallest eigenvalue $\lambda_1^h(A_h)$ rounded to 4 digits after the decimal point:

h	1/20	1/40	1/80	1/160	1/320	1/640
λ_1^h	490.5717	498.0001	499.9198	500.4027	500.5236	500.5534

From here, we conclude also that the convergence of $\lambda_1^h \to \lambda_1$ is of second order.

When solving (67) analytically in the usual way, obtaining the solution by a combination of four exponential functions, you get the eigenvalues as roots of the equation $1 - \cos(\phi)\cosh(\phi) = 0$ (and $\phi = 0$ is not an eigenvalue since the corresponding function satisfying (67) is identically zero).

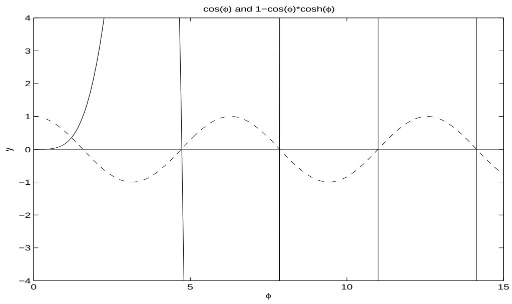

Figure 1. The zeros of $1 - \cos(\phi)\cosh(\phi)$. (Because the y-interval shown is restricted to $|y| \leq 4$, parts of this oscillating function are seen as nearly vertical straight lines.)

The next table exhibits the first zeros of $1 - \cos(\phi)\cosh(\phi)$, for comparison $\frac{2i+1}{2}\pi$ (see also Fig. 1) and the corresponding eigenvalues of (67), along with the first eigenvalues of A_h:

i	ϕ_i	$\frac{2i+1}{2}\pi$	$\lambda_i = \phi_i^4$	$\lambda_i(A_h), h = 1/640$
$i = 1$	4.7300407449	$\underline{4.7}123889804$	500.5639	500.5534
$i = 2$	7.8532046241	$\underline{7.853}9816340$	3803.5371	3803.3684
$i = 3$	10.9956078380	$\underline{10.995}5742876$	14617.6301	14616.5186
$i = 4$	14.1371654913	$\underline{14.13716}69412$	39943.7990	39939.1724

In Theorem 1 we have proved that A_h is positive definite, see (39): $(A_h y, y)_{\overset{\circ}{\omega}_h} \geq 64\|y\|^2_{\overset{\circ}{\omega}_h} > 0$, $0 \neq y \in \mathbb{R}^{N-3}$. Remark that $(\cdot,\cdot)_{\overset{\circ}{\omega}_h}$ is the Euclidean scalar product on $\mathbb{R}^{N-3}$ weighted by h, see (14). This estimate conforms to the λ_1^h-values shown in the above table, but from there and from the additional result that $\lambda_1^h \approx 460.04495$ for $h = 1/9$, $N = 9$ (the smallest number N admitting a full line – of $y_{\bar{x}x\bar{x}x,i}$, not touched by the elimination of the boundary values – when $i = 4$ in the original matrix), $\lambda_1^h \geq 460$ seems possible.

7. Numerical experiments

In our numerical experiments, we shall consider not only the explicit schemes (21)-(24), but besides $\sigma = 0$ also $\sigma = 1/3$. Remember that the latter value guarantees unconditional stability (see Sections 5.1 and 5.2, assuming for the beam equation that the spatial approximation is accomplished by our symmetric matrix) for the approximation of the string and beam equations — without a junction point. For the explicit scheme, however, the Courant numbers must not exceed 1.

First we check the accuracy on an analytical solution to our equations (1)-(8) in the case of a frame with two strings, at $x_{j,1} = \frac{1}{2}$ and $x_{j,2} = \frac{1}{3}$:

$$\begin{aligned}
\text{frame :} \quad & a = 1, \\
u \quad &= \quad \cos\alpha t \cos 2\pi x, \quad 0 \le x \le 1,\, 0 \le t \le T, \quad \alpha := 4\pi^2, \\
\text{string 1 :} \quad & b_1 = \left(\frac{24\pi}{5}\right)^2, \\
v^{(1)}(y,t) \quad &= \quad 2\cos\alpha t \cos \frac{4}{3}\pi(1 - \frac{5}{8}y), \quad 0 \le y \le 1,\, 0 \le t \le T, \\
\text{string 2 :} \quad & b_2 = \left(\frac{12\pi}{5}\right)^2, \\
v^{(2)}(z,t) \quad &= \quad \cos\alpha t \cos \frac{2}{3}\pi(1 + \frac{5}{2}z), \quad 0 \le z \le \frac{1}{2},\, 0 \le t \le T.
\end{aligned}$$

Let us emphasize that not only the compatibility conditions (8) are satisfied in both junction points but also the second time derivatives are equal there, so that (12) holds, too.

