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GENERAL SOLUTION OF A FUNCTIONAL EQUATION ARISEN FROM CHARACTERIZATION PROBLEMS

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Dedicated to Professor Karl-Heinz Indlekofer on his 70th birthday

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Abstract. We give the general solution of the functional equation

$$h_1\left(\frac{x}{\lambda_1(\alpha+y)}\right)\frac{1}{\lambda_1(\alpha+y)}f_Y(y)=h_2\left(\frac{y}{\lambda_2(\beta+x)}\right)\frac{1}{\lambda_2(\beta+x)}f_X(x)$$

for all $(x,y)\in\mathbb{R}_+^2$ with nonnegative functions $h_1,\ h_2,\ f_X,\ f_Y:\mathbb{R}_+\to\mathbb{R}$, such that there exist sets $A_1,\ A_2,\ A_3,\ A_4\subset\mathbb{R}_+$ with positive Lebesgue measure, on which these functions are positive

1. Introduction

In papers [8] and [9] we solved functional equation

$$h_1\left(\frac{x}{\lambda_1(\alpha+y)}\right)\frac{1}{\lambda_1(\alpha+y)}f_Y(y)=h_2\left(\frac{y}{\lambda_2(\beta+x)}\right)\frac{1}{\lambda_2(\beta+x)}f_X(x)\tag{1.1}$$

for almost all $(x,y)\in\mathbb{R}_+^2$ ($\mathbb{R}_+$ is the set of positive real numbers), by reducing it to the same equation satisfied everywhere on $\mathbb{R}_+^2$. To do this we had to suppose

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that the unknown functions in (1.1) are measurable and positive everywhere on their domains.

Then in [6] and [10] we supposed only that the unknown functions in (1.1) are density functions of some random variables (i.e. nonnegative and Lebegue integrable with integral 1). We showed that they are positive almost everywhere on their domains.

Here we give the general solution (without any regularity assumption) of equation (1.1) for all $(x,y) \in \mathbb{R}_+^2$ with nonnegative functions h_1, h_2, f_X, $f_Y : \mathbb{R}_+ \to \mathbb{R}$, such that there exist sets A_1, A_2, A_3, $A_4 \subset \mathbb{R}_+$ with positive Lebesgue measure, on which these functions are positive, respectively. We also suppose that λ_1, $\lambda_2 \in \mathbb{R}_+$ and α, $\beta \geq 0$ are arbitrary constants.

We use the following generalization of Steinhaus' theorem ([2]):

Theorem 1.1. *Let U be an open subset of $\mathbb{R}^2$ and $F : U \to \mathbb{R}$ be a continuously differentiable function with nonvanishing partial derivatives, moreover let A, $B \subset \mathbb{R}$ $(A \times B \subset U)$ be measurable sets with positive Lebesgue measure, then the set $F(A,B)$ has an interior point, i.e. $F(A,B)$ contains a nonvoid open interval.*

2. The reduction of equation (1.1)

We will distinguish three cases:

(i) $\alpha > 0$, $\beta > 0$; (ii) $\alpha = 0$, $\beta > 0$; (iii) $\alpha > 0$, $\beta = 0$.

The case $\alpha = 0$, $\beta = 0$ is skipped now.

(i) In case $\alpha > 0$, $\beta > 0$ equation (1.1) with the substitution $x \to \beta x$, $y \to \alpha y$ and with the notations

$$H_1(t) = h_1\left(\frac{\beta}{\lambda_1 \alpha}\frac{1}{t}\right), \quad H_2(t) = h_2\left(\frac{\alpha}{\lambda_2 \beta}\frac{1}{t}\right) \quad (t \in \mathbb{R}_+),$$

$$F_1(y) = \frac{1}{\lambda_1 \alpha (1+y)} f_Y(\alpha y) \ (y \in \mathbb{R}_+), \ F_2(x) = \frac{1}{\lambda_2 \beta (1+x)} f_X(\beta x) \ (x \in \mathbb{R}_+)$$

gives us, that equation

$$H_1\left(\frac{y+1}{x}\right) F_1(y) = H_2\left(\frac{x+1}{y}\right) F_2(x) \tag{2.1}$$

is satisfied for all $(x,y) \in \mathbb{R}_+^2$.

