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MULTIRESOLUTION IN THE BERGMAN SPACE

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Dedicated to Professor Karl-Heinz Indlekofer
on his seventieth anniversary

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Abstract. In this paper we introduce a new example of sampling set for the Bergman space which can be connected to the Blaschke group operation. Using this set we will generate a multiresolution analysis in the Bergman space and we present properties of the projection operator on the resolution levels. The construction is an analogy with the multiresolution generated by the discrete affine wavelets in the space of the square integrable functions on the real line, and in fact is the discretization of the continuous voice transform generated by a representation of the Blaschke group over the Bergman space.

1. Introduction

The plan of this paper is as follows. First we present some basic results connected to the Bergman space, we give the definition of the voice transform generated by a representation of the Blaschke group on A^2. In the second section we introduce a discrete subset of the Blaschke group and we give a

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sufficient condition from which it follows that this set is a sampling set for the Bergman space. Using this discrete subset of the Blaschke group we construct a multiresolution decomposition in A^2. First the different resolution spaces will be defined using a nonorthogonal basis which shows the analogy between the discrete hyperbolic wavelets in A^2 and the discrete affine wavelets in $L^2(\mathbb{R})$. Applying the Gram–Schmidt orthogonalization we consider the rational orthogonal basis on the n-th multiresolution level V_n. This system is the analogue of the Mamquist-Takenaka system in the Hardy spaces, possesses similar properties and is connected to the contractive zero divisors of a finite set in Bergman space. We prove that the projection operator $P_n f(z)$ on the resolution level V_n is interpolation operator on the set the $\bigcup_{k=0}^n \mathcal{A}_k$, where $\mathcal{A}_k$ is defined by (2.7), and $P_n f(z) \to f(z)$ in norm and uniformly on every compact subset of the unit disc.

1.1. The Bergman space

We will need the following basic results connected to the Bergman spaces. For more detailed exposition see for example in [6]. Let denote by $\mathbb{D} = \{z \in \mathbb{C} \ : |z| < 1 \}$ the unit disc and by $\mathbb{T} = \{z \in \mathbb{C} \ : |z| = 1 \}$ the unit circle.

Recall that if $z = x + iy \in \mathbb{D}$ then the normalized area measure is $dA(z) = \frac{1}{\pi} dxdy$. For $0 < p < \infty$, an analytic function $f : \mathbb{D} \to \mathbb{C}$ belongs to the A^p if

$$\int_{\mathbb{D}} |f(z)|^p \, dA(z) < \infty. \tag{1.1}$$

For $p = 2$ the set A^2 is a reproducing kernel Hilbert space, which is called the Bergman space. The scalar product in $A^2 = A^2(\mathbb{D})$ is given by

$$\langle f, g \rangle := \int_{\mathbb{D}} f(z)\overline{g(z)} \, dA(z). \tag{1.2}$$

The Bergman space $A^2(\mathbb{D})$ is a closed subspace of $L^2(\mathbb{D})$. For each $z \in \mathbb{D}$ the point-evaluation map is a bounded linear functional on $A^2(\mathbb{D})$. Each function $f \in A^2(\mathbb{D})$ has the property

$$|f(z)| \leqq \pi^{-1/2} \delta(z)^{-1} \|f\|_{A^2(\mathbb{D})} \quad (z \in \mathbb{D}),$$

where $\delta(z) = dist(z, \mathbb{T})$. From this it follows that the norm convergence in $A^2(\mathbb{D})$ implies the locally uniform convergence on $\mathbb{D}$. Therefore, by the Riesz

Representation Theorem there is a unique element in $A^2(\mathbb{D})$, denoted by $K(.,z)$, such that

$$f(z) = \langle f, K(.,z)\rangle, \ (f \in A^2(\mathbb{D}), z \in \mathbb{D}).$$

The function

$$K : \mathbb{D} \times \mathbb{D} \to \mathbb{C} \quad \text{with} \quad K(.,z) \in A^2(\mathbb{D})$$

is called the Bergman kernel for $\mathbb{D}$ and it is given by

$$(1.3). \qquad K(\xi, z) = \frac{1}{(1-\overline{z}\xi)^2}.$$

The explicit formula for the kernel function shows that we have the following reproducing formula:

$$(1.4) \quad f(z) = \frac{1}{\pi}\int\limits_{\mathbb{D}} f(\xi)\frac{1}{(1-\bar{\xi}z)^2}d\xi_1 d\xi_2 \quad (f \in A^2(\mathbb{D}),\ z,\xi \in \mathbb{D},\ \xi = \xi_1 + i\xi_2).$$

Applying this formula in particular for $f(z) = (1-\xi\overline{z})^{-2}$ for fixed z in the disc we obtain that

$$\|K(y,.)\|_2^2 = \frac{1}{\pi}\int\limits_{\mathbb{D}} \frac{1}{|1-\bar{\xi}z|^4 d\xi_1 d\xi_2} = \frac{1}{(1-|z|^2)^2} = K(z,z).$$

A sequence $\Gamma = \{z_k : k \in \mathbb{N}\}$ of points in the unit disc is sampling sequence for A^p, where $0 < p < \infty$, if there exist positive constants A and B such that

$$(1.5) \qquad A||f||^p \leq \sum_{k=1}^{\infty} |f(z_k)|^p (1-|z_k|^2)^2 \leq B||f||^p, \quad f \in A^p.$$

For $p = 2$ this is equivalent to the following inequalities:

$$(1.6) \qquad A||f||^2 \leq \sum_{k=1}^{\infty} |\langle f, \varphi_k\rangle|^2 \leq B||f||^2, \quad f \in A^2,$$

where $\varphi_k(z) = K(z,z_k)/\|K(z,z_k)\|$ are the normalized Bergman kernel and localized in z_k in (1.3). These kernel functions are not mutually orthogonal, so far no sequence of distinct points $z_k \in \mathbb{D}$ do the normalized kernel functions form an orthonormal basis. However, this last inequality shows that these functions will constitute a frame for A^2 if and only if $\Gamma = \{z_k : k \in \mathbb{N}\}$ is a sampling set for A^2. A main difference between the Hardy space and the Bergman space is that there is no counterpart of sampling sequences in Hardy spaces. The Bergman spaces A^p do have sampling sequences, but examples are not so easy to construct. Some explicit examples are due to Seip, Duren, Schuster, Horowitz, Luecking (see for ex in [19, 6]). An A^p sampling sequence is never an A^p zero-set. A total characterization of sampling sequences is due to Kristian Seip and can be given with the uniformly discrete property and lower density of the set (see [20, 6]). But the computation of the lower density of a set in general is a difficult task. Duren, Schuster and Vukotic in [7] gave for sampling sufficient conditions based on the pseudohyperbolic metric, that are relatively easy to verify.

The pseudohyperbolic metric is defined by

$$\rho(z,y)=\left|\frac{y-z}{1-\overline{y}z}\right| \quad (y,z\in\mathbb{D}).$$

A sequence $\Gamma=\{z_k\}$ of points in the unit disc is uniformly discrete if

$$\delta(\Gamma)=\inf_{j\neq k}\rho(z_j,z_k)=\delta>0.$$

For $0<\epsilon<1$, a sequence $\Gamma=\{z_k : k\in\mathbb{N}\}$ of points in the unit disc is said to be ϵ-net if each point $z\in\mathbb{D}$ has the property $\rho(z,z_k)<\epsilon$ for some z_k in Γ. An equivalent statement is that

$$\mathbb{D}=\bigcup_{k=1}^{\infty}\Delta(z_k,\epsilon),$$

where $\Delta(z_k,\epsilon)$ denotes a pseudohyperbolic disc.

