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ALGORITHMIC CONSTRUCTION OF SIMULTANEOUS NUMBER SYSTEMS IN THE LATTICE OF GAUSSIAN INTEGERS

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Dedicated to the 70th birthday of Professor Karl-Heinz Indlekofer

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Abstract. In [1] number system constructions were analysed using general block diagonal bases. As a special case simultaneous systems were considered in the lattice of Gaussian integers. Extending the result of G. Nagy [2] it was proved that except 43 cases, the Gaussian integers can always serve as basic blocks for simultaneous number systems using dense digit sets. In this paper we analyze the remaining cases and we give a complete solution for the problem.

1. Introduction

Let Λ be a lattice in $\mathbb{R}^n$, $M : \Lambda \to \Lambda$ be a linear operator such that $\det(M) \neq 0$, and let D be a finite subset of Λ containing 0.

Definition 1.1. The triple (Λ, M, D) is called a *generalized number system* (GNS) if every element x of Λ has a unique, finite representation of the form

$$x = \sum_{i=0}^{l} M^i d_i$$

where $d_i \in D$ and $l \in \mathbb{N}, d_l \neq 0$.

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We may assume that M is integral acting on $\Lambda = \mathbb{Z}^n$. Clearly, Λ is a finitely generated free abelian group with addition. If two elements of Λ are in the same coset of the factor group $\Lambda/M\Lambda$ then they are said to be *congruent modulo* M.

Theorem 1.1 ([3]). *If* (Λ, M, D) *is a number system then*

(1) D *must be a complete residue system modulo* M*,*

(2) M *must be expansive and*

(3) $\det(I - M) \neq \pm 1$.

If a system fulfills these conditions then it is a *radix system* and the operator M is called a *radix base*.

Let $\phi : \Lambda \to \Lambda$, $x \overset{\phi}{\mapsto} M^{-1}(x - d)$ for the unique $d \in D$ satisfying $x \equiv d \pmod{M}$. Since M^{-1} is contractive and D is finite, there exists a norm $\|.\|$ on $\mathbb{R}^n$ and a constant $C \in \mathbb{R}$ such that the orbit of every $x \in \Lambda$ eventually enters the finite set $\{x \in \Lambda \ : \ \|x\| < C\}$ for the repeated application of ϕ. This means that the sequence (path) $x, \phi(x), \phi^2(x), \ldots$ is eventually *periodic* for all $x \in \Lambda$. If a points $p \in \Lambda$ is periodic then $\|p\| \leq L = Kr/(1 - r)$, where $r = \|M^{-1}\| = \sup_{\|x\| \leq 1} \|M^{-1}x\| < 1$ and $K = \max_{d \in D} \|d\|$ (see [6]). Let us denote the set of periodic elements by $\mathcal{P}$. The paths of all periodic elements constitute a finite number of disjoint *cycles* $\mathcal{C}_i$. Then, the number system property is equivalent to $\mathcal{P} = \{0\}$, or with the situation that the system has only one cycle $\mathcal{C}_1 = \{0 \to 0\}$.

In this paper we consider special *block diagonal systems* $(\Lambda_1 \otimes \Lambda_1, M_1 \oplus \oplus M_2, D)$, where $\Lambda_1 = \mathbb{Z}^2$ is the lattice of the the Gaussian integers, M_1 and M_2 are operators of the form $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$ $(a, b \in \mathbb{Z})$, $d_j = (v^T || v^T)^T \in D$ $(v \in \Lambda_1)$, $\otimes, \oplus$ and $||$ denote the direct product, the direct sum, and the concatenation, respectively, furthermore v^T (transpose of v) denotes a row vector. These 4-dimensional systems can be considered as *simultaneous systems of the Gaussian integers*. Simultaneous systems with one dimensional blocks were introduced and investigated by Indlekofer et al. [4]. In our case the digits are in the subspace

$$W = \{(x, y, x, y)^T \ : x, y \in \mathbb{Z}\} \leq \mathbb{Z}^4 \,.$$

It was proved in [1, 2] that the only operators which may serve as bases of simultaneous Gaussian number systems must have the form

$$M_A(a,b) = \begin{pmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & a+1 & -b \\ 0 & 0 & b & a+1 \end{pmatrix}, M_B(a,b) = \begin{pmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & a & -b-1 \\ 0 & 0 & b+1 & a \end{pmatrix}.$$

The first one is called an A-type and the second one is called a B-type base. Moreover, every radix base having one of these forms is a valid base using the dense digit set (a dense digit set consists of elements with the smallest norm from each congruent class, see [5]) except 43 cases. The present paper deals with the remaining cases by constructing appropriate digit sets algorithmically or proving the non-existence of such digit sets.