(ii) In case $\alpha = 0$, $\beta > 0$ equation (1.1) with the substitutions $x \to \beta\, x/y$, $y \to 1/y$ and with the notations

$$H_1(t) = h_1\left(\frac{\beta}{\lambda_1} t\right), \quad H_2(t) = h_2\left(\frac{1}{\lambda_2 \beta}\frac{1}{t}\right) \quad (t \in \mathbb{R}_+),$$

$$F_1(t) = \frac{t}{\lambda_1} f_Y\left(\frac{1}{t}\right), \quad F_2(t) = \frac{1}{\lambda_2 \beta (1+t)} f_X(\beta t) \quad (t \in \mathbb{R}_+)$$

provides us the equation

$$H_1(x)\, F_1(y) = H_2(x+y)\, F_2\left(\frac{x}{y}\right) \tag{2.2}$$

for all $(x, y) \in \mathbb{R}_+^2$.

(iii) In case $\alpha > 0$, $\beta = 0$ equation (1.1) with the substitutions $x \to 1/y$, $y \to \alpha\, x/y$ and with the notations

$$H_1(t) = h_1\left(\frac{1}{\lambda_1 \alpha t}\right), \quad H_2(t) = h_2\left(\frac{\alpha}{\lambda_2} t\right) \quad (t \in \mathbb{R}_+),$$

$$F_1(t) = \frac{1}{\lambda_1 \alpha (1+t)} f_Y(\alpha t), \quad F_2(t) = \frac{t}{\lambda_2} f_X\left(\frac{1}{t}\right) \quad (t \in \mathbb{R}_+)$$

shows that equation

$$H_2(x)\, F_2(y) = H_1(x+y)\, F_1\left(\frac{x}{y}\right) \tag{2.3}$$

is satisfied for all $(x, y) \in \mathbb{R}_+^2$.

Remark 2.1. The definition of F_i, H_i $(i = 1, 2)$ (in all three cases) shows that the functions are nonnegative on $\mathbb{R}_+$ and there exist sets B_1, B_2, B_3, $B_4 \subset \mathbb{R}_+$ with positive Lebesgue measure, on which they are positive.

Remark 2.2. In cases (ii) and (iii), equations (2.2) and (2.3) are dual to each other with the cange of (H_1, F_1) and (H_2, F_2). Here we can use the results of Baker ([1]) and Lajkó ([7]).

From the solutions of (2.1), (2.2) and (2.3) we get the solutions of (1.1) in cases (i), (ii) and (iii), respectively.

3. The general solution of (2.1)

First we prove that the solutions of the everywhere satisfied equation (2.1) are positive everywhere on $\mathbb{R}_+$. To do this we will use Theorem 1.1 with $U = \mathbb{R}_+^2$.

Theorem 3.1. *Let H_1, H_2, F_1, $F_2 : \mathbb{R}_+ \to \mathbb{R}$ be nonnegative functions, satisfying* (2.1) *for all $(x, y) \in \mathbb{R}_+^2$ such that there exist sets B_1, B_2, B_3, $B_4 \subset \mathbb{R}_+$ with positive Lebesgue measure, on which the functions are positive, respectively. Then H_1, H_2, F_1, $F_2 : \mathbb{R}_+ \to \mathbb{R}$ are positive everywhere on $\mathbb{R}_+$.*

Proof. We will use the notation $\{H_1 \neq 0\} = \{x \in \mathbb{R}_+ | H_1(x) \neq 0\}$ and the analogous notations $\{H_2 \neq 0\}$, $\{F_1 \neq 0\}$, $\{F_2 \neq 0\}$.