In [7] it is shown that for $0<p<\infty$, if Γ is a uniformly discrete ϵ-net with

$$\epsilon<\frac{1}{1+\sqrt{\frac{2}{p}}},$$

then is a sampling set for A^p.

Schuster and Varolin [22] improved these sufficient condition. They showed that every uniformly discrete ϵ-net sequence with

$$\epsilon<\sqrt{\frac{p}{p+2}} \tag{1.7}$$

is sampling set for A^p. This last sufficient condition will be used in our proof.

1.2. The continuous voice transform on Bergman space

In signal processing and image reconstruction the wavelet and Gábor transforms play an important role. H. Feichtinger and K. Gröchening unified the

Gábor and wavelet transforms into a single theory. The common generalization of these transforms is the so-called *voice transform* (see [8]). The voice transform on Bergman space is induced by a unitary representation of the Blaschke group on the Bergman space. Results connected the voice transform on Bergman space were published in [16].

Let us denote by

$$B_{\mathbf{a}}(z) := \epsilon \frac{z-b}{1-\bar{b}z} \quad (z \in \mathbb{C}, \mathbf{a} = (b,\epsilon) \in \mathbb{B} := \mathbb{D} \times \mathbb{T}, \bar{b}z \neq 1) \tag{1.8}$$

the so called *Blaschke functions*. If $\mathbf{a} \in \mathbb{B}$, then $B_{\mathbf{a}}$ is an 1-1 map on $\mathbb{T}$ and $\mathbb{D}$ respectively. The restrictions of the Blaschke functions on the set $\mathbb{D}$ or on $\mathbb{T}$ with the operation $(B_{\mathbf{a}_1} \circ B_{\mathbf{a}_2})(z) := B_{\mathbf{a}_1}(B_{\mathbf{a}_2}(z))$ form a group. In the set of the parameters $\mathbb{B} := \mathbb{D} \times \mathbb{T}$ let us define the operation induced by the function composition in the following way: $B_{\mathbf{a}_1} \circ B_{\mathbf{a}_2} = B_{\mathbf{a}_1 \circ \mathbf{a}_2}$. The group $(\mathbb{B}, \circ)$ *is the Blaschke group*. From the definition it follows that this group is isomorphic with the group $(\{B_{\mathbf{a}}, \mathbf{a} \in \mathbb{B}\}, \circ)$. If we use the notations $\mathbf{a}_j := (b_j, \epsilon_j)$, $j \in \{1,2\}$ and $\mathbf{a} := (b,\epsilon) =: \mathbf{a}_1 \circ \mathbf{a}_2$, then

$$b = \frac{b_1\bar{\epsilon}_2 + b_2}{1 + b_1\bar{b}_2\bar{\epsilon}_2} = B_{(-b_2,1)}(b_1\bar{\epsilon}_2), \quad \epsilon = \epsilon_1 \frac{\epsilon_2 + b_1\bar{b}_2}{1+\epsilon_2\bar{b}_1 b_2} = B_{(-b_1\bar{b}_2,\epsilon_1)}(\epsilon_2). \tag{1.9}$$

The neutral element of the group $(\mathbb{B}, \circ)$ is $e := (0,1) \in \mathbb{B}$ and the inverse element of $\mathbf{a} = (b,\epsilon) \in \mathbb{B}$ is $\mathbf{a}^{-1} = (-b\epsilon, \bar{\epsilon})$.

The integral of the function $f : \mathbb{B} \to \mathbb{C}$, with respect to the left invariant Haar measure m of the group $(\mathbb{B}, \circ)$ can be expressed as

$$\int_{\mathbb{B}} f(\mathbf{a})\, dm(\mathbf{a}) = \frac{1}{2\pi} \int_{\mathbb{I}} \int_{\mathbb{D}} \frac{f(b, e^{it})}{(1-|b|^2)^2}\, db_1 db_2 dt, \tag{1.10}$$

where $\mathbf{a} = (b, e^{it}) = (b_1 + ib_2, e^{it}) \in \mathbb{D} \times \mathbb{T}$.

It can be shown that this integral is invariant with respect to the left translation $\mathbf{a} \to \mathbf{a}_0 \circ \mathbf{a}$ and under the inverse transformation $\mathbf{a} \to \mathbf{a}^{-1}$, so Blaschke group is unimodular.

Let consider the following set of functions

$$F_{\mathbf{a}}(z) := \frac{\sqrt{\epsilon(1-|b|^2)}}{1-\bar{b}z} \quad (\mathbf{a} = (b,\epsilon) \in \mathbb{D}, z \in \overline{\mathbb{B}}). \tag{1.11}$$

These functions induce a unitary representation on the space A^2. Namely let define

$$U_{\mathbf{a}} f := [F_{a^{-1}}]^2 f \circ B_{\mathbf{a}}^{-1} \quad (\mathbf{a} \in \mathbb{B}, f \in A^2),$$

then this is a representation of Blaschke group on the A^2, i.e.:

- $U_{x \circ y} = U_x \circ U_y \quad (x, y \in \mathbb{B})$,
- $\mathbb{B} \ni x \to U_x f \in A^2$ is continuous for all $f \in A^2$.

In [16] it was proved that $U_{\mathbf{a}}(\mathbf{a} \in \mathbb{B})$ is an unitary, irreducible square integrable representation of the group $\mathbb{B}$ on the Hilbert space A^2.

The *voice transform* of $f \in A^2$ generated by the representation $U_{\mathbf{a}}$ and by the parameter $g \in A^2$ is the (complex-valued) function on $\mathbb{B}$ defined by

$$(V_g f)(\mathbf{a}) := \langle f, U_{\mathbf{a}} g \rangle \quad (\mathbf{a} \in \mathbb{B}, f, g \in A^2). \tag{1.12}$$

This transform is in same relation with the Blaschke group and the Bergman space as the affine wavelet transform with the affine group and the $L^2(\mathbb{R})$ (see [8], [13], [21]). Indeed let consider the affine group $(G, \circ)$, where

$$G = \{\ell_{(a,b)} \ : \ (a, b) \in \mathbb{R}^* \times \mathbb{R}\},$$

$$\ell_{(a,b)}(x) = ax + b, \ \mathbb{R}^* := \mathbb{R} \setminus \{0\}, \ \ell_1 \circ \ell_2(x) = \ell_1(\ell_2(x)) = a_1 a_2 x + a_1 b_2 + b_1.$$

The representation of the affine group G on $L^2(\mathbb{R})$ is given by

$$U_{(a,b)} f(x) = |a|^{-1/2} f(a^{-1}x - b) = |a|^{-1/2} f(\ell^{-1}_{(a,b)}(x)),$$

where a is the dilatation parameter, and b the translation parameter.