For the explicit scheme we are using, in the junction points, a discretized version of (12) where $\partial_x^4 u \approx u_{\overline{x}x\overline{x}x}$, but for $\partial_x^2 v \approx v_{\overline{x}x}$ (where $v = v^{(1)}(y,t)$ or $v = v^{(2)}(z,t)$, and $h = h_y$ or $h = h_z$, respectively), one-sided approximations of accuracy $O(h^n)$ are employed:

n=1 :

$$(\partial_x^2 v)(0,t) = v_{\overline{x}x,1} + O(h) = \frac{1}{h^2}(v_0 - 2v_1 + v_2) + O(h),$$

n=2 :

$$\begin{aligned}
(\partial_x^2 v)(0,t) &= 2v_{\overline{x}x,1} - v_{\overline{x}x,2} + O(h^2) \\
&= \frac{1}{h^2}(2v_0 - 5v_1 + 4v_2 - v_3) + O(h^2).
\end{aligned}$$

Remark that for both approximations of (12), the coefficient of the value $u_j^k \approx u(x_j, t_k)$ in the joint point is positive, namely (for $h = h_y$ or $h = h_z$)

$$\begin{aligned}
\left(\frac{6a}{h_x^4} + \frac{nb}{h^2}\right) u_j^k = \frac{a}{h_x^4}\left[-u_{j-2}^k + 4(u_{j-1}^k + u_{j+1}^k) - u_{j+2}^k\right] \\
+ \frac{b}{h^2}\left[(n^2+1)v_1^k - n^2 v_2^k + (n-1)v_3^k\right].
\end{aligned} \tag{71}$$

As anticipated, the results for both versions $n = 1, 2$ differ in favour of $n = 2$, but not significantly in the convergence order. Therefore, we show in the next table the maximal errors (rounded to 6 digits) for $n = 1$. The spatial step sizes for the frame (h_x) and the strings (h_y, h_z) were equal and $T = 0.2$. The time step τ was then automatically chosen to satisfy the first condition (62):

h_x	1/18	1/36	1/72	1/144	1/288
$n = 1$:					
frame	0.036266	0.009255	0.002297	0.000569	0.000142
string 1	0.067073	0.017422	0.004238	0.001033	0.000255
string 2	0.036677	0.009639	0.002328	0.000569	0.000142
q_f	-	3.92	4.03	4.04	4.01
q_1	-	3.84	4.11	4.10	4.05
q_2	-	3.81	4.14	4.09	4.01
$n = 2$:					
q_f	-	7.55	5.44	4.69	4.08
q_1	-	6.95	5.94	5.44	4.90
q_2	-	5.46	5.56	5.46	4.79

We add yet that the final maximal errors for $n = 2$, $h_x = 1/288$ were by one order of magnitude better than for $n = 1$:

n=1: frame: 1.4197-4, string 1: 2.5470-4, string 2: 1.4162-4,

n=2: frame: 1.7927-5, string 1: 3.2281-5, string 2: 2.0656-5.

In the table, the numbers q give the quotient of the present and the previous computation (q_f for the frame, q_1, q_2 for the two strings), indicating the convergence rate which in all cases seems to be of second order. The number of subintervals of the frame was chosen so as to be divisible by 3 and 2. By the way, the computing time for the last case $h_x = 1/288$ was about 9 hours (independently of n).