2. The algorithm

In order to be able to examine simultaneous Gaussian systems for the remaning 43 cases we developed a construction algorithm.

The simultaneous Gaussian GNS construction algorithm clearly terminates either with an output that the construction is not possible, or with an appropriate digit set, or with a remark of an unsuccessfully construction attempt.

The FINDPERIODS(M, D) function in lines 4 and 23 can either be the α-type (covering-type, box-type) or the β-type (Brunotte-type) method [6, 7].

The proper work of lines 5–17 is based on the following

Lemma 2.1. *Let C_γ be the coset of the factor group $\mathbb{Z}^4/M\mathbb{Z}^4$ represented by $\gamma \in D$ where M is either an A-type or a B-type base and let $D \subset W = \{(x, y, x, y)^T : x, y \in \mathbb{Z}\}$ be a digit set. If each element $c \in C_\gamma \cap W$ satisfies*

$$(I - M)^{-1}c = \pi_c \in \mathbb{Z}^4$$

then $(\mathbb{Z}^4, M, D)$ can not be a number system for any digit set $D \subset W$.

Proof. Let $M^* \cdot (x, y, x, y)^T = (\alpha_1(x, y), \alpha_2(x, y), \alpha_3(x, y), \alpha_4(x, y))^T$ and let $M^*\gamma = (\beta_1, \beta_2, \beta_3, \beta_4)^T$. Here M^* means the adjoint of M (i.e. the elements are the appropriate sub-determinants). Then the elements of $C_\gamma \cap W$ can be expressed parametrically as common solution of the system of equations

$$\{\alpha_i(x, y) \equiv \beta_i \pmod{\det(M)}, \quad (1 \le i \le 4)\}. \tag{2.1}$$

If each element $c \in C_\gamma \cap W$ satisfy $(I - M)^{-1}c = \pi_c \in \mathbb{Z}^4$ then $\pi_c = c + M\pi_c$, therefore there is always a loop $\{\pi_c \to \pi_c\}$ in the system. In other words in these cases every element $c \in C_\gamma \cap W$ result in a loop, hence it can not be a number system. ■

The algorithm also outputs the representants ($\gamma \in W$) of such congruent classes. Observe that the solution of (2.1) is independent of the choice of γ, i.e. if $\gamma \equiv \gamma_1 \pmod{\det(M)}$ then the solutions are the same.

The **while** cycle in lines 18–25 finds the elements with the maximal norm from each non-trivial cycle C_i and replaces the congruent digits with digits from the same coset that have bigger norms in a minimal measure.

Simultaneous Gaussian GNS Construction Algorithm

Precondition: Given radix base M and dense digit set D

$Limit \leftarrow 15$ ▷ This limit bounds the search space
$LimitCounter \leftarrow 1$
$LoopCounter \leftarrow 0$

$Cycles \leftarrow$ FindPeriods(M, D)
if there are non-trivial loops in $Cycles$ **then**
    for all non-trivial loop $\{\rho \to \rho\}$ **do**
        $C_\gamma \leftarrow$ elements of the factor group $\mathbb{Z}^4/M\mathbb{Z}^4$ represented by γ
        where $\rho \equiv \gamma \pmod{\det(M)}$, $\gamma \in D$
        if all elements c of $C_\gamma \cap W$ satisfy $(I - M)^{-1}c \in \mathbb{Z}^4$ **then**
            Print("C_γ produces loops ", γ)
            $LoopCounter \leftarrow LoopCounter + 1$
        end if
    end for
    if $LoopCounter > 0$ **then**
        Return("GNS construction is not possible")
    end if
end if