The continuous affine wavelet transform is a voice transform generated by this representation of the affine group:

$$W_\psi f(a, b) = |a|^{-1/2} \int_{\mathbb{R}} f(t) \overline{\psi(a^{-1}t - b)} dt = \langle f, U_{(a,b)} \psi \rangle, \quad f, \psi \in L^2(\mathbb{R}).$$

There is a rich bibliography of the affine wavelet theory (see for example [1, 3, 4, 13, 14]). One important question is the construction of the discrete version, i.e., to find ψ so that the discrete translates and dilates

$$\psi_{n,k} = 2^{-n/2} \psi(2^{-n}x - k)$$

form a (orthonormal) basis in $L^2(\mathbb{R})$ and generate a multiresolution (see [3, 4, 14]). Roughly speaking the coefficients of a square integrable function with respect to $\psi_{n,k}$, which are the values of the affine wavelet transform on a special discrete lattice:

$$\langle f, \psi_{n,k} \rangle = W_\psi(2^{-n}, k).$$

The discrete lattice in the affine case is determined by the following discrete subset of the affine group:

$$G_{n,k} = \{\ell_{(2^{-n}, -k)} \ : \ n \in \mathbb{Z}, \, k \in \mathbb{Z}\}.$$

The discretization of the voice transform in general can be achieved using the unified approach of the atomic decomposition elaborated by Feichtinger and Gröchenig [8]. This general description can be applied when the integrability condition of the voice transform is satisfied. In a recent paper [17] it is shown that, the integrability condition in the Bergman space it is not satisfied. This motivates to answer the following question.

Question: Find a discrete subset $\{a_{k\ell} \in \mathbb{B}\}$ of the Blascke group and a function $g \in A^2$ and generate a multiresolutin in the Bergman space using the functions $U_{a_{k\ell}} g$?

2. New results

2.1. Special discrete subsets in $\mathbb{B}$ and their sampling property

Our goal is to construct a sampling set which is a discrete subset of the Blaschke group and to generate a multiresolution analysis based on this set in the Bergman space $A^2(\mathbb{D})$.

The one parameter subgroups

$$\mathbb{B}_1 := \{(r, 1) : r \in (-1, 1)\}, \quad \mathbb{B}_2 := \{(0, \epsilon) : \epsilon \in \mathbb{T}\} \tag{2.1}$$

generate $\mathbb{B}$, i. e.

$$\mathbf{a} = (0, \epsilon_2) \circ (0, \epsilon_1) \circ (r, 1) \circ (0, \bar{\epsilon}_1) \quad (\mathbf{a} = (r\epsilon_1, \epsilon_2),\ r \in [0, 1), \epsilon_1, \epsilon_2 \in \mathbb{T}). \tag{2.2}$$

$\mathbb{B}_1$ is the analogue of the group of dilation, $\mathbb{B}_2$ is the analogue of the group of translation (see [21]).

The group operation $(r, 1) = (r_1, 1) \circ (r_2, 1)$ in $\mathbb{B}_1$ can be expressed using the tangent hyperbolic and its inverse (ath) in the following way

$$r = \frac{r_1 + r_2}{1 + r_1 r_2} = \mathrm{th}(\mathrm{ath}\ r_1 + \mathrm{ath}\ r_2) \quad (r_1, r_2 \in (-1, 1)). \tag{2.3}$$

Let denote $r = \mathrm{th}\,\alpha, r_i = \mathrm{th}\,\alpha_i,\ i = 1, 2$. Then by

$$(r_1, 1) \circ (r_2, 1) = (\mathrm{th}\,\alpha_1, 1) \circ (\mathrm{th}\,\alpha_2, 1) = (\mathrm{th}\,(\alpha_1 + \alpha_2), 1),$$

it follows that $(\mathbb{B}_1, \circ)$ is isomorphic to $(\mathbb{R}, +)$. It is known that $(\mathbb{Z}, +)$ is a subgroup of $(\mathbb{R}, +)$, then $\overline{\mathbb{B}_1} = \{(\mathrm{th}\, k, 1),\ k \in \mathbb{Z}\}$ is an one parameter subgroup of $(\mathbb{B}_1, \circ)$ (see [23]).

Let $a > 1$, denote by

$$\mathbb{B}_3 = \left\{ (r_k, 1) : \ r_k = \frac{a^k - a^{-k}}{a^k + a^{-k}}, \, k \in \mathbb{Z} \right\}. \tag{2.4}$$

It can be proved that $(\mathbb{B}_3, \circ)$ is another subgroup of $(\mathbb{B}, \circ)$, and we have the following composition rule: $(r_k, 1) \circ (r_n, 1) = (r_{k+n}, 1)$. The hyperbolic distance of the points $r_k, \ r_n$ has the following property:

$$\rho(r_k, r_n) := \frac{|r_k - r_n|}{|1 - r_k \overline{r_n}|} = \left| \frac{\frac{a^k - a^{-k}}{a^k + a^{-k}} - \frac{a^n - a^{-n}}{a^n + a^{-n}}}{1 - \frac{a^k - a^{-k}}{a^k + a^{-k}} \frac{a^n - a^{-n}}{a^n + a^{-n}}} \right| = |r_{k-n}|. \tag{2.5}$$

Let $N = N(a, k), \ k \geq 1$ an increasing sequence of natural numbers, $N(a, 0) := 1$, and consider the following set of points $z_{00} := 0$,

$$\mathcal{A} = \{ z_{k\ell} = r_k e^{i \frac{2\pi\ell}{N}}, \ \ \ell = 0, 1, \ldots, N(a, k) - 1, \ \ k = 0, 1, 2, \ldots \} \tag{2.6}$$

and for a fixed $k \in \mathbb{N}$ let the level k be

$$\mathcal{A}_k = \{ z_{k\ell} = r_k e^{i \frac{2\pi\ell}{N}}, \, \ell \in \{0, 1, \ldots, N(a, k) - 1\} \}. \tag{2.7}$$

The radius of the concentric circles are connected to the Blaschke group operation $(r_k, 1) \circ (r_n, 1) = (r_{k+n}, 1)$, this is the analogue property of the dilatation in the affine wavelet case.

We can choose a and $N = N(a, k)$ such that $\mathcal{A}$ will be a set of sampling in the Bergman space.

First we will study the following questions: for which choice of a and $N(a, k)$

1. will be $\mathcal{A}$ uniformly discrete,
2. will be $\mathcal{A}$ an ϵ-net set for some $0 < \epsilon < 1$,
3. will be $\mathcal{A}$ sampling sequence for Bergman spaces A^p?

Theorem 2.1. *Let $a > 1$, and $(N(a, k), k \geq 1)$ a sequence of increasing natural numbers and consider the set of points $\mathcal{A}$ defined by* (2.6). *Suppose that there exists $\alpha = \lim_{k \to \infty} N(a, k) a^{-2k}$.*

1. *If $(N(a, k) a^{-2k}, k \geq 1)$ is increasing sequence and α is finite, then $\mathcal{A}$ is uniformly discrete and the separation constant satisfies*

$$\delta \geq \min \left\{ r_1, \frac{1}{\sqrt{1 + \alpha^2}} \right\}.$$

2. *If $(N(a, k) a^{-2k}, k \geq 1)$ is decreasing and $0 < \alpha < \infty$, then there exists $\epsilon_0 \in (0, 1)$ for which the set $\mathcal{A}$ is ϵ_0-net .*

3. *If* $N(a,k)a^{-2k} = \alpha$, *is a constant for* $k \geq 1$, $0 < \alpha < \infty$ *and*

$$(a - a^{-1})^2 + \pi^2 \frac{a^2}{\alpha^2} < 2p,$$

then $\mathcal{A}$ *is a sampling set for* A^p.

