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ON TRANSFORMATIONS BY DYADIC MARTINGALE STRUCTURE PRESERVING FUNCTIONS

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Dedicated to the 70th birthday of Professor Karl-Heinz Indlekofer

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Abstract. The concept of dyadic martingal structure preserving functions is defined. We show that composition with such functions preserves the classes of UDMD systems, that of $\mathcal{A}_n$-measurable functions, the dyadic function spaces $L^p(\mathbb{I})$, $H^p(\mathbb{I})$, and the Lipschitz classes $\mathrm{Lip}\,(\alpha, \mathbb{I})$.

1. Introduction

Numerous results were published in the last century about the effect of the composition with a Blaschke function on the convergence of the power series of regular functions in a boundary point of the disc $\mathbb{D}$. First, Turán [11] showed, that to any $\zeta \in \mathbb{C}$ $(0 < |\zeta| < 1)$ there is a complex function $f_1(z) = \sum_{n=1}^{\infty} a_n z^n$, regular in $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$, with convergent power-series for $z = 1$, but the power series of $f_2(z) := f_1(B_\zeta(z)) = \sum_{n=1}^{\infty} b_n z^n$ diverges at the corresponding point $z = B_\zeta^{-1}(1)$. $B_\zeta(z)$ denotes the Blaschke

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function with parameter $\zeta \in \mathbb{C}$: $B_\zeta(z) = \frac{z-\zeta}{1-\bar{\zeta}z}$ ($z \in \overline{\mathbb{D}}$). After several results due to Clunie, Schwarz, Halász, Alpár and others, Indlekofer [3] constructed a function f, which is continuous on $\overline{\mathbb{D}}$, its power-series converges for $z = 1$, but the power series of $f^*(z) := f(B_\zeta(z)) = \sum_{n=1}^{\infty} b_n z^n$ diverges at the corresponding point $z = B_\zeta^{-1}(1)$, moreover $\omega(f,h) = O\left(\left(\log \frac{2\pi}{h}\right)^{-1}\right)$ as $h \searrow 0$ holds for the modulus of continuity. He solved hereby the primal conjecture of Turán.

In this paper we consider questions related to the effect of the transformation by composition with a Blaschke function and in general of a dyadic martingal structure preserving function (DMSP-function) on the class of UDMD-systems and on dyadic function classes $L^p(\mathbb{I})$ $(0 < p \leq \infty)$, $H^p(\mathbb{I})$ $(0 < p < \infty)$, $\mathrm{Lip}\,(\alpha, \mathbb{I})$ $(\alpha > 0)$.

Denote by $\mathbb{A} := \{0, 1\}$ the set of bits and by

$$\mathbb{B} := \{a = (a_j, j \in \mathbb{Z}) \mid a_j \in \mathbb{A} \text{ and } \lim_{j \to -\infty} a_j = 0\}$$

the set of bytes. The order of a byte $x \in \mathbb{B}$ is defined in the following way: For $x \neq \theta := (0, 0, \ldots)$ let $\pi(x) = n$ if and only if $x_n = 1$ and $x_j = 0$ for all $j < n$. Set $\pi(\theta) := +\infty$. The norm of a byte x is defined by $\|x\| := 2^{-\pi(x)}$ for $x \in \mathbb{B} \setminus \{\theta\}$, and $\|\theta\| := 0$. By an interval in $\mathbb{B}$ of rank $n \in \mathbb{Z}$ and center $a \in \mathbb{B}$ we mean the set of the form $I_n(a) = \{x \in \mathbb{B} : x_j = a_j \text{ for } j < n\}$. Set $\mathbb{I}_n := I_n(\theta) = \{x \in \mathbb{B} : \|x\| \leqq 2^{-n}\}$ for any $n \in \mathbb{Z}$. The *unit ball* is $\mathbb{I} := \mathbb{I}_0$. Furthermore $\mathbb{S} := \{x \in \mathbb{B} : \|x\| = 1\} = \{x \in \mathbb{B} : \pi(x) = 0\} = \{x \in \mathbb{I} : x_0 = 1\}$ is the unit sphere.