First we will prove that the sets $\{H_1 \neq 0\}$, $\{H_2 \neq 0\}$, $\{F_1 \neq 0\}$, $\{F_2 \neq 0\}$ contain nonvoid open intervals, respectively.

If we use the transformation $x \to (y+1)/x$ we get from (2.1) the functional equation

$$H_1(x) F_1(y) = H_2\left(\frac{x+y+1}{xy}\right) F_2\left(\frac{y+1}{x}\right) \quad (x, y) \in \mathbb{R}_+^2. \tag{3.1}$$

To prove that $\{H_2 \neq 0\}$ contains a nonvoid open interval, we have to check for the function $G_1(x, y) = \frac{x+y+1}{xy}$ $(x, y \in \mathbb{R}_+)$ that

$$\frac{\partial G_1}{\partial x}(x, y) = -\frac{y+1}{x^2 y} < 0, \qquad \frac{\partial G_1}{\partial y}(x, y) = -\frac{x+1}{xy^2} < 0.$$

Since H_1 and F_1 are positive on Lebesgue measurable sets B_1 and B_3 with positive Lebesgue measure, respectively, then by Theorem 1.1 the set $\frac{B_1+B_3+1}{B_1 B_3}$ contains a nonvoid open interval, thus H_2 is different from 0 for all points of the contained nonvoid open interval, thus $\{H_2 \neq 0\}$ contains a nonvoid open interval.

Similarly, to prove that $\{F_2 \neq 0\}$ contains a nonvoid open interval we may use for the function $G_2(x, y) = \frac{y+1}{x}$ $(x, y \in \mathbb{R}_+)$ that the partial derivatives

$$\frac{\partial G_2}{\partial x}(x, y) = -\frac{y+1}{x^2} < 0, \qquad \frac{\partial G_2}{\partial y}(x, y) = \frac{1}{x} > 0$$

for all $(x, y) \in \mathbb{R}_+^2$. By Theorem 1.1 the set $\frac{B_3+1}{B_1}$ contains a nonvoid open interval, thus F_2 is different from zero for all points of this nonvoid open interval, thus the set $\{F_2 \neq 0\}$ contains a nonvoid open interval.

If we use the transformation

$$(T) \qquad u = G_1(x,y) = \frac{x+y+1}{xy}, \quad v = G_2(x,y) = \frac{y+1}{x}, \ (x,y) \in \mathbb{R}_+^2$$

with the inverse transformation

$$(T^{-1}) \qquad x = G_1(u,v) = \frac{u+v+1}{uv}, \quad y = G_2(u,v) = \frac{v+1}{u}, \ (x,y) \in \mathbb{R}_+^2,$$

we get from equation (3.1) the functional equation

$$H_1\left(\frac{u+v+1}{uv}\right) F_1\left(\frac{v+1}{u}\right) = H_2(u) F_2(v) \tag{3.2}$$

for all $(u,v) \in \mathbb{R}_+^2$.

Equations (3.2) and (3.1) are dual if we change (H_1, F_1) and (H_2, F_2), so using equation (3.2) and Theorem 1.1, by precisely the same steps as above, we can prove that the sets $\{H_1 \neq 0\}$ and $\{F_1 \neq 0\}$ contain nonvoid open interval, say $I = (a,b) \subset \mathbb{R}_+$ and $J = (c,d) \subset \mathbb{R}_+$, respectively.

Then we have

$$H_1(x) F_1(y) \neq 0 \text{ for all } x \in (a,b) \ \text{ and } y \in (c,d).$$

Hence by (3.2)

$$H_2(u) \neq 0 \text{ for all } u \in \left\{\frac{x+y+1}{xy} | x \in (a,b),\, y \in (c,d)\right\} = L_1$$

and

$$F_2(v) \neq 0 \text{ for all } v \in \left\{\frac{y+1}{x} | x \in (a,b),\, y \in (c,d)\right\} = M_1.$$