Proof. 1. For simplicity denote by $N = N(a,k)$. We need to consider two types of situations. The pair of points lie on different circles, or they may lie on the same circler of radius r_k. Suppose first that the points $z_{k\ell}, z_{mn}$ lie on two different circles of radius r_k and r_m. Then the generalized triangle inequality for the pseudohyperbolic metric (see [6] pp. 38) implies that

$$\rho(z_{k\ell}, z_{mn}) \geq \left| \frac{r_k - r_m}{1 - r_k r_m} \right| \geq |r_{m-n}| \geq r_1 > 0.$$

Next suppose that the pair of points lie on the same circler of radius r_k, and $\ell \neq n$, then the least pseudohyperbolic distant is attained when $\ell = n+1$, then

$$\rho(z_{k\ell}, z_{kn}) \geq r_k \left| 1 - e^{\frac{2\pi i}{N}} \right| \left| 1 - r_k^2 e^{\frac{2\pi i}{N}} \right|^{-1}$$

$$= 2r_k \sin\frac{\pi}{N} \left[(1 - r_k^2)^2 + 4r_k^2 \sin^2 \frac{\pi}{N} \right]^{-1/2} =$$

$$= \left\{ 1 + [(1 - r_k^2)/(2r_k \sin(\pi/N))]^2 \right\}^{-1/2}.$$

But $\sin(\pi/N) \geq (2/\pi)(\pi/N) = 2/N$, so we deduce that

$$\rho(z_{k\ell}, z_{kn}) \geq \left\{ 1 + [(1 - r_k^2)N/(4r_k)]^2 \right\}^{-1/2}.$$

We observe that

$$(1 - r_k^2)N/(4r_k) = \frac{N}{(a^{2k} - a^{-2k})} = Na^{-2k}[1/(1 - a^{-4k})],$$

and $\rho(z_{k\ell}, z_{kn})$ has a positive lower bound if $\alpha = \lim_{k\to\infty} N(a,k)a^{-2k} < \infty$ and $\rho(z_{k\ell}, z_{kn}) \geq \frac{1}{\sqrt{1+\alpha^2}}$. Combining the two lower bounds we obtain the stated result for the separation constant.

2. For given $z = re^{i\theta} \in \mathbb{D}$ take k and $j \in \{0, 1, \cdots N(a,k) - 1\}$ such that $r_k < r \leq r_{k+1}$, $\theta \in [\frac{2\pi j}{N}, \frac{2\pi(j+1)}{N})$, $\theta_{kj} = \frac{2\pi j}{N}$, then

$$\frac{1}{1 - \rho^2(z, z_{kj})} = \frac{(1 - rr_k)^2 + 4rr_k \sin^2 \frac{\theta - \theta_{kj}}{2}}{(1 - r^2)(1 - r_k^2)} =$$

$$= 1 + \frac{(r - r_k)^2 + 4rr_k \sin^2 \frac{\theta - \theta_{kj}}{2}}{(1 - r^2)(1 - r_k^2)} \leq 1 + \frac{(r - r_k)^2 + 4rr_k \frac{\pi^2}{N^2}}{(1 - r^2)(1 - r_k^2)} =$$

$$= 1 + \frac{(a - a^{-1})^2}{4} + \frac{(a^{2k+2} - a^{-2k-2})(a^{2k} - a^{-2k})}{4} \frac{\pi^2}{N^2}.$$

If $(N(a,k)a^{-2k}, k \geq 1)$ is decreasing and $\alpha = \lim_{k\to\infty} N(a,k)a^{-2k} \in (0,\infty)$, then the last term in the previous inequality is upper bounded by

$$K := 1 + \frac{(a-a^{-1})^2}{4} + \frac{a^2}{4\alpha^2}\pi^2.$$

Then for $\epsilon_0 = \sqrt{1-1/K}$, we have $\rho(z, z_{kj}) < \epsilon_0$.

3. Using (1.7) we have that, if $(a-a^{-1})^2 + \pi^2 a^2/\alpha^2 < 2p$, then $\epsilon_0 < < \sqrt{p/(p+2)}$, consequently $\mathcal{A}$ is sampling sequence for A^p. ■

Remark 2.1. From this theorem we obtain that if $\mathcal{A}$ is a sampling set for A^p, then

$$(a-a^{-1})^2 < 2p,$$

therefore a must be in the interval $(1, \frac{\sqrt{2p}+\sqrt{2p+4}}{2})$. Then we can always choose $N = N(a,k)$ big enough, such that the the sampling condition will be satisfied. From the point of view of computations and to have on every circle the less possible numbers, for $p = 2$ a convenient choice is $a = 2$, and $N(2,k) = 2^{2k+\beta}$ for $k \geq 1$ with β a fixed integer. Then $\alpha = 2^\beta$, and the smallest value for β for which the sampling condition is satisfied is $\beta = 3$, then on the k-th circle we will have $N_1(2,k) = 2^{2k+3}$ equidistant points corresponding to the roots of order 2^{2k+3} of the unity. For $a = \sqrt{2}$ for sampling we need $N_1(\sqrt{2},k) = 2^{k+2}$ points.

2.2. Multiresolution analysis in the Bergman space

We start with the general definition of the affine wavelet multiresolution analysis in $L^2(\mathbb{R})$.

Definition 2.1. Let $V_j,\ j \in \mathbb{Z}$ be a sequence of subspaces of $L^2(\mathbb{R})$. The collections of spaces $\{V_j,\ j \in \mathbb{Z}\}$ is called a multiresolution analysis with scaling function ϕ if the following conditions hold:

1. (nested) $V_j \subset V_{j+1}$
2. (density) $\overline{\cup V_j} = L^2(R)$
3. (separation) $\cap V_j = \{0\}$
4. (basis) The function ϕ belongs to V_0 and the set $\{2^{n/2}\phi(2^n x - k),\ k \in \mathbb{Z}\}$ is a (orthonormal) bases in V_n.

We want to construct an analogue multiresolution decomposition in the Bergman space. In multiresolution analysis, one decomposes a function space in several resolution levels and the idea is to represent the functions from the function space by a low resolution approximation and adding to it the successive details that lift it to resolution levels of increasing detail.

Wavelet analysis couples the multiresolution idea with a special choice of bases for the different resolution spaces and for the wavelet spaces that represent the difference between successive resolution spaces. If V_n are the resolution spaces $V_0 \subset V_1 \subset \ldots \subset V_n \ldots$, then the wavelet spaces W_n are defined by the equality $W_n \bigoplus V_n = V_{n+1}$.

In the construction of affine wavelet multiresolutions the dilatation is used to obtain a higher level resolution ($f(x) \in V_n \Leftrightarrow f(2x) \in V_{n+1}$) and applying the translation we remain on the same level of resolution. This field has also a rich bibliography (see for example [3, 4, 1, 13, 14]).

Using the subgroup $\mathbb{B}_3$ of the Blaschke group, a discrete subgroup of $\mathbb{B}_2$ and the representation U_a we give a similar construction of the affine wavelet multiresolution in the Bergman space. To show the analogy with the affine wavelet multiresolution we first represent the levels V_n by nonorthogonal bases and then we construct an orthonormal bases in V_n and we give also an orthogonal basis in W_n which is orthogonal to V_n. We will show that in the case of this discretization the analogue of the Malmquist–Takenaka systems for Bergman space, will span the resolution spaces and the density property will be fulfilled, i.e., $\overline{\bigcup_{k=1}^{\infty} V_k} = A^2$ in norm. Similar multiresolutin results, based on another discrete subset of the Blaschke group, were obtained by the author in [15] for the Hardy space $\mathcal{H}^2(\mathbb{T})$.