Consider the *Rademacher system* $(r_n, n \in \mathbb{N})$, where $r_n(x) := (-1)^{x_n}$ $(x \in \mathbb{I})$, and the *Walsh-Paley functions*:

$$w_k(x) = \prod_{n=0}^{\infty} r_n(x)^{k_n} = (-1)^{\sum_{j=0}^{+\infty} k_j x_j} \ (x \in \mathbb{I}),$$

with dyadic expansion $k = \sum_{j=0}^{\infty} k_j 2^j \in \mathbb{N}$ $(k_j \in \mathbb{A})$. Set $\varepsilon(t) := \exp(2\pi i t)$ $(t \in \mathbb{R})$. We shall use the product system $(v_m, m \in \mathbb{N})$ generated by the functions

$$v_{2^n}(x) := \varepsilon\left(\frac{x_n}{2} + \frac{x_{n-1}}{2^2} + \cdots + \frac{x_0}{2^{n+1}}\right) \qquad (x \in \mathbb{I}, n \in \mathbb{N}).$$

Then $v_m(x) = \prod_{j=0}^{\infty} (v_{2^j}(x))^{m_j}$ $(m \in \mathbb{N})$. We will use the notation $\circ$ for composition of functions.

A *metric* is defined on $\mathbb{B}$ as follows:

$$\rho(x, y) := \begin{cases} 0, & \text{if } x = y \\ 2^{-n}, & \text{if } x \neq y,\ n := \min\{k \in \mathbb{Z} : x_k \neq y_k\}. \end{cases}$$

A *measure* can be defined on $\mathbb{B}$ in the following way:

$$\mu(I_n(a)) := 2^{-n} \quad (a \in \mathbb{B}, \quad n \in \mathbb{Z}). \tag{1}$$

Extend μ to the ring $\mathcal{R}$ of sets formed by finite unions of intervals so that μ is finitely additive. Then, μ is countably additive on $\mathcal{R}$. By the Caratheodory extension theorem follows, that there is a measure (denoted also by μ) defined on the σ-ring of Borel sets $\mathcal{B}_\mu$ which satisfies (1).

The concept of *UDMD systems* is due to Schipp [5]. Denote by $\mathcal{A}$ the σ-algebra generated by the intervals $I_n(a)$ $(a \in \mathbb{I}, n \in \mathbb{N})$. $\mathbb{I}$, $\mathcal{A}$, and the restriction of the measure μ on $\mathbb{I}$ form a probability measure space $(\mathbb{I}, \mathcal{A}, \mu)$. Let $\mathcal{A}_n$ be the sub-σ-algebra of $\mathcal{A}$ generated by the intervals $I_n(a)$ $(a \in \mathbb{I})$. Let $L(\mathcal{A}_n)$ denote the set of $\mathcal{A}_n$-measurable functions on $\mathbb{I}$. The *conditional expectation* of an $f \in L^1(\mathbb{I})$ with respect to $\mathcal{A}_n$ is of the form

$$(\mathcal{E}_n f)(x) := \frac{1}{\mu(I_n(x))} \int\limits_{I_n(x)} f d\mu \quad (x \in \mathbb{I}).$$

A sequence of functions $(f_n, n \in \mathbb{N})$ is called a *dyadic martingale* if each f_n is $\mathcal{A}_n$-measurable and $\mathcal{E}_n f_{n+1} = f_n$ $(n \in \mathbb{N})$. The *sequence of martingale differences* of $(f_n, n \in \mathbb{N})$ is the sequence $\phi_n := f_{n+1} - f_n$ $(n \in \mathbb{N})$.

The martingale difference sequence $(\phi_n, n \in \mathbb{N})$ is called a *unitary dyadic martingale difference sequence* or a *UDMD sequence*, if $|\phi_n(x)| = 1$ $(n \in \mathbb{N})$. According to Schipp [5], $(\phi_n, n \in \mathbb{N})$ is a UDMD sequence if and only if

$$\phi_n = r_n g_n, \; g_n \in L(\mathcal{A}_n), \; |g_n| = 1 \quad (n \in \mathbb{N}). \tag{2}$$

The *dyadic maximal operator* and for $0 < p < \infty$ *the H_p norm* is defined by

$$\mathcal{E}^*(f) := \sup_{n \in \mathbb{N}} |\mathcal{E}_n f| \quad (f \in L^1(\mathbb{I}))$$
$$\|f\|_{H^p} := \|\mathcal{E}^* f\|_p \quad (f \in L^1(\mathbb{I})).$$