while $LimitCounter \leq Limit$ and $Cycles \neq \big\{\{0 \to 0\}\big\}$ **do**
    for all periodic cycle $Cyc \in Cycles$ except $\{0 \to 0\}$ **do**
        Find the element $p \in Cyc$ with maximal norm
        Replace $d \in D$ with $d^* \in S$ in D for which $d \equiv d^* \equiv p \pmod{M}$,
        $\|d\| \leq \|d^*\|$ and $\|d^*\|$ is the smallest possible
    end for
    $Cycles \leftarrow$ FindPeriods(M, D)
    $LimitCounter \leftarrow LimitCounter + 1$
end while

if $LimitCounter \leq Limit$ **then**
    Print("GNS has been found, the new digit set is: ")
    Draw(D)
else
    Print("Unable to construct GNS with the given limit of attempts.")
end if

We implemented the algorithm in Maple language. The simultaneous Gaussian GNS construction algorithm terminates successfully in all but one cases. For this $M_A(-2, 1)$ case instead of raising the *Limit* parameter we applied a randomized digit searching algorithm, where the elements of the full residue system were chosen randomly from the set $\{(x,y,x,y)^T, -3 \le x,y \le 3\}$.

Figures 1–3 show the GNS results, i.e., when the digit sets with the appropriate operators form simultaneous Gaussian number systems. The marked points $(x,y) \in S$ in the pictures denote that the vecor $(x,y,x,y)^T$ belongs to the appropriate digit set. Tables 1–2 show the systems which can not be simultaneous Gaussian number systems for any digit set D. The tables contain the γ residue class representants as well.

The complete computation time on a single laptop was approximately 5 minutes.

Example 2.1. Consider the base $M = M_A(1,1)$. Then the elements of a dense digit set are

$$D = \big\{(0,0,0,0)^T, (1,0,1,0)^T, (0,1,0,1)^T, (1,1,1,1)^T, (-1,0,-1,0)^T,\\ (0,-1,0,-1)^T, (-1,-1,-1,-1)^T,\\ (-1,1,-1,1)^T, (1,-1,1,-1)^T, (-1,2,-1,2)^T\big\}.$$

In line 4 FindPeriods(M,D) gives the following non-trivial loops (we used the α-type method):

- $(-1,-1,0,1)^T \to (-1,-1,0,1)^T$
- $(-1,1,-1,0)^T \to (-1,1,-1,0)^T$
- $(1,1,0,1)^T \to (1,1,0,1)^T$
- $(1,-1,1,0)^T \to (1,-1,1,0)^T$