Let formulate the analogue of the multiresolution for the Bergman spaces:

Definition 2.2. Let $V_j,\ j \in N$ be a sequence of subspaces of A^2. The collections of spaces $\{V_j,\ j \in N\}$ is called a multiresolution if the following conditions hold:

1. (nested) $V_j \subset V_{j+1}$,
2. (density) $\overline{\cup V_j} = A^2$
3. (analogue of dilatation) $U_{(r_1,1)^{-1}}(V_j) \subset V_{j+1}$
4. (basis) There exist $\psi_{j\ell}$ (orthonormal) bases in V_j.

Let consider $a > 1$, denote by $r_k = \frac{a^k - a^{-k}}{a^k + a^{-k}}$, $k \in \mathbb{N}$, $N = N_k = N(a,k)$ a sequence of natural numbers such that $\alpha = N(a,k)a^{-2k}$ satisfies

$$0 < \alpha < \infty, \quad (a - a^{-1})^2 + \pi^2 \frac{a^2}{\alpha^2} < 4,$$

and let consider $z_{k\ell} = r_k e^{i\frac{2\pi\ell}{N}}$, $\ell = 0, 1, \ldots, N(a,k) - 1$. If these conditions are satisfied then, due to Theorem 2.1, $\mathcal{A}$ given by (2.6) is a sampling set for A^2.

This implies that the set of normalized kernels

$$\left\{\varphi_{kl}(z)=\frac{(1-r_k^2)}{(1-\overline{z_{k\ell}}z)^2},\ k=0,1,\cdots,\ \ell=0,1,\cdots N_k-1\right\}$$

will constitute a frame system for A^2. From the frame theory (see for example in [10]), or from the atomic decomposition results (see Theorem 3 of [26]), follows that every function f from A^2 can be represented

$$f(z)=\sum_{(k,\ell)} c_{k\ell}\varphi_{kl}(z)$$

for some $\{c_{k\ell}\}\in\ell^2$, with the series converging in A^2 norm. The determination of the coefficients it is related to the construction of the inverse frame operator (see [10]), which is not an easy task in general. This is the reason why we try to construct other approximation process for $f\in A^2$ such that the determination of the coefficients follow an exactly defined algorithmic scheme. We note that Feichtinger and Onchis in the case of the spline-type spaces developed a new numerical approach for the computation of the dual systems and the coefficients (see [9]).

Let us consider the function $\varphi_{00}=1$ and let $V_0=\{c,\ c\in\mathbb{C}\}$. Let us consider the nonorthogonal hyperbolic wavelets at the first level

$$\varphi_{1,\ell}(z)=(U_{(z_{1\ell},1)^{-1}}p_0)(z)=\frac{(1-r_1^2)}{(1-\overline{z_{1\ell}}z)^2},\quad \ell=0,1,\cdots,N(a,1)-1. \tag{2.8}$$

They can be obtained from $\varphi_{1,0}$ using the analogue of translation operator which in the unit disc is a multiplication by a unimodular complex number, and from $\varphi_{0,0}$ using first the representation operator $U_{(r_1,1)^{-1}}$ followed by the translation operator:

$$\varphi_{1,\ell}(z)=\varphi_{1,0}(ze^{-\frac{2\pi i\ell}{N(a,1)}})=(U_{(r_1,1)^{-1}}\varphi_{0,0})(ze^{-\frac{2\pi i\ell}{N(a,1)}}). \tag{2.9}$$

Let us define the first resolution level as follows

$$V_1=\left\{f:\mathbb{D}\to\mathbb{C},\ f(z)=\sum_{k=0}^{1}\sum_{\ell=0}^{N(a,k)-1}c_{k,\ell}\varphi_{k,\ell},\ c_{k,\ell}\in\mathbb{C}\right\}. \tag{2.10}$$

Let us consider the nonorthogonal wavelets on the n-th level

$$\varphi_{n,\ell}(z)=(U_{(z_{n\ell},1)^{-1}}p_0)(z)=\frac{(1-r_n^2)}{(1-\overline{z_{n\ell}}z)^2},\quad \ell=0,1,\ldots,N(a,n)-1, \tag{2.11}$$

which can be obtained from $\varphi_{n,0}$ using the translation operator, and from $\varphi_{0,0}$ using the representation $U_{((r_{n-1},1)\circ(r_1,1))^{-1}}$, and the translations

$$\varphi_{n,\ell}(z)=(U_{((r_{n-1},1)\circ(r_1,1))^{-1}}p_0)(ze^{-i\frac{2\pi\ell}{N(a,n)}}). \tag{2.12}$$

Let us define the n-th resolution level by

$$V_n = \left\{ f : \mathbb{D} \to \mathbb{C},\ f(z) = \sum_{k=0}^{n} \sum_{\ell=0}^{N(a,k)-1} c_{k,\ell}\varphi_{k,\ell},\ c_{k,\ell} \in \mathbb{C} \right\}. \tag{2.13}$$

The closed subset V_n is spanned by

$$\{\varphi_{k,\ell},\ \ell = 0, 1, \ldots, N(a,k) - 1,\ k = 0, \ldots, n\}. \tag{2.14}$$

Continuing this procedure we obtain a sequence of closed, nested subspaces of A^2 for $z \in \mathbb{D}$

$$V_0 \subset V_1 \subset V_2 \subset \ldots..V_n \subset \ldots. A^2. \tag{2.15}$$

Due to Theorem 2.1 the normalized kernels

$$\{\varphi_{kl}(z) = \frac{(1 - r_k^2)}{(1 - \overline{z_{k\ell}}z)^2},\ k = 0, 1, \cdots,\ \ell = 0, 1, \cdots N(a,k) - 1\}$$

form a frame system for A^2 which implies that this is a complete and a closed set in norm, i.e.,

$$\overline{\bigcup_{n\in\mathbb{N}} V_n} = A^2, \tag{2.16}$$

consequently the density property it is satisfied.

For $a = 2$ and $N(2,n) = 2^{2n+3}$, if a function $f \in V_n$, then $U_{(r_1,1)^{-1}} f \in V_{n+1}$. This is the analogue of the dilation. For this it is sufficient to show that

$$U_{(r_1,1)^{-1}}(\varphi_{k,\ell})(z) = U_{(r_1,1)^{-1}}[(U_{(r_k,1)^{-1}} p_0)](ze^{-i\frac{2\pi\ell}{2^{2k+3}}}) = \tag{2.17}$$

$$= [(U_{(r_{k+1},1)^{-1}} p_0)](ze^{-i\frac{2\pi 4\ell}{2^{2(k+1)+3}}}) \in V_{n+1}, \quad k = 1, .., n,\ \ell = 1, \ldots 2^{2k+3} - 1.$$

From now on for simplicity we will deal with this case.

Since the set $\mathcal{A}$ is a sampling set it follows that is a set of uniqueness for A^2, which means that every function $f \in A^2$ is uniquely determined by the values $\{f(z_{k\ell})\}$. In the paper of Kehe Zhu [27] described in general how can be recaptured a function from a Hilbert space when the values of the function on a set of uniqueness are known and developed in details this process in the Hardy space. At the beginning we will follow the steps of the recapturation process but we will combine this with the multiresolutin analysis. The elements of the set

$$\left\{ \frac{1}{(1 - \overline{z_{k\ell}}z)^2},\ \ell = 0, 1, \ldots, 2^{2k+3} - 1,\ k = 0, 1, \ldots, n. \right\} \tag{2.18}$$

are linearly independent and constitute a nonorthogonal basis in V_n.

