

## SUMS OF CONTINUED FRACTIONS TO THE NEAREST INTEGER

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*Dedicated to Professor Dr. K.-H. Indlekofer  
at the occasion of his 70th birthday*

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**Abstract.** Let  $b$  be a positive integer. We prove that every real number can be written as sum of an integer and at most  $\lfloor \frac{b+1}{2} \rfloor$  continued fractions to the nearest integer each of which having partial quotients at least  $b$ .

### 1. Introduction and statement of the main results

In 1947, Hall [7] proved that every real number can be written as a sum of an integer and two regular continued fractions each of which having partial quotients less than or equal to four. Denoting by  $F(b)$  the set of those real numbers  $x$  having a regular continued fraction expansion  $x = [a_0, a_1, a_2, \dots, a_n, \dots]$  with arbitrary  $a_0 \in \mathbb{Z}$  and partial quotients  $a_n \leq b$  for  $n \in \mathbb{N}$  (with  $\mathbb{N} := \{1, 2, \dots\}$ ), where  $b$  is a positive integer, Hall's theorem can be stated as  $F(4) + F(4) = \mathbb{R}$ ; here the sumset  $A + B$  is defined as the set of all pairwise sums  $a + b$  with  $a \in A$  and  $b \in B$  (also called Minkowski sum in some literature). There have been several generalizations of Hall's remarkable result. For example, Cusick

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[4] and Diviš [6] showed independently that  $F(3) + F(3) \neq \mathbb{R}$ ; Hlavka [8] obtained  $F(3) + F(4) = \mathbb{R}$  as well as  $F(2) + F(4) \neq \mathbb{R}$ ; Astels [2] proved among other things that  $F(5) \pm F(2) = \mathbb{R}$  and, quite surprisingly,  $F(3) - F(3) = \mathbb{R}$ . Actually, Astels' general approach [1] yields a powerful tool for any kind of related questions with respect to regular continued fractions.

On the contrary, one may ask what one can get by adding continued fractions where all partial quotients are larger than a given quantity. For this purpose Cusick [3] defined for  $b \geq 2$  the set  $S(b)$  consisting of all  $x = [0, a_1, a_2, \dots, a_n, \dots] \leq b^{-1}$  containing no partial quotient less than  $b$  and proved  $S(2) + S(2) = [0, 1]$ . \* In [5], Cusick & Lee extended this result by proving

$$(1.1) \quad bS(b) = [0, 1] \quad \text{for any integer } b \geq 2,$$

where the left hand-side is defined as the sumset of  $b$  copies of  $S(b)$ . The result of Cusick & Lee is best possible as the following example illustrates:

$$\left(\frac{7}{12}, \frac{3}{5}\right) \notin 2S(3) \subset [0, \frac{2}{3}].$$

Here we are concerned about an analogue of this result for continued fractions to the nearest integer.

Given a real number  $x \in [-\frac{1}{2}, \frac{1}{2})$ , its continued fraction to the nearest integer is of the form

$$x = \frac{\epsilon_1}{a_1} + \frac{\epsilon_2}{a_2} + \dots + \frac{\epsilon_n}{a_n} + \dots,$$

resp.  $x = [0, \epsilon_1/a_1, \epsilon_2/a_2, \dots, \epsilon_n/a_n, \dots]$  for short. The partial quotients  $a_n$  and signs  $\epsilon_n = \pm 1$  are determined by the map

$$x \mapsto T(x) = \frac{1}{|x|} - \left\lfloor \frac{1}{|x|} + \frac{1}{2} \right\rfloor \quad \text{for } x \neq 0$$

and  $T(0) = 0$  on  $[-\frac{1}{2}, \frac{1}{2})$  by setting  $\epsilon_n = \pm 1$  according to  $T^{n-1}(x)$  being positive or not, and

$$a_n := \left\lfloor \frac{\epsilon_n}{T^{n-1}(x)} + \frac{1}{2} \right\rfloor,$$

where  $T^k = T \circ T^{k-1}$  denotes the  $k$ th iteration of  $T$  and  $T^0$  is the identity. This continued fraction expansion to the nearest integer was first introduced by Minnigerode [10]. Notice that

$$(1.2) \quad a_n + \epsilon_{n+1} \geq 2$$

for  $n \in \mathbb{N}$ . For further details we refer to Perron's monograph [11].