Next we show maximal errors and numerical convergence rates under similar circumstances as before ($h_x = h_y = h_z$, $T = 0.2$) but now for $\tau = T \cdot (2h_z)$ and, more importantly, the schemes were now implicit with $\sigma = 1/3$ and realized by matrices corresponding to the implicit discrete systems for the frame and the two strings, separately. After each time step, the values $u^k_{j,1}$, $u^k_{j,2}$ at the junction points on the frame were calculated by (71), $n = 1$, anew (based on the new values of $u, v^{(1)}, v^{(2)}$ on the frame and the strings, respectively) and used to overwrite also $v_0^{(1,k)}$, $v_0^{(2,k)}$:

h_x	1/18	1/36	1/72	1/144	1/288	1/576	1/1152
frame	3.697-2	4-191-3	1.194-3	4.487-4	1.508-4	4.307-5	1.113-5
string 1	0.3590	8.320-2	1.724-2	4.402-3	1.088-3	2.700-4	6.778-5
string 2	0.1912	0.2498	0.1459	8.414-2	3.912-2	1.930-2	9.555-3
q_f	-	8.82	3.51	2.66	2.98	3.50	3.87
q_1	-	4.31	4.83	3.92	4.04	4.03	3.98
q_2	-	0.77	1.71	1.73	2.15	2.03	2.02

Here, the convergence rate q_2 shows first order behaviour, but the computations of the last case $h_x = 1/1152$ took less than one minute using the implicit schemes.

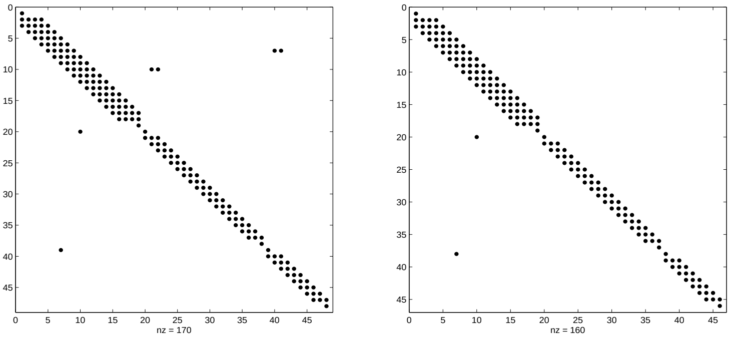

Figure 2. left: the compound matrix corresponding to (21)-(27), (71), right: the compound matrix corresponding to (21)-(27), (65)-(66).

We now compare the possibility to connect the frame and the two strings by either condition (71), $n = 1$, (a discretized version of (12)) or (65)-(66), using the implicit scheme, again for $\sigma = 1/3$. In both cases we get one compound matrix, and, for the frame, apply the boundary conditions of Theorem 1. First we show, see Fig. 2, the two matrices (one for connection (71) and one for (65)-(66)), multiplying the new, searched values of the solutions u, $v^{(1)}, v^{(2)}$ (including also the boundary values). There is an essential difference between the two parts of the figure: in the right, the rows and columns corresponding to $v_0^{(\ell)} (= u_j)$ of the strings, $\ell = 1, 2$, have been deleted, in other words: the discrete string equations start from v_1, of course taking into account (66), and only when drawing the whole string, the corresponding v_0-value is added.

The values of a and $b^{(1)}, b^{(2)}$ are those of the above exact solutions, and $h_x = h_y = h_z = 1/18$, $\tau = 0.2 \cdot (2h_z)$.

When using the considered two compound matrices for larger discretization numbers, decisive becomes an appropriate scaling to reach smaller (estimates of the) condition numbers. In the table below we report the estimated condition numbers. Here, the following scaling has been found appropriate for the rows containing the boundary conditions or the connections (71) or (65)-(66), by looking at the matrices and the results of a series of experiments:

rows corresponding to first kind boundary conditions are multiplied by $\sqrt{a\tau}/h_x^2$ (the factor for (28)-(29)), and by $b^{(\ell)}\tau^2/h_\ell^2$, where $\ell = 1, 2$, $h_1 = h_y$, $h_2 = h_z$, respectively,

the rows corresponding to (71) are multiplied by $\sigma\tau^2$,

the rows corresponding to (66) are multiplied by $2\sigma\tau^3/h_\ell$, $\ell = 1, 2$,

rows corresponding to $u_{j,\ell} = v_0^{(\ell)}$, $\ell = 1, 2$, are multiplied by $b^{(\ell)}\sigma/h_\ell^2$.