2. The effect of transformations by a DMSP-function

Definition 1. We call a function $B : \mathbb{I} \to \mathbb{I}$ a *dyadic martingale structure preserving* function or shortly a *DMSP-function* if it is generated by a system of bijections $(\psi_n, n \in \mathbb{N})$, $\psi_n : \mathbb{A} \to \mathbb{A}$, and an arbitrary system $(\varphi_n, n \in \mathbb{N}^*)$, $\varphi_n : \mathbb{A}^n \to \mathbb{A}$ in the following way:

$$\begin{aligned} &(B(x))_0 := \psi_0(x_0), \\ &(B(x))_n := \psi_n(x_n) + \varphi_n(x_0, x_1, \ldots, x_{n-1}) \pmod 2 \quad (n \in \mathbb{N}^*). \end{aligned} \tag{3}$$

Proposition. *For each generating systems* $(\psi_n, n \in \mathbb{N})$ *and* $(\varphi_n, n \in \mathbb{N}^*)$, *$B$ is a bijection on* $\mathbb{I}$ *and its inverse function,* B^{-1} *is also a DMSP-function.*

The question, which function systems can be transformed by composition with a DMSP-function into a UDMD system, has a simple answer: exactly the UDMD systems.

Lemma 1. *Let* $B : \mathbb{I} \to \mathbb{I}$ *be a DMSP-function. Then, for each* $n \in \mathbb{N}$ *we have*

(4)
a) $r_n \circ B = r_n \cdot h_n$ *with some* $h_n \in L(\mathcal{A}_n), |h_n| = 1$,

b) $L(\mathcal{A}_n)$ *is invariant with respect to the composition with a DMSP function.*

Proof. a) By the definition of $y = B(x)$ we have

$$r_n(B(x)) = (-1)^{y_n} = (-1)^{\psi_n(x_n)}(-1)^{\varphi_n(x_0,\cdots,x_{n-1})} = \\ = r_n(x)(-1)^{\psi_n(0)}(-1)^{\varphi_n(x_0,\cdots,x_{n-1})} = r_n(x)h_n(x). \tag{5}$$

Obviously, $h_n(x) := (-1)^{\psi_n(0)}(-1)^{\varphi_n(x_0,\cdots,x_{n-1})} \in L(\mathcal{A}_n)$ and $|h_n| = 1$.

b) The statement is a simple consequence of the definitions. ■

Theorem 1. *Let* $B : \mathbb{I} \to \mathbb{I}$ *be a DMSP-function. The function system* $(f_n, n \in \mathbb{N})$ *is a UDMD system on* $\mathbb{I}$, *if and only if* $(f_n \circ B, n \in \mathbb{N})$ *is a UDMD system on* $\mathbb{I}$.

Proof. Let B be a DMSP-function. If $(f_n, n \in \mathbb{N})$ is a UDMD system, then by (2) there are functions $g_n \in L(\mathcal{A}_n)$ with $|g_n| = 1$ such that $f_n(x) = r_n(x)g_n(x)$ $(x \in \mathbb{I})$. It follows $r_n(B(x)) = r_n(x)h_n(x)$ for some $h_n \in L(\mathcal{A}_n)$, $|h_n| = 1$. Since $g_n \in L(\mathcal{A}_n)$ we have by Lemma 1, that $g_n \circ B \in L(\mathcal{A}_n)$. Consequently,

$$h_n\,(g_n \circ B) \in L(\mathcal{A}_n), \quad |h_n\,(g_n \circ B)\,| = 1, \quad \text{and}$$

$$f_n(B(x)) = r_n(B(x))g_n(B(x)) = r_n(x)\underbrace{h_n(x)g_n(B(x))}_{\in L(\mathcal{A}_n)} \qquad (x \in \mathbb{I}).$$

Thus $(f_n \circ B, n \in \mathbb{N})$ fulfills the requirements of a UDMD system formulated in (2).

Since the inverse of a DMSP-function is also a DMSP-function, it follows that if for any given system $(f_n, n \in \mathbb{N})$ the system $(g_n := f_n \circ B, n \in \mathbb{N})$ is a UDMD system, then the original one $(f_n = g_n \circ B^{-1}, n \in \mathbb{N})$ is also a UDMD system. ■

Remarks

1. As the Walsh-Paley functions $w_n (n \in \mathbb{N})$ and the functions v_n $(n \in \mathbb{N})$ are UDMD-product systems on $\mathbb{I}$, their composition with a DMSP-function result a UDMD-product system. For a precise statement see Remark 3.

