POWER SERIES – THE STRUCTURE
OF H.-O.-GAPS

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Dedicated to an excellent mathematician
and my real good friend Karl-Heinz Indlekofer

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Abstract. Suppose that a power series $f(z) = \sum_{\nu=0}^{\infty} a_\nu (z - z_0)^\nu$ with positive radius of convergence has a sequence of H.-O.-gaps. Then there exists a neighborhood $U(z_0)$ of $z_0$, such that the rearrangement $f(z) = \sum_{\nu=0}^{\infty} b_\nu (z - \zeta)^\nu$ also has H.-O.-gaps for all $\zeta \in U(z_0)$.

1. Introduction

Let be given a power series $\sum_{\nu=0}^{\infty} a_\nu (z - z_0)^\nu$ with radius of convergence $R$, where $0 < R < \infty$. We say that this series has a sequence $\{p_k, q_k\}$ of Hadamard–Ostrowski-gaps (H.-O.-gaps for short) if the following conditions hold:

\begin{itemize}
  \item $p_k, q_k$ are natural numbers with $p_1 < q_1 < p_2 < q_2 < \ldots$,
  \item there exists $\lambda > 1$ such that $\frac{q_k}{p_k} > \lambda$ for all $k \in \mathbb{N}$,
  \item for $I := \bigcup_{k=1}^{\infty} [p_k, q_k]$ we have $\lim_{\nu \to \infty} \left| a_\nu \right|^{1/\nu} < \frac{1}{R}$.
\end{itemize}

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In the case that
\[
\frac{q_k}{p_k} \to \infty \quad \text{and} \quad \lim_{\nu \to \infty} |a_\nu|^{1/\nu} = 0
\]
we say that this series has Ostrowski-gaps (O.-gaps for short).

Approximately 90 years ago it was shown by Ostrowski ([3], [4], [5], [6]) that there exists a deep interdependence with the occurrence of those gaps and the phenomenon of overconvergence i.e. the convergence of a subsequence of the partial sums
\[
s_n(z) = \sum_{\nu=0}^{n} a_\nu (z - z_0)^\nu
\]
in a bigger domain than the circle of convergence.

Ostrowski’s main results on overconvergence are the following.

**Theorem O₁** Suppose that the power series
\[
f(z) = \sum_{\nu=0}^{\infty} a_\nu (z - z_0)^\nu
\]
with radius of convergence \(R, 0 < R < \infty\) possesses H.-O.-gaps \(\{p_k, q_k\}\) and let the function \(f\) be holomorphic at \(z_1, |z_1 - z_0| = R\).

Then there exists a neighborhood \(U(z_1)\) of \(z_1\) such that the sequence \(\{s_{p_k}(z)\}\) of partial sums converges compactly on \(U(z_1)\).

If the power series has O.-gaps \(\{p_k, q_k\}\), then the function \(f\) is holomorphic in a simply connected domain \(G(f) \supset \{z : |z - z_0| < R\}\) and \(\{s_{p_k}(z)\}\) converges compactly on \(G(f)\).

**Theorem O₂** Suppose that the power series
\[
f(z) = \sum_{\nu=0}^{\infty} a_\nu (z - z_0)^\nu
\]
has radius of convergence \(R, 0 < R < \infty\) and that \(\{s_{p_k}(z)\}\) is an overconvergent subsequence of its partial sums.

Then the series has a sequence of H.-O.-gaps of the type \(\{p_k, q_k\}\).

For a good treatise on the theory of overconvergence we refer to Hille’s book [2], section 16.7.

In recent years the investigation of overconvergence has got a revival since its connection with universal properties of power series has been detected. For details we refer to the excellent survey of Große-Erdmann [1], where also a synopsis of the relevant literature is given.

In this paper we are dealing with a long outstanding problem: If \(f(z) = \sum_{\nu=0}^{\infty} a_\nu (z - z_0)^\nu\) has a sequence of H.-O.-gaps, does there exist a neighborhood \(U(z_0)\) of \(z_0\), such that the series \(f(z) = \sum_{\nu=0}^{\infty} b_\nu (z - \zeta)^\nu\) has also H.-O.-gaps for all \(\zeta \in U(z_0)\)?