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\*The reader shan't be confused by our use of rectangular brackets for closed intervals and continued fractions. It'll always be clear from the context what is meant.

We denote by  $\mathcal{L}(b)$  the set of all real numbers  $x \in [-\frac{1}{2}, \frac{1}{2})$  having a continued fraction to the nearest integer with all partial quotients  $a_n$  being larger than or equal to  $b$ , where  $b$  is a positive integer. Following Cusick [3] it is not difficult to show that  $\mathcal{L}(b)$  is a Cantor set and, in particular, of Lebesgue measure zero (see also Rockett & Szűsz [12], Chapter V). The following theorem extends the theorem of Cusick & Lee (1.1) to continued fractions to the nearest integer:

**Theorem 1.1.** *Let  $b$  be a positive integer. Every real number can be written as sum of an integer and at most  $\lfloor \frac{b+1}{2} \rfloor$  continued fractions to the nearest integer each of which having partial quotients at least  $b$ . Moreover, if  $b \geq 3$ , then*

$$\left\lfloor \frac{b+1}{2} \right\rfloor \mathcal{L}(b) = \left[ - \left\lfloor \frac{b+1}{2} \right\rfloor \beta, \left\lfloor \frac{b+1}{2} \right\rfloor \beta \right],$$

with  $\beta = \frac{1}{2}(b - \sqrt{b^2 - 4})$ , and the interval on the right hand-side has length larger than one. The result is best possible in the following sense: if  $m < \lfloor \frac{b+1}{2} \rfloor$ , then  $m\mathcal{L}(b) \subset [-m\beta, m\beta]$  and the interval on the right has length less than one.

This result is well-known in the case  $b = 2$  (and the proof follows already from Lemma 2.1 below). Notice that  $\beta \sim \frac{1}{b}$ . Thus, comparing with the theorem of Cusick & Lee (1.1), it follows that for general  $b$  only about half of the continued fractions are needed when those are built with respect to the nearest integer. This factor *one half* is a consequence of the fact that continued fractions to the nearest integer have *two* signs. Moreover,

$$\frac{4}{15} = \frac{1}{3} + \frac{1}{1+3} = \frac{1}{4} + \frac{-1}{4};$$

hence, this number is an element of  $\mathcal{L}(4)$  but not of  $\mathcal{S}(4)$ . This already indicates that continued fractions to the nearest integer 'avoid' very small partial quotients. A last remark: whenever  $\lfloor \frac{b+1}{2} \rfloor \geq 2$ , that is  $b \geq 3$ , the assertion of the theorem implies also that there is a representation of any real number as a difference of an integer and suitable continued fractions to the nearest integer. For instance,

$$\mathcal{L}(3) - \mathcal{L}(3) = [\sqrt{5} - 3, 3 - \sqrt{5}] = [-0.76393\dots, 0.76393\dots]$$

in contrast to the aforementioned results [4, 6, 2] for regular continued fractions with bounded partial quotients. The reason behind is the symmetry of continued fractions to the nearest integer with respect to zero (simply by changing the first sign  $\epsilon_1$  in the corresponding expansion).

Theorem 1.1 will be proved in Section 3. The case of complex continued fractions to the nearest Gaussian integer will be discussed in the final section. However, we start with some preliminaries.

## 2. Preliminaries

In the sequel we sometimes denote the  $n$ th partial quotient and the  $n$ th sign in the continued fraction expansion to the nearest integer of  $x$  by  $a_n(x)$  and  $\epsilon_n(x)$ , respectively.