We shall prove that

$$\begin{cases} L_1 = \left(\dfrac{b+d+1}{bd}, \dfrac{a+c+1}{ac}\right) = (a', b'), \\[2ex] M_1 = \left(\dfrac{c+1}{b}, \dfrac{d+1}{a}\right) = (c', d'). \end{cases} \tag{3.3}$$

We get for the Jacobian of T that

$$J = \begin{vmatrix} -\dfrac{y+1}{x^2 y} & -\dfrac{x+1}{xy^2} \\ -\dfrac{y+1}{x^2} & \dfrac{1}{x} \end{vmatrix} = -\frac{y+1}{x^3 y}\left(1 + \frac{x+1}{y}\right) < 0,$$

thus T is regular.

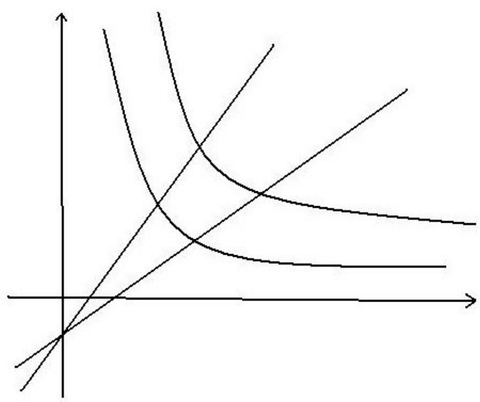
It is well known that under regular transformation the image of an open connected domain is again an open connected domain (see [3]). Thus the image of the open rectangle $(a,b)\times(c,d)$ is an open connected domain. Further the image of the closed rectangle $[a,b]\times[c,d]$ is a closed domain and the boundary of this domain is the image of the boundary of the closed rectangle.

One can easily see that the images of the lines

$$x=a,\quad x=b,\quad y=c \text{ and } y=d$$

are the curves

$$\frac{u+v+1}{uv}=a,\qquad \frac{u+v+1}{uv}=b,$$
$$\frac{v+1}{u}=c,\qquad \frac{v+1}{u}=d.$$



By simple calculation we get (3.3).

Now using (3.2) and the fact, that $H_2(u)\,F_2(v)\neq 0$ for all $u\in(a',b')$ and for all $v\in(c',d')$ and $T^{-1}=T$, by the same argument as above, we have

$$H_1(x)\neq 0 \text{ for all } x\in\left(\frac{b'+d'+1}{b'd'},\frac{a'+c'+1}{a'c'}\right)=(a_1,b_1)\,,$$

$$F_1(y)\neq 0 \text{ for all } y\in\left(\frac{c'+1}{b'},\frac{d'+1}{a'}\right)=(c_1,d_1)\,.$$

It is easy to show, that

$$(a_1,b_1)=\left(\frac{a+c+1+c\,(d+1)+ac}{(a+c+1)\,(d+1)}a,\frac{b+d+1+d\,(c+1)+bd}{(b+d+1)\,(c+1)}b\right),$$

$$(c_1, d_1) = \left(\frac{b+c+1}{a+c+1}\frac{a}{b}c, \frac{a+d+1}{b+d+1}\frac{b}{a}d\right).$$

It is valid moreover, that $a_1 < a$, $b < b_1$, $c_1 < c$ and $d < d_1$.

We have showed that

$$H_1(x)\, F_1(y) \neq 0 \text{ for all } x \in (a, b),\ y \in (c, d)$$

implies

$$H_1(x)\, F_1(y) \neq 0 \text{ for all } x \in (a_1, b_1),\ y \in (c_1, d_1).$$

Now define positive sequences $\langle a_k \rangle$, $\langle b_k \rangle$, $\langle c_k \rangle$ and $\langle d_k \rangle$ by

(a) $$a_{n+1} = \frac{a_n + c_n + 1 + c_n(d_n + 1) + a_n c_n}{(a_n + c_n + 1)(d_n + 1)} a_n,$$

(b) $$b_{n+1} = \frac{b_n + d_n + 1 + d_n(c_n + 1) + b_n d_n}{(b_n + d_n + 1)(c_n + 1)} b_n,$$