It is obvious that the corresponding problem cannot hold for power series with Hadamard-gaps or Fabry-gaps.
2. Power series with H.-O.-gaps

We first deal with power series having H.-O.-gaps and prove the following result.

**Theorem 1.** Suppose that the power series \( f(z) = \sum_{\nu=0}^{\infty} a_\nu(z - z_0)^\nu \) with radius of convergence \( R, 0 < R < \infty \) has a sequence of H.-O.-gaps. Then there is a neighborhood \( U(z_0) \) of \( z_0 \) such that \( f(z) = \sum_{\nu=0}^{\infty} b_\nu(z - \zeta)^\nu \) has also H.-O.-gaps for all \( \zeta \in U(z_0) \).

**Proof.** Without loss of generality we may assume that \( z_0 = 0 \) and \( R = 1 \) and that \( z = 1 \) is a singularity of the function \( f \). We denote by \( \{p_k, q_k\} \) the sequence of H.-O.-gaps and in addition we can suppose that \( a_\nu = 0 \) for \( p_k \leq \nu \leq q_k \).

We choose another sequence \( \{p_k^*, q_k^*\} \) of H.-O.-gaps and constants \( \gamma > 1 \) and \( \lambda^* > 1 \) in the following way

\[
p_k < p_k^* < q_k^* < q_k ; \quad \frac{p_k^*}{p_k} \geq \gamma ; \quad \frac{q_k^*}{q_k} \geq \gamma ; \quad \frac{q_k^*}{p_k} \geq \lambda^* .
\]

The partial sums of the considered power series are denoted by

\[
s_n(z) = \sum_{\nu=0}^{n} a_\nu z^\nu .
\]

Let be given a point \( \zeta \in \mathbb{D} \), then the expansion of \( f \) around \( \zeta \):

\[
f(z) = \sum_{\nu=0}^{\infty} b_\nu(z - \zeta)^\nu
\]

has radius of convergence at most \( |1 - \zeta| \).

1. For \( p_k \leq n \leq q_k \) we have

\[
s_n(z) = \sum_{\nu=0}^{n} b_\nu^{(\zeta)}(z - \zeta)^\nu
\]

and we obtain for \( 0 < r < 1 - |\zeta| \) and \( p_k \leq \nu \leq q_k \)