**Lemma 2.1.** *Given  $j, n \in \mathbb{N}$ , we have  $a_n(x) = \pm j$  if, and only if,*

$$T^{n-1}(x) \in \left\{ \left[ \frac{-1}{j-\frac{1}{2}}, \frac{-1}{j+\frac{1}{2}} \right) \cup \left( \frac{1}{j+\frac{1}{2}}, \frac{1}{j-\frac{1}{2}} \right] \right\} \cap \left[-\frac{1}{2}, \frac{1}{2}\right).$$

More precisely, for positive  $T^{n-1}(x)$ , we have  $\epsilon_n(x) = +1$  and

$$\begin{aligned} a_n(x) = j \geq 3 &\iff T^{n-1}(x) \in \left( \frac{1}{j+\frac{1}{2}}, \frac{1}{j-\frac{1}{2}} \right], \\ a_n(x) = 2 &\iff T^{n-1}(x) \in \left( \frac{2}{5}, \frac{1}{2} \right). \end{aligned}$$

while, for negative  $T^{n-1}(x)$ , we have  $\epsilon_n(x) = -1$  and

$$\begin{aligned} a_n(x) = j \geq 3 &\iff T^{n-1}(x) \in \left( \frac{-1}{j-\frac{1}{2}}, \frac{-1}{j+\frac{1}{2}} \right], \\ a_n(x) = 2 &\iff T^{n-1}(x) \in \left[-\frac{1}{2}, -\frac{2}{5}\right). \end{aligned}$$

A partial quotient equal to 1 is impossible.

This indicates a symmetry in the distribution of partial quotients with respect to zero for the interior of the intervals. Furthermore, the lemma implies Condition (1.2). Another trivial consequence is  $\mathcal{L}(2) = [-\frac{1}{2}, \frac{1}{2})$ ; hence, every real number has a continued fraction expansion to the nearest integer with all partial quotients being larger than or equal to two which is an assertion of the theorem for  $b = 2$ . (See Figure 1 for an illustration.)

**Proof.** Writing

$$x = \frac{\epsilon_1(x)}{\frac{1}{|x|}} = \frac{\epsilon_1(x)}{\lfloor \frac{1}{|x|} + \frac{1}{2} \rfloor + \frac{1}{|x|} - \lfloor \frac{1}{|x|} + \frac{1}{2} \rfloor} = \frac{\epsilon_1(x)}{a_1(x) + T(x)},$$

we find  $a_1(x) = j$  if, and only if,

$$|x| \in \left( \frac{1}{j+\frac{1}{2}}, \frac{1}{j-\frac{1}{2}} \right] \cap \left[-\frac{1}{2}, \frac{1}{2}\right),$$

where the intersection on the right is with respect to the condition  $x \in [-\frac{1}{2}, \frac{1}{2})$ . The corresponding intervals may or may not lie completely inside  $[-\frac{1}{2}, \frac{1}{2})$ . In

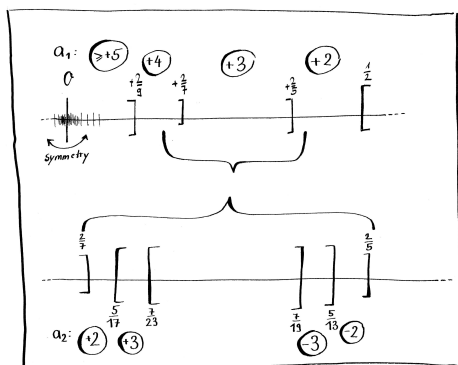


Figure 1. Distribution of the partial quotients  $a_1$  and  $a_2$ ; the structure is fractal.

order to obtain precise intervals for the partial quotients we observe that on the positive real axis

$$\left( \frac{1}{j + \frac{1}{2}}, \frac{1}{j - \frac{1}{2}} \right] \subset \left[ -\frac{1}{2}, \frac{1}{2} \right),$$

provided  $j \geq 3$ ; the partial quotient 2 is assigned to the interval  $(\frac{2}{5}, \frac{1}{2})$ , and a partial quotient 1 is impossible. The case of negative  $x$  follows from symmetry by switching the sign  $\epsilon_1$ . Replacing  $x$  in the previous lemma by some iterate  $T_\alpha^{n-1}(x)$ , the formulae of the lemma follow. ■

The following lemma is about a certain continued fraction to the nearest integer which is involved in the statement of Theorem 1.1 and in many estimates needed for its proof.