2. Gát [1], [2] constructed a generalisation of the UDMD-systems, the so called Vilenkin-like systems on a more general space G_m. Extending the definition of the DMSP-functions to G_m, similar statement holds which is a consequence of Lemma 1, b) and Remark 3.

3. Schipp [7], [8] defined a general concept of systems, the adapted conditionally orthonormal systems or AC-ONS with respect to a regular sequence of weights. An AC-ONS on $\mathbb{I}$ is transformed by composition with a DMSP-function into an AC-ONS, which is a consequence of Lemma 1, b) and (11).

4. As UDMD-systems are taken into UDMD-systems by a DMSP-transformation, it follows by [4] that a.e. convergence and $(C, 1)$-summation of functions $f \in L^1(\mathbb{I})$ are also preserved by this kind of transformation.

We will show that the function classes $L^p(\mathbb{I})$ $(0 < p \leq \infty)$ and $H^p(\mathbb{I})$ $(0 < p < \infty)$ are invariant under the composition with a DMSP-function.

Lemma 2. *Let $B : \mathbb{I} \to \mathbb{I}$ be a DMSP-function and $n \in \mathbb{N}$. Then*

$$B(I_n(x)) = I_n(B(x)) \qquad (x \in \mathbb{I}). \tag{6}$$

Proof. If $t \in I_n(x)$, then $t_0 = x_0,\ t_1 = x_1, \ldots, t_{n-1} = x_{n-1}$. For $k < n$ we have $\psi_k(t_k) + \varphi_k(t_0, t_1, \ldots, t_{k-1}) = \psi_k(x_k) + \varphi_k(x_0, x_1, \ldots, x_{k-1})$, that is, $(B(t))_k = (B(x))_k$ $(k < n)$. Thus $B(t) \in I_n(B(x))$ $(t \in I_n(x))$, so

$$B(I_n(x)) \subseteq I_n(B(x)) \quad (x \in \mathbb{I}). \tag{7}$$

In particular (7) holds for the DMSP-function B^{-1} and $x = B(y)$. Thus by

$$B^{-1}(I_n(B(y))) \subseteq I_n(y) \qquad (y \in \mathbb{I})$$

follows $I_n(B(y)) \subseteq B(I_n(y))$ $(y \in \mathbb{I})$, which completes the proof together with (7). ∎

From (6) follows that $\mu(B(I_n(x))) = \mu(I_n(B(x))) = 2^{-n} = \mu(I_n(x))$, so $\mu(B(E)) = \mu(E)$ holds for each $E \in \mathcal{A}_n$. Thus

$$\mu(B(E)) = \mu(E) \qquad (E \in \mathcal{A}). \tag{8}$$

Consequently, $B : \mathbb{I} \to \mathbb{I}$ is measure preserving, i.e.

$$\int_{\mathbb{I}} f \circ B d\mu = \int_{\mathbb{I}} f d\mu \qquad (f \in L^1(\mathbb{I})). \tag{9}$$

Theorem 2. *Composition with a DMSP-function preserves* $L^p(\mathbb{I})$ $(0 < p \leq \leq \infty)$ *and the dyadic Hardy space* $H^p(\mathbb{I})$ $(0 < p < \infty)$. *Moreover,*

$$\|f \circ B\|_p = \|f\|_p \ (0 < p \leq \infty), \qquad \|f \circ B\|_{H^p} = \|f\|_{H^p} \ (0 < p < \infty). \tag{10}$$

Proof. For $0 < p < \infty$ and $f \in L^p(\mathbb{I})$, we have by (9) that $\|f \circ B\|_p = = \|f\|_p < \infty$. Hence $f \circ B \in L^p(\mathbb{I})$.

If $f \in L^\infty(\mathbb{I})$, then for $M := \|f\|_\infty \in \mathbb{R}$, we have $|f(x)| \leq M$ for a.e. $x \in \mathbb{I}$. By (8) follows that

$$\begin{aligned}\mu(\{x \in \mathbb{I} : \ |(f \circ B)(x)| > M \}) &= \mu(\{B(x) \in \mathbb{I} : \ |f(B(x))| > M \}) = \\ &= \mu(\{y \in \mathbb{I} : \ |f(y)| > M \}) = 0.\end{aligned}$$

Hence $f \circ B \in L^\infty(\mathbb{I})$ and $\|f \circ B\|_\infty \leq \|f\|_\infty$. As this holds also for DMSP function B^{-1} instead of B and $f \circ B$ instead of f, the first equality in (10) follows.