\[
b_\nu - b_\nu^{(\zeta)} = \frac{1}{2\pi i} \int_{|z-\zeta|=r} \frac{f(z) - s_\nu(z)}{(z - \zeta)^{\nu+1}} \, dz = \frac{1}{2\pi i} \int_{|z-\zeta|=r} \frac{f(z) - s_{q_k}(z)}{(z - \zeta)^{\nu+1}} .
\]
We choose $0 < \varepsilon < \min\{1 - |\zeta| - r, r^{1/\gamma} - r\}$ and for all sufficiently great $k$ it follows
\[
|b_\nu - b^{(\zeta)}_\nu| \leq \frac{1}{r^{\nu}} \cdot \max_{|z - \zeta| \leq r} |f(z) - s_{q_k}(z)| \leq \frac{1}{r^{\nu}} \cdot \max_{|z| \leq |\zeta| + r} |f(z) - s_{q_k}(z)| \leq \frac{1}{r^{\nu}} \cdot (|\zeta| + r + \varepsilon)^{q_k}.
\]
Therefore, we have for $p_k \leq \nu \leq q_k$
\[
|b_\nu - b^{(\zeta)}_\nu|^{1/\nu} \leq \frac{1}{r} \cdot (|\zeta| + r + \varepsilon)^{q_k} \leq \frac{1}{r} \cdot (|\zeta| + r + \varepsilon)^{q_k} \leq \frac{1}{r} \cdot (|\zeta| + r + \varepsilon)^{q_k} \leq \frac{1}{r} \cdot (|\zeta| + r + \varepsilon)^{q_k} \leq 1.
\]
For $\zeta = 0$ we have $(r + \varepsilon)^{q_k} < r$ and by continuity there exists $\zeta_1 \neq 0$ such that
\[
(1 + |\zeta|)^2 \cdot (|\zeta| + r + \varepsilon)^{q_k} < r
\]
for all $|\zeta| < |\zeta_1|$. We therefore get for $0 < |\zeta| < |\zeta_1|$ and sufficiently great $k$
\[
|b_\nu - b^{(\zeta)}_\nu|^{1/\nu} \leq \frac{1}{(1 + |\zeta|)^2} \leq \frac{1}{1 + |\zeta|} \leq \frac{1}{1 - |\zeta|}.
\]
2. For $p_k \leq \nu \leq q_k$ and $R > 1$ we have
\[
b^{(\zeta)}_\nu = \frac{1}{2\pi i} \int_{|z - \zeta| = R} \frac{s_\nu(z)}{(z - \zeta)^{\nu + 1}} \, dz = \frac{1}{2\pi i} \int_{|z - \zeta| = R} \frac{s_{p_k}(z)}{(z - \zeta)^{\nu + 1}} \, dz.
\]
If we choose $\zeta \neq 0$ and $\varepsilon > 0$ so small that $R + |\zeta| + \varepsilon < R^2$, we get for all sufficiently large $k$
\[
|b^{(\zeta)}_\nu| \leq \frac{1}{R^{\nu}} \cdot \max_{|z - \zeta| \leq R} |s_{p_k}(z)| \leq \frac{1}{R^{\nu}} \cdot \max_{|z| \leq |\zeta| + R} |s_{p_k}(z)| \leq \frac{1}{R^{\nu}} \cdot (R + |\zeta| + \varepsilon)^{q_k}.
\]
Therefore, we have for $p_k \leq \nu \leq q_k$
\[
|b^{(\zeta)}_\nu|^{1/\nu} \leq \frac{1}{R} \cdot (R + |\zeta| + \varepsilon)^{q_k} \leq \frac{1}{R} \cdot (R + |\zeta| + \varepsilon)^{q_k} \leq \frac{1}{R} \cdot (R + |\zeta| + \varepsilon)^{q_k} \leq (R + |\zeta| + \varepsilon)^{1/\gamma}
\]
and again as above we find a $\zeta_2 \neq 0$ such that for $0 < |\zeta| < |\zeta_2|$ and all sufficiently great $k$

$$|b^{(k_0)}_\nu|^{1/\nu} < \frac{1}{(1 + |\zeta|)^2} < \frac{1}{1 + |\zeta|} \leq \frac{1}{|1 - \zeta|}.$$ 

3. It follows that for $I^* = \bigcup_{k=1}^{\infty} [p_k^*, q_k^*]$ and all $\zeta$ with

$$0 < |\zeta| < \min \{|\zeta_1|, |\zeta_2|\} = \{|\zeta_0|\},$$

so, that the series $f(z) = \sum_{\nu=0}^{\infty} b_\nu (z - \zeta)^\nu$ has H.-O.-gaps.

**Remark 2.1.** By slightly changing the proof one can choose $\zeta_0 \neq 0$ so small, that for $m \geq 2$

$$\lim_{\nu \to \infty} \frac{|b_\nu|^{1/\nu}}{\nu} \leq \frac{1}{(1 + |\zeta|)^m}$$

for all $\zeta$ with $0 < |\zeta| < |\zeta_0|$. Therefore, the coefficients in the intervals $[p_k^*, q_k^*]$ become arbitrary small, when $\zeta$ is near to zero.

3. **Power series with O.-gaps**

We now consider power series $f(z) = \sum_{\nu=0}^{\infty} a_\nu (z - z_0)^\nu$ with radius of convergence $R, 0 < R < \infty$ and O.-gaps. It is well known [4] that such a series defines a function $f$, which is holomorphic in a simply connected domain $G(f)$ and every boundary point of $G(f)$ is a singularity for $f$.

We prove the following result.

**Theorem 2.** Suppose that the power series $f(z) = \sum_{\nu=0}^{\infty} a_\nu (z - z_0)^\nu$ with radius of convergence $R, 0 < R < \infty$, has a sequence of O.-gaps.

Then the power series $f(z) = \sum_{\nu=0}^{\infty} b_\nu (z - \zeta)^\nu$ has also O.-gaps for all $\zeta \in G(f)$.