**Lemma 2.2.** For  $3 \leq b \in \mathbb{N}$ , denote by

$$\beta := [0, +1/b, \overline{-1/b}] := [0, +1/b, -1/b, -1/b, \dots]$$

the infinite eventually periodic continued fraction to the nearest integer with all partial quotients  $a_n = b$  and signs  $\epsilon_1 = +1 = -\epsilon_{n+1}$  for  $n \in \mathbb{N}$ . Then,

$$\beta = \frac{1}{2}(b - \sqrt{b^2 - 4}) \sim \frac{1}{b}.$$

For  $b = 2$  the formula yields  $\beta = 1$ , however, the expansion is not the continued fraction expansion for 1 since Condition (1.2) is not fulfilled; fortunately, this case of the theorem is already proved by the previous lemma. For  $b \geq 3$ , however, Condition (1.2) is satisfied and  $\beta$  is represented by the above continued fraction expansion to the nearest integer; in all these cases  $\beta$  is an irrational number inside  $[-\frac{1}{2}, \frac{1}{2})$ .

**Proof.** In view of the definition of  $\beta$ ,

$$\beta = \frac{1}{b + \frac{-1}{b + \frac{-1}{b + \dots}}} = \frac{1}{b - \beta};$$

hence,  $\beta$  is the positive root of the quadratic equation  $\beta^2 - b\beta + 1 = 0$ . The asymptotic formula for  $\beta$  follows easily from the Taylor expansion

$$\beta = \frac{b}{2} \left( 1 - \sqrt{1 - \frac{4}{b^2}} \right) = \frac{1}{b} + \frac{1}{b^3} + \frac{2}{b^5} + O\left(\frac{1}{b^7}\right).$$

The lemma is proved. ■

The next and final lemma is due to Cusick & Lee [5]. It is a generalization of Hall's interval arithmetic for the addition of Cantor sets which is the core of his method. We denote the length of an interval  $I$  by  $|I|$ .

**Lemma 2.3.** *Let  $I_0, I_1, \dots, I_n$  be disjoint bounded closed intervals of real numbers. Suppose that an open interval  $G$  is removed from the middle of  $I_0$ , leaving two closed intervals  $L$  and  $R$  on the left and right, respectively. If*

$$(2.1) \quad |G| \leq (m - 1) \min\{|L|, |R|, |I_1|, \dots, |I_n|\}$$

for some positive integer  $m$ , then

$$m \left( L \cup R \cup \bigcup_{j=1}^n I_j \right) = m \bigcup_{j=0}^n I_j.$$

Hence, if a sufficiently small interval is removed from the middle of some interval in a certain disjoint union, still the  $m$ -folded sum of the shrunk union adds up to the  $m$ -folded sum of the complete union. For the straightforward proof we refer to Cusick & Lee [5].

### 3. A Cusick & Lee-type theorem

The method of proof is along the lines of Hall's original paper [7] and Cusick & Lee [5] as well. Since the case  $b = 2$  has already been solved in the previous section, we may suppose  $b \geq 3$ .

Assume  $x = [0, \epsilon_1/a_1, \epsilon_2/a_2, \dots, \epsilon_n/a_n, \dots] \in \mathcal{L}(b)$ , then, by Lemma 2.1, the condition  $a_1 \geq b$  on the first partial quotient implies

$$-\frac{1}{b - \frac{1}{2}} \leq x \leq \frac{1}{b - \frac{1}{2}}.$$

In view of the second partial quotient  $a_2 \geq b$  we further find by a simple calculation

$$-\frac{1}{b + b - \frac{1}{2}} \leq x \leq \frac{1}{b + b - \frac{1}{2}}.$$

Going on, we find via  $a_n \geq b$  the inequality

$$(3.1) \quad -\beta \leq x \leq \beta$$

with  $\beta = [0, +1/b, -1/b, -1/b, \dots] = \frac{1}{2}(b - \sqrt{b^2 - 4})$  as in Lemma 2.2. Hence,  $\mathcal{L}(b) \subset [-\beta, \beta]$  and a necessary condition to find a representation of an arbitrary real number as a sum of an integer and  $m$  continued fractions to the nearest integer each of which having no partial quotient less than  $b$  is that  $m\mathcal{L}(b)$  covers an interval of length at least one. We thus obtain the necessary inequality

$$|[-m\beta, m\beta]| = 2m\beta \geq 1.$$

In view of  $\beta \sim b^{-1}$  by Lemma 2.2 we thus may expect  $m$  to be about  $\frac{b}{2}$ . However,  $m = \lfloor \frac{b-1}{2} \rfloor$  will not suffice since

$$m = \left\lfloor \frac{b-1}{2} \right\rfloor < \frac{1}{2\beta} = \frac{b + \sqrt{b^2 - 4}}{4} < \frac{b}{2},$$

as a simple computation shows.