(c) $$c_{n+1} = \frac{b_n + c_n + 1}{a_n + c_n + 1}\frac{a_n}{b_n} c_n,$$

(d) $$d_{n+1} = \frac{a_n + d_n + 1}{b_n + d_n + 1}\frac{b_n}{a_n} d_n,$$

for $n = 1, 2, 3, \dots$

By repeating the above argument and using induction we find

$$H_1(x)\, F_1(y) \neq 0 \text{ if } x \in (a_n, b_n) \text{ and } y \in (c_n, d_n)$$

for every $n = 1, 2, 3, \dots$

We show that $\bar{a} = \lim_{n\to\infty} a_n = 0$, $\bar{b} = \lim_{n\to\infty} b_n = +\infty$, $\bar{c} = \lim_{n\to\infty} c_n = 0$, $\bar{d} = \lim_{n\to\infty} d_n = +\infty$.

It is easy to see that $\langle a_k \rangle$ and $\langle c_k \rangle$ are strictly decreasing and $\langle b_k \rangle$, $\langle d_k \rangle$ are strictly increasing, further $\bar{b} > 0$, $\bar{d} > 0$, $\bar{a} \geq 0$, $\bar{c} \geq 0$.

From (b) we get

$$\bar{b} = \frac{\bar{b} + \bar{d} + 1 + \bar{d}(\bar{c} + 1) + \bar{b}\bar{d}}{\left(\bar{b} + \bar{d} + 1\right)(\bar{c} + 1)} \bar{b},$$

which implies $\bar{b} = +\infty$ or in case $0 < \bar{b} < +\infty$

$$\frac{\bar{b} + \bar{d} + 1 + \bar{d}(\bar{c} + 1) + \bar{b}\bar{d}}{\left(\bar{b} + \bar{d} + 1\right)(\bar{c} + 1)} = 1 \quad \Longleftrightarrow \quad \bar{d}\left(\bar{b} + 1\right) = \bar{c}\left(\bar{b} + 1\right) \quad \Longleftrightarrow \quad \bar{d} = \bar{c},$$

but this is impossible, because $c_1 < d_1$ and $\langle c_k \rangle$ is strictly decreasing, $\langle d_k \rangle$ is strictly increasing. So $\bar{b} = +\infty$.

From (c) we get

$$c_{n+1} = \frac{1 + \frac{c_n+1}{b_n}}{a_n + c_n + 1} a_n c_n,$$

and by $\bar{b} = +\infty$ this implies that

$$\bar{c} = \frac{1}{\bar{a} + \bar{c} + 1} \bar{a}\bar{c},$$

which provides that $\bar{c} = 0$ or

$$\frac{\bar{a}}{\bar{a} + \bar{c} + 1} = 1 \quad \Longleftrightarrow \quad \bar{c} + 1 = 0,$$

but this is a contradiction, because $\bar{c} \geq 0$. Thus $\bar{c} = 0$.

From (a) by $\bar{c} = 0$ we get that

$$\bar{a} = \frac{\bar{a} + 1}{(\bar{a} + 1)\left(\bar{d} + 1\right)} \bar{a}$$

holds, which gives that $\bar{a} = 0$ or

$$\frac{\bar{a} + 1}{(\bar{a} + 1)\left(\bar{d} + 1\right)} = 1 \quad \Longleftrightarrow \quad \bar{d} + 1 = 0,$$

but $\bar{d} > 0$ gives that it is impossible. Thus $\bar{a} = 0$.

From (d) we get

$$d_{n+1} = \frac{1 + \frac{d_n+1}{a_n}}{1 + \frac{d_n+1}{b_n}} d_n,$$

and this implies that

$$\bar{d} = \frac{1 + \frac{\bar{d}+1}{\bar{a}}}{1 + \frac{\bar{d}+1}{\bar{b}}} \bar{d},$$

which is valid if $\bar{d} = +\infty$ or $\frac{1}{\bar{a}} = \frac{1}{\bar{b}}$, but this is a contradiction, because $\bar{a} = 0$, $\bar{b} = +\infty$. Thus $\bar{d} = +\infty$.