**Proof.** Similar to the situation in the proof of Theorem 1 we again may assume that $z_0 = 0$ and $R = 1$, so that $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^{\nu}$. The sequence of O.-gaps is denoted by $\{p_k, q_k\}$ and we may suppose that $a_\nu = 0$ for $p_k \leq \nu \leq q_k$, where $\frac{p_k}{q_k} \to \infty$ for $k \to \infty$. 


We choose $p_k^*, q_k^*$ with $p_k < p_k^* < q_k^* < q_k$ in such way that
\[
\frac{p_k^*}{p_k} \to \infty, \quad \frac{q_k^*}{q_k} \to \infty, \quad \frac{q_k}{q_k^*} \to \infty \quad \text{for } k \to \infty.
\]
Let be given a point $\zeta \in G(f)$, consider the power series
\[
f(z) = \sum_{\nu=0}^{\infty} b_\nu (z - \zeta)^\nu
\]
and suppose that its radius of convergence is $2r$.

1. We consider
\[
s_{q_k^*}(z) = \sum_{\nu=0}^{q_k^*} a_\nu z^\nu = \sum_{\nu=0}^{q_k^*} b_\nu^{(q)}(z - \zeta)^\nu.
\]
For all sufficiently great $k$ we get with a constant $\vartheta, 0 < \vartheta < 1$
\[
\max_{|z-\zeta| \leq r} |f(z) - s_{q_k^*}(z)| = \max_{|z-\zeta| \leq r} |f(z) - s_{q_k}(z)| \leq \vartheta^{q_k^*}.
\]
For $p_k \leq \nu \leq q_k$ we have
\[
b_\nu - b_\nu^{(q)} = \frac{1}{2\pi i} \int_{|z-\zeta|=r} \frac{f(z) - s_{q_k}(z)}{(z - \zeta)^{\nu+1}} \, dz
\]
and it follows for $p_k^* \leq \nu \leq q_k^*$
\[
|b_\nu - b_\nu^{(q)}|^{1/\nu} \leq \frac{1}{r} \cdot \vartheta^{\frac{q_k}{p_k}}
\]
which implies that for $I^* = \bigcup_{k=1}^{\infty} [p_k^*, q_k^*]$
\[
\lim_{\nu \to \infty} \nu \leq I^*|b_\nu - b_\nu^{(q)}|^{1/\nu} = 0.
\]
2. For $R > 1$ and $p_k^* \leq \nu \leq q_k^*$ we have for sufficiently great $k$
\[
b_\nu^{(q)} = \frac{1}{2\pi i} \int_{|z-\zeta|=R} \frac{s_{q_k^*}(z)}{(z - \zeta)^{\nu+1}} \, dz = \frac{1}{2\pi i} \int_{|z-\zeta|=R} \frac{s_{p_k}(z)}{(z - \zeta)^{\nu+1}} \, dz
\]
and therefore we get for sufficiently great $k$
\[
|b_\nu^{(q)}|^{1/\nu} \leq \frac{1}{R} \cdot \max_{|z| \leq |\zeta|+R} |s_{p_k}(z)|^{1/\nu} \leq \frac{1}{R} \cdot (|\zeta| + R + 1)^{\frac{2k}{p_k}} \leq \frac{1}{R} \cdot (|\zeta| + R + 1)^{\frac{2k}{p_k}}
\]
which implies
\[ \lim_{\nu \to \infty} |b_{\nu}^{(\zeta)}|^{1/\nu} \leq \frac{1}{R} \]
and, since \( R > 1 \) was arbitrary, it follows
\[ \lim_{\nu \to \infty} |b_{\nu}^{(\zeta)}|^{1/\nu} = 0. \]

3. It follows that for all \( \zeta \in G(f) \) the power series \( f(z) = \sum_{\nu=0}^{\infty} b_{\nu}(z - \zeta)^{\nu} \) has a sequence of O.-gaps \( \{p_{k}^*, q_{k}^*\} \) (which is independent of \( \zeta \)). In addition the sequence \( s_{p_{k}^*}(z) = \sum_{\nu=0}^{p_{k}^*} b_{\nu}^{(\zeta)}(z - \zeta)^{\nu} \) converges to \( f(z) \) compactly on \( G(f) \). ■

References


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