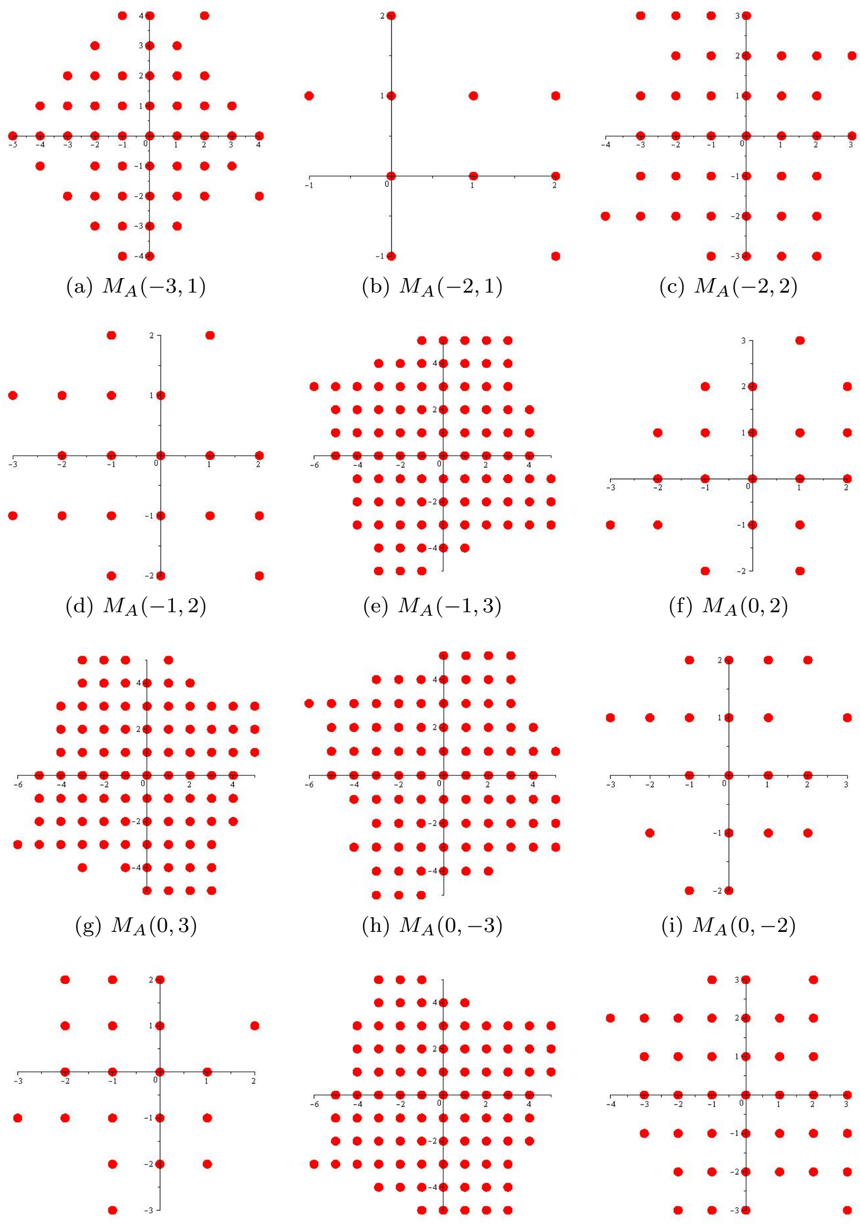
The first loop element congruent with the digit $(-1,1,-1,1)^T$. The set of equations (2.1) is

$$\{-5\,x + 5\,y = 10, -2\,x + 4\,y = 6, 4\,x + 2\,y = -2, 5\,x + 5\,y = 0\}\ .$$

The common solution is $x = 4k_1 + k_2 + 3, y = 3k_2 + 2k_1 + 3$ $(k_1, k_2 \in \mathbb{Z})$. Then

$$(I-M)^{-1}(x,y,x,y)^T = (-3k_2-2k_1-3, 4k_1+k_2+3, -3k_1-2k_2-3, k_1-k_2)^T \in \mathbb{Z}^4.$$

The computations are similar for the other three loops. Hence, the system $(\mathbb{Z}^4, M, D)$ can not be a simultaneous Gaussian number system for any digit set $D \subset W$.