Then it follows that H_1 and F_1 are different from 0 everywhere on $\mathbb{R}_+$ and therefore H_2 and F_2 are different from 0 everywhere on $\mathbb{R}_+$ as well.

Therefore H_1, H_2, F_1, F_2 are positive everywhere on $\mathbb{R}_+$. ■

Now we are ready to prove the following general result for (2.1).

Theorem 3.2. *If the nonnegative functions* H_i, $F_i : \mathbb{R}_+ \to \mathbb{R}$ $(i = 1, 2)$ *satisfy* (2.1) *for all* $(x, y) \in \mathbb{R}_+^2$ *and they are positive on sets* B_1, B_2, B_3, $B_4 \subset \mathbb{R}_+$ *with positive Lebesgue measure, then*

$$H_1(x) = \exp\left(-l_1(x) + l_2(x) + l_3(x+1) + d_2\right) \qquad (x \in \mathbb{R}_+),$$

$$H_2(x) = \exp\left(l_1(x) + l_2(x) + l_3(x+1) + d_1\right) \qquad (x \in \mathbb{R}_+),$$

$$F_1(x) = \exp\left(l_1\left(\frac{x+1}{x}\right) - l_2(x(x+1)) - l_3(x) - d_3\right) \qquad (x \in \mathbb{R}_+),$$

$$F_2(x) = \exp\left(-l_1\left(\frac{x+1}{x}\right) - l_2(x(x+1)) - l_3(x) - d_4\right) \qquad (x \in \mathbb{R}_+),$$

where $l_i : \mathbb{R}_+ \to \mathbb{R}$ $(i = 1, 2, 3)$ *satisfies the Cauchy logarithmic equation*

$$l_i(xy) = l_i(x) + l_i(y) \qquad (x, y \in \mathbb{R}_+)$$

and d_1, d_2, d_3, $d_4 \in \mathbb{R}$ *are arbitrary constants with* $d_1 + d_3 = d_2 + d_4$.

Proof. By Theorem 3.1 H_i, F_i $(i = 1, 2)$ are positive. Taking the logarithm of (2.1), we get the functional equation

$$\ln\left(H_2\left(\frac{x+1}{y}\right)\right) + \ln\left(\frac{1}{F_1(y)}\right) = \ln\left(H_1\left(\frac{y+1}{x}\right)\right) + \ln\left(\frac{1}{F_2(x)}\right)$$

for all $(x, y) \in \mathbb{R}_+^2$. Thus the functions

$$G_1 = \ln \circ H_2, \quad G_2 = \ln \circ H_1, \quad \overline{F}_1 = \ln \circ \frac{1}{F_1}, \quad \overline{F}_2 = \ln \circ \frac{1}{F_2}$$

satisfy the functional equation

$$G_1\left(\frac{x+1}{y}\right) + \overline{F}_1(y) = G_2\left(\frac{y+1}{x}\right) + \overline{F}_2(x) \qquad \left((x, y) \in \mathbb{R}_+^2\right).$$

Using Theorem 6 from paper of Glavosits and Lajkó ([4]) and that

$$H_2 = \exp \circ G_1, \quad H_1 = \exp \circ G_2, \quad F_1 = \exp \circ \left(-\overline{F}_1\right), \quad F_2 = \exp \circ \left(-\overline{F}_2\right),$$

we get immediately the statement of our Theorem. ■

4. The general solution of (2.2) and (2.3)

Using the main results of papers Baker ([1]) and Lajkó ([7]) for the general solution of the so-called Olkin-Baker functional equation

$$f(x)\,g(y) = p(x+y)\,q\left(\frac{x}{y}\right) \qquad \left((x,y) \in \mathbb{R}_+^2\right),$$

we get the following theorems for equation (2.2) and (2.3).