For $f \in H^p(\mathbb{I})$ $(0 < p < \infty)$ we have by definition that $\|\mathcal{E}^* f\|_p < \infty$. By (6) follows that $1_{I_n(x)}(t) = 1_{I_n(B(x))}(B(t))$ $(t \in \mathbb{I})$. Hence by (9) we obtain

$$\begin{aligned}\mathcal{E}_n\left(f \circ B\right)(x) &= \frac{1}{\mu(I_n(x))} \int\limits_{I_n(x)} f(B(t)) d\mu(t) = \\ &= 2^n \int\limits_{\mathbb{I}} f(B(t)) \cdot 1_{I_n(x)}(t) d\mu(t) = \\ &= 2^n \int\limits_{\mathbb{I}} f(B(t)) \cdot 1_{I_n(B(x))}\left(B(t)\right) d\mu(t) = \\ &= \frac{1}{\mu(I_n(B(x)))} \int\limits_{I_n(B(x))} f(t) d\mu(t) \\ &= \mathcal{E}_n\left(f\right)\left(B(x)\right).\end{aligned} \tag{11}$$

Thus

$$\mathcal{E}^*(f \circ B) := \sup_{n \in \mathbb{N}} |\mathcal{E}_n\left(f \circ B\right)| = \sup_{n \in \mathbb{N}} |\left(\mathcal{E}_n f\right) \circ B| = (\mathcal{E}^* f) \circ B.$$

Then by the first equality in (10) we have

$$\|\mathcal{E}^*(f \circ B)\|_p = \|\left(\mathcal{E}^* f\right) \circ B\|_p = \|\mathcal{E}^* f\|_p < \infty.$$

Consequently, $f \circ B \in H^p(\mathbb{I})$ and $\|f \circ B\|_{H^p} = \|f\|_{H^p}$ $(0 < p < \infty)$. ∎

Remark. From (10) and (11) follows that

$$\|f\circ B\|_{BMO} = \sup_{n\in\mathbb{N}} \| \left(\mathcal{E}_n|f-\mathcal{E}_nf|^2\right)^{\frac{1}{2}} \circ B\|_\infty = \|f\|_{BMO}.$$

Thus the space of bounded dyadic mean oscillation BMO and the space of vanishing dyadic mean oscillation VMO are also preserved under composition with a DMSP-function. For more on these spaces see Schipp [5].

Recall, that for $\alpha > 0$ the function class $\text{Lip}\,(\alpha, \mathbb{B})$ denotes the collection of functions $f : \mathbb{I} \to \mathbb{R}$ which satisfy

$$|f(y) - f(x)| \le c\ \rho(x,y)^\alpha \quad (x, y \in \mathbb{B})$$

for some constant $c \in \mathbb{R}$ which depends only on f.

Theorem 3. *Composition with a DMSP-function preserves* $\text{Lip}\,(\alpha, \mathbb{I})$ $(\alpha > 0)$.

Proof. For $x, y \in \mathbb{I},\ x \ne y$ consider $m := \min\{n : x_n \ne y_n\}$. Then $\rho(x,y) = 2^{-m}$ and m is the largest number in $\mathbb{N}$ for which $x \in I_m(y)$. It follows from (6), that $B(x) \in I_m\,(B(y))$ and m is the largest integer with this property. Thus

$$\rho\,(B(x), B(y)) = 2^{-m} = \rho\,(x,y) \quad (x, y \in \mathbb{I}).$$

For $f \in \text{Lip}\,(\alpha, \mathbb{I})$ we have

$$|f(B(y)) - f(B(x))| \le c\ \rho\,(B(x), B(y))^\alpha = c\ \rho\,(x,y)^\alpha$$

for some $c \in \mathbb{R}$. That is, $f \circ B \in Lip(\alpha, \mathbb{I})$. ∎

3. Examples of DMSP-functions

Consider the 2-series (or logical) field $(\mathbb{B}, \overset{\circ}{+}, \circ)$ and the 2-adic (or arithmetical) field $(\mathbb{B}, \overset{\bullet}{+}, \bullet)$.