Using Gram–Schmidt orthogonalization process they can be orthogonalized. Denote by $\psi_{k,\ell}$ the resulting functions. They can be seen as the analogue of the Malmquist–Takenaka system in the Hardy space. This functions can be obtained using the following two methods. The first arises from the orthogonalization procedure. To describe this let reindex the points of the set $\mathcal{A}$ as follows $a_1 = z_{00}$, $a_2 = z_{10}$, $a_3 = z_{11}$, $a_4 = z_{12}$, ..., $a_{33} = z_{1,31}$, $a_{34} = z_{2,0}$, ..., $a_m = z_{k\ell} \dots, k = 0, 1, \dots, \ell = 0, 1, \dots, 2^{2k+3} - 1$, and denote by $K(z, z_{k\ell}) = \frac{1}{(1-\overline{z_{k\ell}}z)^2} := K(z, a_m)$, then the resulted orthonormal system is

$$\phi_{00}(z) = \phi(a_1, z) = \frac{K(z, a_1)}{\|K(., a_1)\|}, \ \phi_{k\ell}(z) = \phi(a_1, a_2, \dots, a_m, z) =$$

$$= K(z, a_m) - \sum_{i=1}^{m-1} \phi(a_1, a_2, \dots, a_i, z) \frac{\langle K(., a_m), \phi(a_1, a_2, \dots, a_i, .)\rangle}{\|\phi(a_1, a_2, \dots, a_i, .)\|^2}.$$

Thus the normalized functions

$$\left\{ \psi_{k\ell}(z) = \frac{\phi_{k\ell}(z)}{\|\phi_{k\ell}\|}, \, k = 1, 2, \dots, \ell = 0, 1, \dots 2^{2k+3} \right\}$$

became an orthonormal system. Applying similar construction in Hardy space we get in this way the Malmquist–Takenaka system. They can be written in a nice closed form using the Blaschke products. Unfortunately in our situation this is not the case and the properties of the system can be not seen from the previous construction.

Another approach is given by Zhu in [27]. He proved that the result of the Gram–Schmidt process are connected to some reproducing kernels and the contractive divisors on Bergman spaces. Let denote $A_m = \{a_1, a_2, \dots a_m\}$ a set of distinct points in the unit disc. Let H_{A_m} the subspace of A^2 consisting of all functions in A^2 which vanish on A_m. H_{A_m} is a closed subspace of A^2 and denote by K_{A_m} the reproducing kernel of H_{A_m}. These reproducing kernels satisfies the following recursion formula

$$(2.19) \qquad K_{A_{m+1}}(z, w) = K_{A_m}(z, w) - \frac{K_{A_m}(z, a_{m+1}) K_{A_m}(a_{m+1}, w)}{K_{A_m}(a_{m+1}, a_{m+1})}, m \geq 0,$$

$$K_{A_0} := K(z, w) = \frac{1}{(1 - \overline{w}z)^2}.$$

The result of the Gram–Schmidt process can be expressed as

$$\frac{K(z, a_1)}{\sqrt{K(a_1, a_1)}}, \ \frac{K_{A_1}(z, a_2)}{\sqrt{K_{A_1}(a_2, a_2)}}, \ \dots, \ \frac{K_{A_{m-1}}(z, a_m)}{\sqrt{K_{A_{m-1}}(a_m, a_m)}}, \ \dots.$$

Then

$$\psi_{k\ell}(z) = \frac{K_{A_{m-1}}(z, a_m)}{\sqrt{K_{A_{m-1}}(a_m, a_m)}} \tag{2.20}$$

and is the solution of the following problem

$$\sup\{Ref(a_m) : f \in H_{A_{m-1}}, \|f\| \leq 1\}.$$

This extremal functions in the context of the Bergman spaces have been studied extensively in recent years by Hedenmalm [11]. The main result in [11] is that the function

$$\frac{K_{A_{m-1}}(z, a_m)}{\sqrt{K_{A_{m-1}}(a_m, a_m)}}$$

is a contractive divisor on the Bergman space, vanishes on A_{m-1}, and if $\mathcal{A}$ is not a zero set for A^2, as is in our case, the functions converge to 0 as $m \to \infty$. In Hardy space the partial products of a Blaschke product corresponding to a nonzero set own all these nice properties.

From the Gram–Schmidt orthogonalization process it follows that

$$V_n = span\{\psi_{k,\ell},\ \ell = 0, 1, \ldots, 2^{2k+3} - 1,\ k = \overline{0, n}\}. \tag{2.21}$$

The wavelet space W_n is the orthogonal complement of V_n in V_{n+1}. We will prove that

$$W_n = span\{\psi_{n+1,\ell}, \quad \ell = 0, 1, \ldots, 2^{2n+5} - 1\}. \tag{2.22}$$

If $f \in V_n$, one has $f(z) = \sum_{k=0}^{n} \sum_{\ell=0}^{2^{2k+3}-1} c_{k,\ell}\varphi_{k,\ell} \subset A^2$, then using (1.4) we obtain that

$$\langle \psi_{n+1j}, f\rangle = \sum_{k=0}^{n} \sum_{\ell=0}^{2^{2k+3}-1} c_{k,\ell}\langle \psi_{n+1,j}, \varphi_{k,\ell}\rangle =$$

$$\sum_{k=0}^{n} \sum_{\ell=0}^{2^{2k+3}-1} c_{k,\ell}(1 - r_k^2)\psi_{n+1,\ell}(z_{k\ell}) = 0,\ j = 0, 1, \ldots 2^{2n+5} - 1.$$

We have proved that for $f \in V_n$

$$\langle f, \psi_{n+1,j}\rangle = 0, \tag{2.23}$$

which is equivalent with

$$\psi_{n+1,j} \perp V_n, \quad (j = 0, 1, \ldots, 2^{2n+5} - 1). \tag{2.24}$$

From

$$V_{n+1} = V_n \bigoplus span\{\varphi_{n+1,j},\ j = 0, 1, \dots, 2^{2n+5} - 1\} \tag{2.25}$$

it follows that W_n is an 2^{2n+5} dimensional space and

$$W_n = span\{\psi_{n+1,\ell},\quad \ell = 0, 1, \dots, 2^{2n+5} - 1\}. \tag{2.26}$$

2.3. The projection operator corresponding to the n-th resolution level

Let us consider the orthogonal projection operator of an arbitrary function $f \in A^2$ on the subspace V_n given by

$$P_n f(z) = \sum_{k=0}^{n} \sum_{\ell=0}^{2^{2k+3}-1} \langle f, \psi_{k,\ell}\rangle \psi_{k,\ell}(z). \tag{2.27}$$

This operator is called the projection of f at scale or resolution level n and $P_n f(z)$ is a rational function.

Theorem 2.2. *For $f \in A^2$ the projection operator $P_n f$ is an interpolation operator in the points $z_{k\ell} = r_k e^{i\frac{2\pi\ell}{2^{2k+3}}}$, $(\ell = 0, \dots, 2^{2k+3} - 1,\quad k = 0, \dots, n)$, is norm convergent in A^2 to f, i.e.*

$$\|f - P_n f\| \to 0, \quad n \to \infty,$$

uniformly convergent inside the unit disc on every compact subset, and is the solution of a minimal norm interpolation problem.