For a start we remove from the complete interval  $[-\frac{1}{2}, \frac{1}{2})$  the intervals  $[-\frac{1}{2}, -\beta)$  and  $(\beta, \frac{1}{2})$  according to Condition (3.1); obviously, the *two* signs  $\epsilon_1 = \pm 1$  are responsible for removing intervals on *both* sides. Notice that  $0 < \beta \leq \frac{1}{2}(3 - \sqrt{5}) < \frac{1}{2}$  for any  $b \geq 3$ . In the remaining closed interval  $J_0 := [-\beta, \beta]$  all real numbers  $x = [0, \epsilon_1/a_1, \epsilon_2/a_2, \dots, \epsilon_n/a_n, \dots]$  have a first partial quotient  $a_1 \geq b$  as already explained above. Now consider all such  $x$  having sign  $\epsilon_1 = \epsilon$  for some fixed  $\epsilon \in \{\pm 1\}$  and partial quotient  $a_1 = a$  for some  $a \geq b$ . Clearly, the set of those  $x$  forms an interval  $I_1(\epsilon/a)$ , say. Since each element of  $I_1(\epsilon/a)$  is of the form

$$x = [0, \epsilon/a + T(x)] = \frac{\epsilon}{a + T(x)}$$

with  $T(x) \in [-\frac{1}{2}, \frac{1}{2})$ , we have either

$$I_1(-1/a) = \{x = [0, -1/a + t] : t \in [-\frac{1}{2}, \frac{1}{2})\} = \left[[0, -1/a - \frac{1}{2}], [0, -1/a + \frac{1}{2}]\right)$$

or

$$I_1(+1/a) = \{x = [0, +1/a + t] : t \in [-\frac{1}{2}, \frac{1}{2})\} = \left([0, +1/a + \frac{1}{2}], [0, +1/a - \frac{1}{2}]\right]$$

according to the sign  $\epsilon = \pm 1$ . In view of the condition  $a_2 \geq b$  we remove from any such  $I_1(\epsilon/a)$  in the next step two semi-open intervals with boundary points

$[0, \pm 1/a \pm \frac{1}{2}]$  and  $[0, \pm 1/a \pm \beta]$  on both sides. Consequently, the remaining intervals are

$$J_1(-1/a) := [[0, -1/a - \beta], [0, -1/a + \beta]]$$

and

$$J_1(+1/a) := [[0, +1/a + \beta], [0, +1/a - \beta]].$$

In general, we consider an interval  $J_n(\mathbf{a})$  consisting of those real numbers  $x$  having a prescribed continued fraction expansion to the nearest integer; denote by  $\mathbf{a}$  an arbitrary admissible sequence of signs and partial quotients  $\epsilon_1/a_1, \dots, \epsilon_n/a_n$ , namely positive integers  $a_j \geq b$  and  $\epsilon_j \in \{\pm 1\}$ , then  $J_n(\mathbf{a})$  is the closed interval

$$J_n(\mathbf{a}) := [[0, \epsilon_1/a_1, \dots, \epsilon_n/a_n - \beta], [0, \epsilon_1/a_1, \dots, \epsilon_n/a_n + \beta]].$$

Here and in the sequel it may happen that in an interval  $[A, B]$  or  $(A, B)$  we have the relation  $A > B$  for the boundary points in which case the interval is meant to be equal to  $[B, A]$ , resp.  $(B, A)$ . From such an interval  $J_n(\mathbf{a})$  we remove the open intervals of the form

$$G_{n+1}(\mathbf{a}') := ([0, \epsilon_1/a_1, \dots, \epsilon_n/a_n, \epsilon/a + 1 - \beta], [0, \epsilon_1/a_1, \dots, \epsilon_n/a_n, \epsilon/a + \beta])$$

for any  $a \geq b$  and  $\epsilon = \pm 1$ , where

$$\mathbf{a}' := \mathbf{a}, \epsilon/a := \epsilon_1/a_1, \dots, \epsilon_n/a_n, \epsilon/a$$

(by adding  $\epsilon/a$  to  $\mathbf{a}$  at the end). This leads to further intervals of the form  $J_{n+1}(\mathbf{a}')$ . Following Cusick & Lee [5], we call this the Cantor dissection process (see Figure 2 for an illustration).