The *2-series (or logical) sum* $a \overset{\circ}{+} b$ and *product* $a \circ b$ of elements $a, b \in \mathbb{B}$ is defined by

$$a \overset{\circ}{+} b := (a_n + b_n \ (\text{mod}\ 2),\ n \in \mathbb{Z})$$
$$a \circ b := (c_n, n \in \mathbb{Z}), \quad \text{where} \quad c_n := \sum_{k\in\mathbb{Z}} a_k b_{n-k}\ (\text{mod}\ 2) \quad (n \in \mathbb{Z}).$$

The *2-adic* (*or arithmetical*) *sum* $a \overset{\bullet}{+} b$ of elements $a = (a_n, n \in \mathbb{Z})$, $b = (b_n, n \in \mathbb{Z}) \in \mathbb{B}$ is defined by $a \overset{\bullet}{+} b := (s_n, n \in \mathbb{Z})$ where the bits $q_n, s_n \in \mathbb{A}$ $(n \in \mathbb{Z})$ are obtained recursively as follows:

$$\begin{aligned} &q_n = s_n = 0 \quad \text{ for } n < m := \min\{\pi(a), \pi(b)\}, \\ &\text{and } \; a_n + b_n + q_{n-1} = 2q_n + s_n \quad \text{ for } n \geq m. \end{aligned}$$

The *2-adic* (*or arithmetical*) *product* of $a, b \in \mathbb{B}$ is $a \bullet b := (p_n, n \in \mathbb{Z})$, where the sequences $q_n \in \mathbb{N}$ and $p_n \in \mathbb{A}$ $(n \in \mathbb{Z})$ are defined recursively by

$$\begin{aligned} &q_n = p_n = 0 \quad (n < m := \pi(a) + \pi(b)) \\ &\text{and } \sum_{j=-\infty}^{\infty} a_j b_{n-j} + q_{n-1} = 2q_n + p_n \quad (n \geq m). \end{aligned}$$

The *reflection* x^- of a byte $x = (x_j, j \in \mathbb{Z})$ is defined by

$$(x^-)_j := \begin{cases} x_j, & \text{for } j \leqq \pi(x) \\ 1 - x_j, & \text{for } j > \pi(x). \end{cases}$$

$e := (\delta_{n0}, n \in \mathbb{Z})$, where δ_{nk} is the Kronecker-symbol. We will use the following notation: $a \overset{\bullet}{-} b := a \overset{\bullet}{+} b^-$.

1) The following functions are trivial DMSP-functions on $(\mathbb{B}, \overset{\circ}{+}, \circ)$ and $(\mathbb{B}, \overset{\bullet}{+}, \bullet)$:

$$\begin{aligned} &B(x) := x \overset{\circ}{+} a, && B(x) := x \overset{\bullet}{+} a && (a \in \mathbb{I}), \\ &B(x) := x \circ a, && B(x) := x \bullet a && (a \in \mathbb{S}), \\ &B(x) := x, && B(x) := x^{-1} && (x \in \mathbb{I}). \end{aligned}$$

The last one follows from the recursive expansion of x^{-1} in [6] pp. 41–42.

2) If $c_n \in \mathbb{I}$ satisfies $\pi(c_n) = n$ $(n \in \mathbb{N}^*)$, then the function

$$B(x) := \prod_{j=1}^{\infty} (e + c_j)^{x_j} = \prod_{j=1}^{\infty} (e + x_j c_j) \tag{12}$$

can be obtained by a simple recursion Therefore, it is a DMSP-function from $\mathbb{I}_1$ to $\mathbb{S}$. See Schipp [6], pp 51-53. Its importance lies in the consequence, that the multiplicative digits of a given byte $y \in \mathbb{S}$ with respect to a sequence $(b_n = e + c_n, n \in \mathbb{N}^*)$, $\pi(c_n) = n$ can be obtained from its additive digits.

3) The dyadic Blaschke functions, introduced by the author in [10] are also DMSP-functions:

For $a \in \mathbb{I}_1$ *the logical Blaschke function* on $(\mathbb{I}, \overset{\circ}{+}, \circ)$ is defined by

$$B_a(x) := (x \overset{\circ}{+} a) \circ (e \overset{\circ}{+} a \circ x)^{-1} = \frac{x \overset{\circ}{+} a}{e \overset{\circ}{+} a \circ x} \qquad (x \in \mathbb{I}).$$

With $y = B_a(x)$ we have $y = x \overset{\circ}{+} a \overset{\circ}{+} y \circ a \circ x$. So,

$$\begin{cases} y_n = 0, \text{ for } n < 0, \\ y_n = x_n + a_n + (y \circ a \circ x)_n \pmod 2, \text{ for } n \geqq 0. \end{cases}$$

Since the n-th digit of $y \circ a \circ x$ depends only on a and x_k-s with $k < n$, we have that the logical Blaschke function is a DMSP-function.