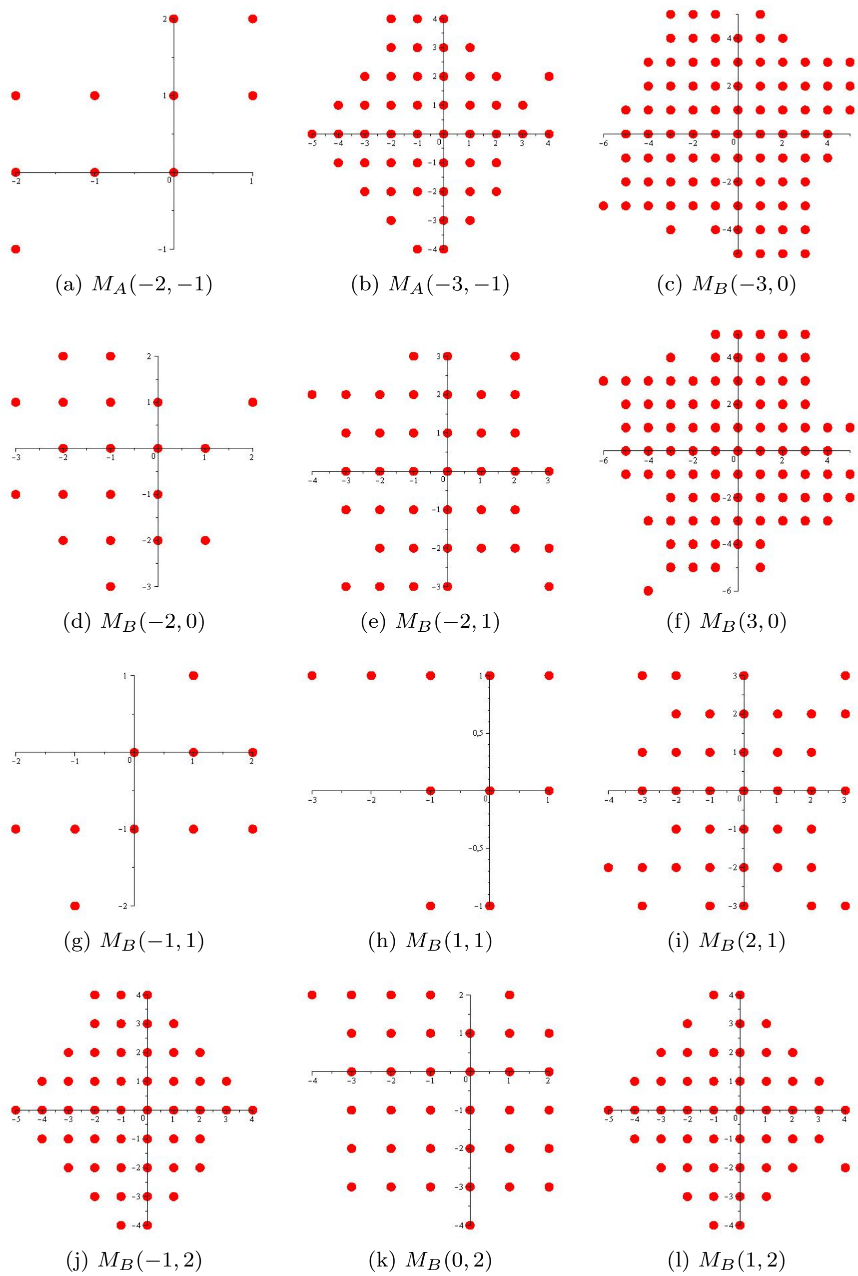
(a) $M_A(-3,1)$ (b) $M_A(-2,1)$ (c) $M_A(-2,2)$

(d) $M_A(-1,2)$ (e) $M_A(-1,3)$ (f) $M_A(0,2)$

(g) $M_A(0,3)$ (h) $M_A(0,-3)$ (i) $M_A(0,-2)$

(j) $M_A(-1,-2)$ (k) $M_A(-1,-3)$ (l) $M_A(-2,-2)$

Figure 1: Digit sets of simultaneous Gaussian number systems



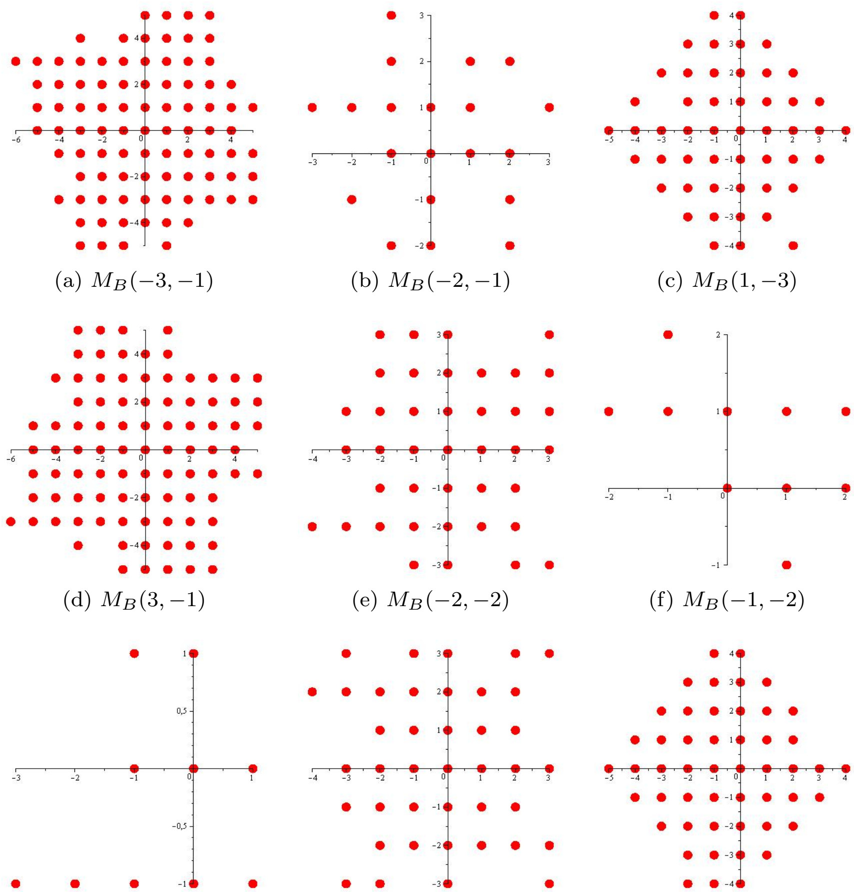
(a) $M_A(-2,-1)$ (b) $M_A(-3,-1)$ (c) $M_B(-3,0)$