Theorem 4.1. *If the nonnegative functions H_1, H_2, F_1, $F_2 : \mathbb{R}_+ \to \mathbb{R}$ satisfy (2.2) for all $(x,y) \in \mathbb{R}_+^2$ and they are positive on sets B_1, B_2, B_3, $B_4 \subset \mathbb{R}_+$ with positive Lebesgue measure, then*

$$\begin{aligned} H_1(x) &= \exp\left(a(x) + l_1(x) + d_1\right), \\ H_2(x) &= \exp\left(a(x) + l_2(x) + d_3\right), \\ F_1(x) &= \exp\left(a(x) + l_2(x) - l_1(x) + d_2\right), \\ F_2(x) &= \exp\left(l_2(x) - l_1(x+1) + d_4\right) \end{aligned}$$

for all $x \in \mathbb{R}_+$, where $a : \mathbb{R} \to \mathbb{R}$ is additive, l_1, $l_2 : \mathbb{R}_+ \to \mathbb{R}$ are logarithmic functions and d_1, d_2, d_3, $d_4 \in \mathbb{R}$ are arbitrary constants with $d_1 + d_3 = d_2 + d_4$.

Theorem 4.2. *If the nonnegative functions H_1, H_2, F_1, $F_2 : \mathbb{R}_+ \to \mathbb{R}$ satisfy (2.3) for all $(x,y) \in \mathbb{R}_+^2$ and they are positive on sets B_1, B_2, B_3, $B_4 \subset \mathbb{R}_+$ with positive Lebesgue measure, then*

$$\begin{aligned} H_1(x) &= \exp\left(a(x) + l_2(x) + d_3\right), \\ H_2(x) &= \exp\left(a(x) + l_1(x) + d_1\right), \\ F_1(x) &= \exp\left(a(x) + l_2(x) - l_1(x) + d_2\right), \\ F_2(x) &= \exp\left(l_2(x) - l_1(x+1) + d_4\right) \end{aligned}$$

for all $x \in \mathbb{R}_+$, where $a : \mathbb{R} \to \mathbb{R}$ is additive, l_1, $l_2 : \mathbb{R}_+ \to \mathbb{R}$ are logarithmic functions and d_1, d_2, d_3, $d_4 \in \mathbb{R}$ are arbitrary constants with $d_1 + d_3 = d_2 + d_4$.

5. Main results for equation (1.1)

Using Theorems 3.2, 4.1 and 4.2 and the definitions of H_i, F_i $(i = 1, 2)$ in cases (i), (ii) and (iii), respectively, we get easily the following theorems for the general solution of equation (1.1).

Theorem 5.1. *If the nonnegative functions h_1, h_2, f_X, $f_Y : \mathbb{R}_+ \to \mathbb{R}$ satisfy equation* (1.1) *in case* (i) *for all $(x,y) \in \mathbb{R}_+^2$ and they are positive on the sets A_1, A_2, A_3, $A_4 \subset \mathbb{R}_+$ with positive Lebesgue measure, then*

$$h_1(x) = \exp\left(-l_1\left(\frac{\beta}{\lambda_1\alpha}\frac{1}{x}\right) + l_2\left(\frac{\beta}{\lambda_1\alpha}\frac{1}{x}\right) + l_3\left(\frac{\beta}{\lambda_1\alpha}\frac{1}{x} + 1\right) + d_2\right),$$

$$h_2(x) = \exp\left(l_1\left(\frac{\alpha}{\lambda_2\beta}\frac{1}{x}\right) + l_2\left(\frac{\alpha}{\lambda_2\beta}\frac{1}{x}\right) + l_3\left(\frac{\alpha}{\lambda_2\beta}\frac{1}{x} + 1\right) + d_1\right),$$

$$f_X(x) = \lambda_2(\beta + x)\exp\left(-l_1\left(\frac{x+\beta}{x}\right) - l_2\left(\frac{x(x+\beta)}{\beta^2}\right) - l_3\left(\frac{x}{\beta}\right) - d_4\right),$$