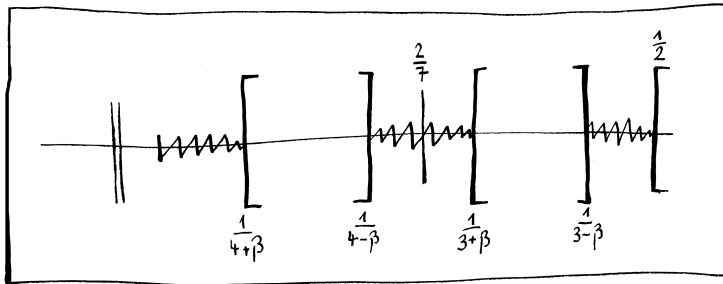


Figure 2. The first step of the Cantor dissection process for  $b = 3$ ; for instance, the number  $\frac{2}{7} = [0, +1/4 - \frac{1}{2}]$  is excluded from  $\mathcal{L}(b)$ .



The main idea now is applying Lemma 2.3 to this dissection process over and over again. In the beginning (when  $n = 0$ ) we have  $J_0 = [-\beta, \beta]$  and we remove step by step all open intervals of the form

$$G_1(+1/a) = ([0, +1/a + 1 - \beta], (0, +1/a + \beta])$$

for all  $a \geq b$  and their counterparts on the negative real axis. In fact, these are two closed intervals  $L := [-\beta, [0, +1/a + \beta]]$  and  $R := [[0, +1/a - \beta], \beta]$  on the left and right of  $G_1(+1/a)$ . Lemma 2.3 implies

$$m(L \cup R) = m(L \cup G_1(+1/a) \cup R) = mJ_0,$$

provided Condition (2.1) for the lengths of the intervals of type  $G_1$  and  $L, R$  is fulfilled. It is an easy computation to prove that the start of the Cantor dissection process gives no obstruction to the general case which we shall consider below. In view of the symmetry the situation on the left is similar.

In the general case, we have to find the least positive integer  $m$  satisfying

$$(3.2) \quad |G_{n+1}(\mathbf{a}, \pm 1/a_{n+1})| \leq (m-1) \min_{a_{n+1} \geq b} |J_{n+1}(\mathbf{a}, \pm 1/a_{n+1})|$$

with arbitrary  $a_{n+1} \geq b$ . If this quantity  $m$  is found, then it follows from Lemma 2.3 in combination with  $J_0 = [-\beta, \beta]$  that  $m\mathcal{L}(b) = [-m\beta, m\beta]$  and we are done, provided  $2m\beta \geq 1$  in order to cover an interval of length at least one.

For this aim we compute the lengths of the corresponding intervals by the standard continued fraction machinery as follows. Firstly, any continued fraction to the nearest integer can be written as a convergent

$$x = [0, \epsilon_1/a_1, \dots, \epsilon_n/a_n] = \frac{p_n}{q_n}$$

with coprime  $p_n$  and  $q_n > 0$ . The numerators and denominators  $p_n, q_n$  satisfy certain recursion formulae (as in the case of regular continued fractions; see [11], Kapitel I), which leads to

$$[0, \epsilon_1/a_1, \dots, \epsilon_n/a_n \pm \beta] = [0, \epsilon_1/a_1, \dots, \epsilon_n/a_n, \pm 1/b - \beta] = \frac{(b - \beta)p_n \pm p_{n-1}}{(b - \beta)q_n \pm q_{n-1}},$$

as well as

$$[0, \epsilon_1/a_1, \dots, \epsilon_n/a_n, \pm 1/a_{n+1} + 1 - \beta] = \frac{(a_{n+1} + 1 - \beta)p_n \pm p_{n-1}}{(a_{n+1} + 1 - \beta)q_n \pm q_{n-1}},$$