Proof. Let consider $N = 1 + 2^5 + \dots + 2^{2n+3}$ and the corresponding kernel function of the projection operator

$$\mathbf{K}_N(z, \xi) = \sum_{k=0}^{n} \sum_{\ell=0}^{2^{2k+3}-1} \overline{\psi_{k,\ell}(\xi)} \psi_{k,\ell}(z) =$$

$$= \sum_{m=1}^{N} \frac{K_{A_{m-1}}(z, a_m)}{\sqrt{K_{A_{m-1}}(a_m, a_m)}} \overline{\left(\frac{K_{A_{m-1}}(\xi, a_m)}{\sqrt{K_{A_{m-1}}(a_m, a_m)}} \right)} = \tag{2.28}$$

$$= \sum_{m=1}^{N} \frac{K_{A_{m-1}}(z, a_m) K_{A_{m-1}}(a_m, \xi)}{K_{A_{m-1}}(a_m, a_m)}.$$

From the recursion relation (2.19) it follows that

$$\mathbf{K}_N(z,\xi) = \sum_{m=1}^{N} (K_{A_{m-1}}(z,\xi) - K_{A_m}(z,\xi)) = K(z,\xi) - K_{A_N}(z,\xi) \tag{2.29}$$

From this relation it follows that the values of the kernel-function in the points $z_{k\ell}$, $(\ell = 0, \ldots, 2^{2k+3}-1, \quad k = 0, \ldots, n)$ are equal to

$$K(z_{kl},\xi) = \frac{1}{(1 - z_{k\ell}\bar{\xi})^2}. \tag{2.30}$$

Using again formula (1.4) we have

$$P_n f(z_{k\ell}) = \int\limits_{\mathbb{D}} \frac{f(w)}{(1 - \overline{w} z_{mj})^2} dA(w) = f(z_{mj}) \tag{2.31}$$

for $(j = 0, \ldots, 2^{2m+3}-1, \quad m = 0, \ldots, n)$. We obtain that $P_n f$ is interpolation operator for every $f \in A^2$ on the set $\cup_{m=0}^{n} \mathcal{A}_m$.

Because of 2.16 and 2.21 $\{\psi_{k,\ell},\ k = 0, \cdots, \infty,\ \ell = 0, 1, \ldots, 2^{2k+3}-1\}$ is a closed set in the Hilbert space A^2, we have that that $\|f - P_n f\| \to 0$ as $n \to \infty$. Since convergence in A^2 norm implies uniform convergence on every compact subset inside the unit disc, we conclude that $P_n f(z) \to f(z)$ uniformly on every compact subset of the unit disc. From Theorem 5.3.1 of [18] there exists a unique $\hat{f}_n \in V_n$ with minimal norm such that

$$\hat{f}_n(z_{mj}) = f(z_{mj}), \quad (j = 0, \ldots, 2^{2m+3}-1, \quad m = 0, \ldots, n), \tag{2.32}$$

$\hat{f}_n$ is uniquely determined by the interpolation conditions and is equal to the orthogonal projection of f on V_n, thus $\hat{f}_n(z) = P_n f(z)$. ■

2.4. Reconstruction algorithm

In what follows we propose a computational scheme for the best approximant in the wavelet base $\{\psi_{k,\ell},\ \ell = 0, 1, \ldots, 2^{2k+3}-1,\ k = 0, \ldots, n\}$.

The projection of $f \in A^2$ onto V_{n+1} can be written in the following way:

$$P_{n+1} f = P_n f + Q_n f, \tag{2.33}$$

where

$$Q_n f(z) := \sum_{\ell=0}^{2^{2n+5}-1} \langle f, \psi_{n+1,\ell} \rangle \psi_{n+1,\ell}(z). \tag{2.34}$$

This operator has the following properties

$$Q_n f(z_{k\ell}) = 0, \quad k = 1,\dots,n,\ \ell = 0,1,\dots,2^{2n+3}-1. \tag{2.35}$$

Consequently P_n contains information on low resolution, i.e., until the level $\mathcal{A}_n$, and Q_n is the high resolution part. After n steps

$$P_{n+1}f = P_1 f + \sum_{k=1}^{n} Q_n f. \tag{2.36}$$

Thus

$$V_{n+1} = V_0 \bigoplus W_0 \bigoplus W_1 \bigoplus \dots \bigoplus W_n. \tag{2.37}$$

The set of coefficients of the best approximant $P_n f$

$$\{b_{k\ell} = \langle f, \psi_{k,\ell}\rangle,\ \ell = 0.1,\dots 2^{2k+3}-1 \ \ k = 0,1,\dots,n\} \tag{2.38}$$

is the (discrete) hyperbolic wavelet transform of the function $f \in A^2$. Thus it is important to have an efficient algorithm for the computation of the coefficients.

The coefficients of the projection operator $P_n f$ can be computed if we know the values of the functions on $\bigcup_{k=0}^{n} \mathcal{A}_k$. For this reason we express first the function $\psi_{k,\ell}$ using the bases $\{\varphi_{k',\ell'}\ \ell' = 0,1,\dots 2^{2k'+3}-1,\ k' = 0,\dots,k\}$, i.e. we write the partial fraction decomposition of $\psi_{k\ell}$:

$$\psi_{k,\ell} = \sum_{k'=0}^{k-1} \sum_{\ell'=0}^{2^{2k'+3}-1} c_{k',\ell'} \frac{1}{(1-\overline{z_{k'\ell'}}\xi)^2} + \sum_{j=0}^{\ell} c_{k,j} \frac{1}{(1-\overline{z_{kj}}\xi)^2}. \tag{2.39}$$

Using the orthogonality of the functions

$$\{\psi_{k',\ell'}\ \ell' = 0,1,\dots 2^{2k'+3}-1,\ k' = 0,\dots,k\}$$

and the formula (1.4) we obtain that

$$\begin{gathered}\delta_{kn}\delta_{\ell m} = \langle \psi_{nm}, \psi_{k\ell}\rangle = \\ = \sum_{k'=0}^{k-1} \sum_{\ell'=0}^{2^{2k'+3}-1} \overline{c_{k',\ell'}} \psi_{n,m}(z_{k'\ell'}) + \sum_{j=0}^{\ell} \overline{c_{k,j}} \psi_{n,m}(z_{kj}), \\ (m = 0,1,\dots 2^{2n+3}-1,\ n = 0,\dots,k).\end{gathered} \tag{2.40}$$

If we order these equalities so that we write first the relations (2.40) for $n = k$ and $m = \ell, \ell-1, \dots, 0$ respectively, then for $n = k-1$ and $m = 2^{2(k-1)+3}-1$,

$2^{2(k-1)+3}-2,\dots,0$, etc., this is equivalent to

(2.41)

$$\begin{pmatrix}1\\0\\0\\ \cdot\\ \cdot\\ \cdot\\0\end{pmatrix}=\begin{pmatrix}\psi_{k,\ell}(z_{k,\ell}) & 0 & 0 & \dots & 0\\ \psi_{k,\ell-1}(z_{k,\ell}) & \psi_{k,\ell-1}(z_{k,\ell-1}) & 0 & \dots & 0\\ \psi_{k,\ell-2}(z_{k,\ell}) & \psi_{k,\ell-2}(z_{k,\ell-1} & 0 & \dots & 0\\ \vdots & & & \vdots & \\ \psi_{00}(z_{k,\ell}) & \psi_{00}(z_{k,\ell-1}) & \psi_{00}(z_{k,\ell-2}) & \dots & \psi_{00}(z_{00})\end{pmatrix}\begin{pmatrix}\overline{c_{k,\ell}}\\ \overline{c_{k,\ell-1}}\\ \overline{c_{k,\ell-2}}\\ \vdots\\ \overline{c_{00}}\end{pmatrix}.$$

Because on the main diagonal the elements of the matrix are different from zero this system has a unique solution $(\overline{c_{k,\ell}},\overline{c_{k,\ell-1}},\overline{c_{k,\ell-2}},\dots,\overline{c_{00}})^T$. If we determine this vector, then we can compute the exact value of $\langle f,\psi_{k,\ell}\rangle$ knowing the values of f on the set $\bigcup_{k=0}^{n}\mathcal{A}_k$.