$$f_Y(x) = \lambda_1(\alpha + x)\exp\left(l_1\left(\frac{x+\alpha}{x}\right) - l_2\left(\frac{x(x+\alpha)}{\alpha^2}\right) - l_3\left(\frac{x}{\alpha}\right) - d_3\right)$$

for all $x \in \mathbb{R}_+$, where l_1, l_2, $l_3 : \mathbb{R}_+ \to \mathbb{R}$ are logarithmic functions and d_1, d_2, d_3, $d_4 \in \mathbb{R}$ are arbitrary constants with $d_1 + d_3 = d_2 + d_4$.

Theorem 5.2. *If the nonnegative functions h_1, h_2, f_X, $f_Y : \mathbb{R}_+ \to \mathbb{R}$ satisfy equation* (1.1) *in case (ii) for all $(x,y) \in \mathbb{R}_+^2$ and they are positive on the sets A_1, A_2, A_3, $A_4 \subset \mathbb{R}_+$ with positive Lebesgue measure, then*

$$h_1(x) = \exp\left(a\left(\frac{\lambda_1}{\beta}x\right) + l_1\left(\frac{\lambda_1}{\beta}x\right) + d_1\right),$$

$$h_2(x) = \exp\left(a\left(\frac{1}{\lambda_2\beta}\frac{1}{x}\right) + l_2\left(\frac{1}{\lambda_2\beta}\frac{1}{x}\right) + d_2\right),$$

$$f_X(x) = \lambda_2(\beta + x)\exp\left(l_1\left(\frac{x}{\beta}\right) - l_2\left(\frac{x+\beta}{\beta}\right) + d_4\right),$$

$$f_Y(x) = \lambda_1 x\exp\left(a\left(\frac{1}{x}\right) + l_2\left(\frac{1}{x}\right) - l_1\left(\frac{1}{x}\right) + d_3\right)$$

for all $x \in \mathbb{R}_+$, where $a : \mathbb{R} \to \mathbb{R}$ is additive, l_1, $l_2 : \mathbb{R}_+ \to \mathbb{R}$ are logarithmic functions and d_1, d_2, d_3, $d_4 \in \mathbb{R}$ are arbitrary constants with $d_1 + d_3 = d_2 + d_4$.

Theorem 5.3. *If the nonnegative functions h_1, h_2, f_X, $f_Y : \mathbb{R}_+ \to \mathbb{R}$ satisfy equation* (1.1) *in case (iii) for all $(x,y) \in \mathbb{R}_+^2$ and they are positive on the sets A_1, A_2, A_3, $A_4 \subset \mathbb{R}_+$ with positive Lebesgue measure, then*

$$h_1(x) = \exp\left(a\left(\frac{1}{\lambda_1\alpha}\frac{1}{x}\right) + l_2\left(\frac{1}{\lambda_1\alpha}\frac{1}{x}\right) + d_2\right),$$

$$h_2(x) = \exp\left(a\left(\frac{\lambda_2}{\alpha}x\right) + l_1\left(\frac{\lambda_2}{\alpha}x\right) + d_1\right),$$

$$f_X(x) = \lambda_2 x\exp\left(a\left(\frac{1}{x}\right) + l_2\left(\frac{1}{x}\right) - l_1\left(\frac{1}{x}\right) + d_3\right),$$

$$f_Y(x) = \lambda_1(\alpha + x)\exp\left(l_1\left(\frac{x}{\alpha}\right) - l_2\left(\frac{x+\alpha}{\alpha}\right) + d_4\right)$$

for all $x \in \mathbb{R}_+$, where $a : \mathbb{R} \to \mathbb{R}$ is additive, l_1, $l_2 : \mathbb{R}_+ \to \mathbb{R}$ are logarithmic functions and d_1, d_2, d_3, $d_4 \in \mathbb{R}$ are arbitrary constants with $d_1 + d_3 = d_2 + d_4$.

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