For $a \in \mathbb{I}_1$ *the arithmetical Blaschke function* on $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$ is defined by

$$B_a(x) := (x \overset{\bullet}{-} a) \bullet (e \overset{\bullet}{-} a \bullet x)^{-1} = \frac{x \overset{\bullet}{-} a}{e \overset{\bullet}{-} a \bullet x} \qquad (x \in \mathbb{I}).$$

The same recursion method holds for the arithmetical Blaschke function, too. See Simon [10] or [9]. So, it is also a DMSP-function.

Remark. As the additive and multiplicative characters of $\mathbb{I}$ on both fields can be obtained recursively, their compositions with a DMSP-function result in a UDMD-product system.

For $n \in \mathbb{N}^*$ *let* $j := \max\{k \in \mathbb{N} : \ n \geq 2^k\}$. *Then,*

$$w_n \circ B = w_n \cdot g_j \text{ with some } g_j \in L(\mathcal{A}_j), |g_j| = 1,$$
$$v_n \circ B = v_n \cdot g_j \text{ with some } g_j \in L(\mathcal{A}_j), |g_j| = 1.$$

The statements hold obviously for $n = j = 0$*, too.*

Proof. We have $n = \sum_{i=0}^{j} n_i 2^i$. By (5) follows

$$w_n(B(x)) = \prod_{i=0}^{j} r_i^{n_i}(B(x)) = \prod_{i=0}^{j} r_i^{n_i}(x) h_i^{n_i}(x) = $$
$$= w_n(x) g_j(x) \qquad (n \in \mathbb{N}^*),$$

where $h_i \in L(\mathcal{A}_i)$ and $|h_i| = 1$ $(i \in \{0, 1, \ldots, j\})$. Thus $g_j := \prod_{i=0}^{j} h_i^{n_i} \in L(\mathcal{A}_j)$ and $|g_j| = 1$.

The statement for $(v_n, n \in \mathbb{N})$ follows analogously. ∎

References

[1] **Gát, G.,** Orthonormal systems on Vilenkin groups, *Acta Math. Hungar.,* **58(1–2)** (1991), 193–198.

[2] **Gát, G.,** Convergence and summation with respect to Vilenkin-like systems, Recent Developments in: *Abstract Harmonic Analysis with Applications in Signal Processing*, Nauka, Belgrade and Elektronski fakultet, Nis, 1996, pp. 137–146.

[3] **Indlekofer, K.-H.,** Bemerkungen über äquivalante Potenzreihen von Funktionen mit gewissem Stetigkeitsmodul, *Monatsh. Math.*, **76** (1972), 124–129.

[4] **Schipp, F.,** Universal contractive projections and a.e. convergence, in: *Probability Theory and Applications, Essays to the Memory of József Mogyoródi*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1992, pp. 47–75.

[5] **Schipp, F., W.R. Wade, P. Simon and J. Pál,** *Walsh Series, An Introduction to Dyadic Harmonic Analysis*, Adam Hilger, Ltd., Bristol and New York, (1990).

[6] **Schipp, F. and W.R. Wade,** *Transforms on Normed Fields*, Pécs (1995), available also at `http://numanal.inf.elte.hu/~schipp/Publications/Tr_NFields.pdf`

[7] **Schipp, F.**, On L_p-norm convergence of series with respect to product systems, *Analysis Math.*, **2** (1976), 49–64.

[8] **Schipp, F.,** On orthonormal product systems, *Acta Mathematica Academiae Paedagogicae Nyíregyháziensis*, **20** (2004), 185-206.

[9] **Simon, I.,** The characters of the Blaschke-group of the arithmetic field, *Studia Univ. "Babes-Bolyai", Mathematica*, **54(3)** (2009), 149–160.

[10] **Simon, I.,** Discrete Laguerre functions on the dyadic fields, *PUMA*, **17** (2006), 459–468.

[11] **Turán, P.,** A remark concerning the behavior of power series on the periphery of its convergence circle, *Publ. Inst. Math. (Beograd)*, **12** (1958), 19–26.

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