In the table below, cond1 shows the estimates of the condition numbers corresponding to the left matrix in Fig. 2 (but for different $h_x = h_y = h_z$, $\tau = 0.2 \cdot (2h_z)$), and cond2 denotes the estimates corresponding to the right matrix there. Further, cond3 gives the estimates for the frame matrix alone. Finally, cond4 denotes the condition number estimate for the case (rejected in Section 2) that there is no special condition for the junction points at all, only the corresponding frame values are taken over as boundary value for the strings (but otherwise applying the above scaling).

h_x	1/18	1/36	1/72	1/144	1/288	1/576	1/1152
cond1 (71)	7.32+2	2.17+3	7.54+3	2.80+4	1.06+5	4.09+5	1.60+6
cond2 (65)-(66)	2.57+2	1.23+3	5.55+3	2.48+4	1.00+5	4.01+5	1.45+6
cond3 frame only	3.00+2	1.23+3	5.01+3	2.00+4	7.96+4	3.23+5	1.29+6
cond4, no condition	1.66+3	1.60+4	1.60+5	1.76+6	1.96+7	2.20+8	2.48+9

This table may convince the Reader that it is worth, also numerically, to think on special junction point conditions.

8. Conclusion

Though our paper does not give a final solution of the problem to model mathematically and numerically the junction point between frame and string, it shows several possibilities which are to be investigated further. As a by-product, a theorem is proved on difference boundary conditions for the discrete frame equation which make the whole matrix symmetric and positive definite.

Acknowledgement. The author has been supported by the Faculty of Informatics, ELTE Budapest, under contract FEOR 2410.

References

[1] **Bank B.,** *Physics-based sound synthesis of string instruments including geometric nonlinearities*, PhD thesis, TU Budapest, Dept. Measurements and Information Systems, 2006.

[2] **Berthaut, J., M.N. Ichchou and L. Jézéquel,** Piano soundboard: structural behavior, numerical and experimental study in the modal range, *Applied Acoustics*, **64(11)** (2003), 1113–1136.

[3] **Gáspár C.,** Multi-level biharmonic and bi-Helmholtz interpolation with application to the boundary element method, *Engineering Analysis with Boundary Elements*, **24** (2000), 559–573.

[4] **Grossmann, C. and H.-G. Roos,** *Numerische Behandlung partieller Differentialgleichungen*, 3rd revised and expanded ed., Teubner Studienbücher Mathematik. Teubner, Wiesbaden 2005 (German). English translation: C. Grossmann, H.-G. Roos and M. Stynes, *Numerical Treatment of Partial Differential Equations*, Springer-Verlag, Berlin 2007.

[5] **Mamou-Mani, A., J. Frelat and C. Benainou,** Numerical simulation of a piano soundboard under downbearing, *J. Acoust. Soc. Amerika*, **123(4)** (2008), 2401–2406.

[6] **Monk, P.,** *Finite Element Methods for Maxwell's Equations*, Oxford University Press, Oxford 2003.

[7] **Moskal'kov, M.N.,** On the accuracy of difference schemes which approximate the wave equation with piecewise smooth solutions, *Zhurn. Vychisl. Math. Math. Physics*, **14(2)** (1974), 390–401 (Russian).

[8] **Nishiguchi, I.** Recent research on the acoustics of the piano, *Acoust. Sci. & Technology*, **25** (2004), 413–418.

[9] **Potemski, D.,** *Mathematical models for string instruments, and their implementation*, Diploma Thesis, ELTE IK, 2012.

[10] **Samarskij, A.A. and V.B. Gulin,** *Stability of Difference Schemes*, Nauka, Moscow, 1973 (Russian).

[11] **Samarskij, A.A. and V.B. Andrejev,** *Solution of Elliptic Equations*, Nauka, Moscow, 1976 (Russian).

[12] **Samarskij, A.A., P.N. Vabishchevich and P.P. Matus,** *Difference Schemes with Operator Coefficients*, ZOTZH, Minsk 1998 (Russian).

[13] **Stoyan G. and G. Takó,** *Numerical Methods II*, 2nd corr. ed., Typotex, Budapest 2012 (Hungarian).

[14] **Stoyan G. and G. Takó,** *Numerical Methods III*, 3rd corr. ed., Typotex, Budapest 2012 (Hungarian).

[15] **Zeidler, E.,** *Nonlinear Analysis and its Applications*, Springer-Verlag, New York 1990.

G. Stoyan
Department of Numerical Analysis
Eötvös Loránd University
Pázmány P. sétány 1/C.
H-1117 Budapest, Hungary
gisbert.stoyan@t-online.hu