Indeed, using again the partial fraction decomposition of $\psi_{k,\ell}$ and the reconstruction formula formula we get that

$$\begin{aligned}\langle f,\psi_{k,\ell}\rangle &= \sum_{k'=0}^{k-1}\sum_{\ell'=0}^{2^{2k'}-1}\overline{c_{k',\ell'}}\left\langle f(\xi),\frac{1}{(1-\overline{z_{k'\ell'}}\xi)^2}\right\rangle+\\ &+\sum_{j=0}^{\ell}\overline{c_{k,j}}\left\langle f(\xi),\frac{1}{(1-\overline{z_{kj}}\xi)^2}\right\rangle=\\ &=\sum_{k'=0}^{k-1}\sum_{\ell'=0}^{2^{2k'}-1}\overline{c_{k',\ell'}}f(z_{k',\ell'})+\sum_{j=0}^{\ell}\overline{c_{k,j}}f(z_{k,j}).\end{aligned}\tag{2.42}$$

Conclusion We have introduced a new sampling set for A^p which is connected to the Blaschke group operation. We have generated a multiresolution in A^2 and we have constructed a rational orthogonal wavelet system which generates the levels of the multiresolution. Measuring the values of the function f in the points of the set $\mathcal{A}=\bigcup_{k=0}^{n}\mathcal{A}_k\subset\mathbb{D}$ we can write the operator $(P_nf,\ n\in\mathbb{N})$ which is convergent in A^2 norm to f, is the minimal norm interpolation operator on the set the $\bigcup_{k=0}^{n}\mathcal{A}_k$ and $P_nf(z)\to f(z)$ uniformly on every compact subset of the unit disc. We described a computational algorithm for the determination of the wavelet coefficients.

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References

[1] **Chui, C.K.,** *An Introduction to Wavelets*, Academic Press, New York, London, Toronto, Sydney, San Francisco, 1992.

[2] **Chui, C.K. and G. Chen,** *Signal Processing and Systems Theory*, Springer-Verlag, Series in Information Sciences, **26** (1992).

[3] **Daubechies, I.,** Orthonormal bases of compactly supported wavelets, *Comm. Pure Appl. Math.*, **41** (1988), 909–996.

[4] **Daubechies, I.,** *Ten Lectures on Wavelets*, CBMS-NSF Regional Conference Series in Applied Mathematics, Vol. 61, SIAM, Society for Industrial and Applied Mathematics; Philadelphia, ISBN: 0-89871-274-2.

[5] **Dudley, W.N.F. and J.R. Partington,** Robust identification in the disc algebra using rational wavelets and orthonormal basis functions, *Int. J. Control*, **64(3)** (1996), 409–423.

[6] **Duren, P. and A. Schuster,** *Bergman Spaces*, American Mathematical Society, Providence, RI, (2004).

[7] **Duren P., A. Schuster and D. Vukotic,** On uniformly discrete sequences in the disc, in: Quadrature Domains and Their Applications (P. Ebenfelt, B. Gustafsson, D. Khavinson, and M. Putinar, editors), Operator Theory: Advances and Applications 156 (2005), 105124.

[8] **Feichtinger, H.G. and K. Gröchenig,** *A unified approach to atomic decompositions trough integrable group representations* , Functions Spaces and Applications, M. Cwinkel e. all. eds. Lecture Notes in Math. 1302, Springer-Verlag (1989), 307-340.

[9] **Feichtinger H.G. and D. Onchis,** Constructive realization of dual systems for generators of multi-window spline-type spaces, *J. Comput. Appl. Math.*, **234(12)**: 3467–3479.

[10] **Gröchenig, K.**, *Fondation of Time-Frequency Analysis*, Birkhöser Boston, 2001, ISBN 0-8176-4022-3.

[11] **Hedenmalm, H.,** A factorization for square area-integrable analytic functions, *J. Reine Angew. Math.*, **422** (1991), 45–68.

[12] **Hendenmalm, H., B. Korenblum and K. Zhu,** *Theory of Bergman Spaces*, Graduate Text in Mathematics 199, Springer-Verlag, New York, (2000).

[13] **Heil C.E. and D.F. Walnut,** Continuous and discrete wavelet transforms, *SIAM Review*, **31**, No. **4**, December (1989), 628–666.

[14] **Mallat, S.,** Theory of multiresolution signal decomposition: The wavelet representation, *IEEE Trans. Pattern. Anal. Math. Intell.*, **11(7)** (1989), 674–693.

[15] **Pap, M.,** Hyperbolic wavelets and multiresolution in $H^2(\mathbb{T})$, *J. Fourier Anal. Appl.*, **17(5)** (2011), 755–776, DOI: 10.1007/s00041-011-9169-2.

[16] **Pap, M. and F. Schipp,** The voice transform on the Blaschke group III., *Publ. Math.*, **75(1-2)** (2009), 263–283.

[17] **Pap, M.,** Properties of the voice transform of the Blaschke group and connection with atomic decomposition results in the weighted Bergman spaces, *Journal of Mathematical Analysis and Applications*, **389** (2012), 340–350, DOI 10.1016/j.jmaa.2011.11.060.

[18] **Partington, J.,** *Interpolation, Identification and Sampling*, volume 17 of London Mathematical Society Monographs, Oxford University Press, 1997.

[19] **Seip, K.,** *Interpolattion and Identification in Spaces of Analytic Functions*, American Mathemathical Society, Univ. Lect. Series vol. 33, 2004, ISBN: 0-8218-3554-8.

[20] **Seip, K.,** Berling type density theorems in the unit disc, *Invent. Math.*, **113** (1994), 21–39.

[21] **Schipp, F.,** Wavelets on the disc, in: Proc. Workshop on System and Control Theory, Budapest, Univ. Technology and Economics, Comp. and Automation Research Institute of HAS, 2009, 101-109.

[22] **Schuster, A. and D. Varoline,** Interpolation and sampling on Riemann surfaces, *Rev. Mat. Iberoamericana*, **24(2)** (2008), 499–530. arXiv:math/0310174v2.

[23] **Soumelidis, A., J. Bokor and F. Schipp,** Signal and system representations on hyperbolic groups: beyond rational Orthogonal Bases, ICC 2009 7th IEEE, International Conference on Computational Cybernetics ICCC 2009. Palma de Mallorca, (Length: 14 page).

[24] **Schipp, F. and W.R. Wade,** *Transforms on Normed Fields*, Leaflets in Mathematics, Janus Pannonius University Pécs, 1995.

[25] **Schipp, F.,** Wavelet like transform on the Blaschke group, *Walsh and dyadic Analysis*, Proc. of the workshop, NIS. (2007), 85–93.

[26] **Zhu, K.,** Evaluation operators on the Bergman space, *Math. Proc. Comb. Phil. Soc.*, **117** (1995), 513–523.

[27] **Zhu, K.,** Interpolating and recapturating in reproducing Hilbert spaces, *BHKMS*,**1** (1997), 21–33.

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