and

$$[0, \epsilon_1/a_1, \dots, \epsilon_n/a_n, \pm 1/a_{n+1} + \beta] = \frac{(a_{n+1} + \beta)p_n \pm p_{n-1}}{(a_{n+1} + \beta)q_n \pm q_{n-1}}$$

after a short computation. Using this in combination with

$$p_{n+1}q_n - p_nq_{n+1} = (-1)^n \prod_{j=1}^{n+1} \epsilon_j = \pm 1,$$

yields

$$|G_{n+1}(\mathbf{a}')| = \frac{1 - 2\beta}{((a_{n+1} + \beta)q_n \pm q_{n-1})((a_{n+1} + 1 - \beta)q_n \pm q_{n-1})},$$

as well as

$$|J_{n+1}(\mathbf{a}')| = \frac{a_{n+1} + 2\beta - b}{((b - \beta)q_n \pm q_{n-1})((a_{n+1} + \beta)q_n \pm q_{n-1})},$$

and

$$|J_{n+1}(\mathbf{a}')| = \frac{a_{n+1} + 1 - b}{((a_{n+1} + 1 - \beta)q_n \pm q_{n-1})((b - \beta)q_n \pm q_{n-1})},$$

depending on  $J_{n+1}(\mathbf{a}')$  lying on the left or on the right of  $G_{n+1}(\mathbf{a}')$ . Plugging this into (3.2), leads to

$$m - 1 \geq \max \left\{ \frac{1 - 2\beta}{a_{n+1} + 2\beta - b} \cdot \frac{(b - \beta)q_n \pm q_{n-1}}{(a_{n+1} + 1 - \beta)q_n \pm q_{n-1}}, \frac{1 - 2\beta}{a_{n+1} + 1 - b} \cdot \frac{(b - \beta)q_n \pm q_{n-1}}{(a_{n+1} + \beta)q_n \pm q_{n-1}} \right\}.$$

In view of  $a_{n+1} \geq b$  we deduce the condition

$$m \geq 1 + \frac{1 - 2\beta}{2\beta} = \frac{1}{2\beta} = \frac{b + \sqrt{b^2 - 4}}{4}$$

Hence, we may choose  $m = \lfloor \frac{b+1}{2} \rfloor$  as another short computation shows. This proves the theorem.

#### 4. Complex continued fractions

We conclude with some observations for complex continued fractions. Given a complex number  $z = x + iy$ , where  $i = \sqrt{-1}$  denotes, as usual, the imaginary unit in the upper half-plane, we may apply results for real continued fractions to both, the real and the imaginary part of  $z$  separately. A short computation shows

$$i \left( a_0 + \frac{1}{a_1 + \frac{1}{\alpha_2}} \right) = ia_0 + \frac{1}{-ia_1 + i\alpha_2},$$

where  $a_0, a_1$  are integers and  $\alpha_2$  is real (such that the expression on the left makes sense). Hence, the theorem of Cusick & Lee (1.1) immediately implies that every complex number  $z$  can be written as the sum of a Gaussian integer and  $2b$  regular continued fractions, where  $b$  of them have real partial quotients  $a_n \geq b$  while the others have partial quotients of the form  $\pm ia_n$  with integral  $a_n \geq b$ . Here the set of partial quotients  $\mathbb{Z}$  is replaced by the set of Gaussian integers  $\mathbb{Z}[i]$ . Using Theorem 1.1 we may deduce in the same way a comparable result for continued fractions to the nearest integer in the complex case:

**Corollary 4.1.** *Every complex number  $z$  can be written as the sum of a Gaussian integer and  $2\lfloor \frac{b+1}{2} \rfloor$  continued fractions to the nearest Gaussian integer, where half of them have real partial quotients  $a_n \geq b$  while the other half have partial quotients of the form  $\pm ia_n$  with integral  $a_n \geq b$ .*

In future work we shall consider complex methods for continued fractions to the nearest Gaussian integer as introduced by A. Hurwitz [9]. We expect that a careful analysis of the complex case will allow representations with less complex continued fractions. Of course, the partial quotients will not necessarily carry as much structure as in the above application; they just will be 'random' Gaussian integers of absolute value at least  $b$ .

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Happy  
Birthday!

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