(d) $M_B(-2,0)$ (e) $M_B(-2,1)$ (f) $M_B(3,0)$

(g) $M_B(-1,1)$ (h) $M_B(1,1)$ (i) $M_B(2,1)$

(j) $M_B(-1,2)$ (k) $M_B(0,2)$ (l) $M_B(1,2)$

Figure 2: Digit sets of simultaneous Gaussian number systems



(a) $M_B(-3,-1)$ (b) $M_B(-2,-1)$ (c) $M_B(1,-3)$

(d) $M_B(3,-1)$ (e) $M_B(-2,-2)$ (f) $M_B(-1,-2)$

(g) $M_B(1,-2)$ (h) $M_B(2,-2)$ (i) $M_B(-1,-3)$

Figure 3: Digit sets of simultaneous Gaussian number systems

Base	γ	Base	γ
$M_A(1,1)$	$(-1,1,-1,1)^T$	$M_A(2,0)$	$(2,2,2,2)^T$
	$(1,1,1,1)^T$		$(2,0,2,0)^T$
	$(1,-1,1,-1)^T$		$(2,-2,2,-2)^T$
	$(-1,-1,-1,-1)^T$		$(0,2,0,2)^T$
$M_A(1,2)$	$(-4,2,-4,2)^T$		$(0,-2,0,-2)^T$
$M_A(2,1)$	$(1,3,1,3)^T$		$(-2,2,-2,2)^T$
	$(3,-1,3,-1)^T$		$(-2,0,-2,0)^T$
	$(-1,-3,-1,-3)^T$		$(-2,-2,-2,-2)^T$
	$(-3,1,-3,1)^T$	$M_A(3,0)$	$(-6,-6,-6,-6)^T$
$M_A(1,-1)$	$(1,1,1,1)^T$		$(-6,0,-6,0)^T$
	$(-1,-1,-1,-1)^T$		$(0,-6,0,-6)^T$
	$(1,-1,1,-1)^T$	$M_A(2,-1)$	$(3,1,3,1)^T$
	$(-1,1,-1,1)^T$		$(-1,3,-1,3)^T$
$M_A(1,-2)$	$(2,-4,2,-4)^T$		$(1,-3,1,-3)^T$
			$(-3,-1,-3,-1)^T$

Table 1: The A-type bases for which there does not exist any digit set constituting a Gaussian GNS. The γ values represents the cosets for which the elements produce loops in the system

Base	γ	Base	γ
$M_B(2,0)$	$(1,1,1,1)^T$	$M_B(2,-1)$	$(2,2,2,2)^T$
	$(1,-1,1,-1)^T$		$(-2,2,-2,2)^T$
	$(2,0,2,0)^T$		$(2,-2,2,-2)^T$
	$(-1,-1,-1,-1)^T$		$(-2,-2,-2,-2)^T$
	$(0,-2,0,-2)^T$		$(1,3,1,3)^T$
	$(0,2,0,2)^T$		$(3,-1,3,-1)^T$
	$(-3,1,-3,1)^T$		$(-1,-3,-1,-3)^T$
	$(-2,0,-2,0)^T$		$(-1,3,-1,3)^T$
	$(-1,1,-1,1)^T$		$(-3,1,-3,1)^T$

Table 2: The B-type bases for which there do not exist any digit set constituting a Gaussian GNS. The γ values represents the cosets for which the elements produces loops in the system

Example 2.2. Consider the base $M = M_B(-1,1)$. The dense digit set for this operator is

$$D = \big\{(0,0,0,0)^T, (1,0,1,0)^T, (0,1,0,1)^T, (1,1,1,1)^T, (-1,0,-1,0)^T, (0,-1,0,-1)^T, (-1,-1,-1,-1)^T, (-1,1,-1,1)^T, (1,-1,1,-1)^T, (-2,-1,-2,1)^T\big\}.$$

Now FindPeriods(M, D) in line 4 gives the following non-trivial periods:

- $(0, 1, 0, 0,)^T \to (0, -2, 0, 1)^T \to (-2, 1, -1, 0)^T \to (1, 0, 0, 0)^T \to (-1, -2, 0, 1)^T \to (0, 1, 0, 0)^T$
- $(-1, -1, 0, -1)^T \to (-1, 1, -1, 0)^T \to (1, 1, 0, 1)^T \to (1, -1, 1, 0)^T \to (-1, -1, 0, -1)^T$

In the iteration 18–25 the values are

$$p = (-2, 1, -1, 0)^T, \quad d = (-1, 0, -1, 0)^T \text{ and } d^* = (2, -1, 2, -1)^T$$

for the first cycle and

$$p = (-1, -1, 0, -1)^T, \; d = (-1, 1, -1, 1)^T \;\text{ and }\; d^* = (2, 0, 2, 0)^T$$

for the second one. Then FindPeriods(M, D) provides the non-trivial cycle

- $(0, -1, 0, -0)^T \to (-1, -1, 0, -0)^T \to (1, 2, 0, 1)^T \to (0, -1, 0, 0)^T$

Now we have $p = (1, 2, 0, 1)^T$, $d = (0, 1, 0, 1)^T$, and $d^* = (-1, -2, -1, -2)^T$. The method FindPeriods(M, D) shows that the system is a simultaneous Gaussian number system. The newly constructed digit set can be seen in Figure 2 (g).

3. Summary

In this paper together with [1] and [2] we proved the following

Theorem 3.1. *Every radix base $M_A(a, b)$ or $M_B(a, b)$ $(a, b \in \mathbb{Z})$ may serve as a base of a simultaneous Gaussian number system except the cases $M_A(1, 1)$, $M_A(1, 2)$, $M_A(2, 1)$, $M_A(1, -1)$, $M_A(1, -2)$, $M_A(2, 0)$, $M_A(3, 0)$, $M_A(2, -1)$, $M_B(2, 0)$, $M_B(2, -1)$.*

Applying the result of [1] Theorem 3.1 can also be reformulated and refined:

Theorem 3.1a *Let $G_1, G_2 \in \mathbb{Z}[i]$ such that $G_2 = G_1 + g_j$ $(j = 1, 2)$ where $g_1 = 1$ or $g_2 = i$. Then (G_1, G_2, D) is a simultaneous number system of the Gaussian integers for some digit set D except the following cases:*

- *If $G_1 = -2i$ or $-3i$ then $(G_1, G_1 + 1, D)$ is a simultaneous number system for some digit set D.*

- *If* $G_1 = -1 + i, -2$ *or* -3 *then* $(G_1, G_1 + i, D)$ *is a simultaneous number system for some digit set* D.
- *If* $G_1 = 0, \pm 1, \pm i$, 2, $2 - i$ *or* $-1 - i$ *then* $(G_1, G_1 + g, D)$ $(g = 1$ *or* $g = i)$ *can not be a simultaneous number system for any digit set* D.
- *If* $G_1 = 1 \pm i$, $1 \pm 2i$, $2 + i$, *or* 3 *then* $(G_1, G_1 + 1, D)$ *can not be a simultaneous number system for any digit set* D.

Theorem 3.1a (or Theorem 3.1) enumerates all simultaneous number system bases in the lattice of the Gaussian integers by which the description is complete.

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