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EXISTENCE OF MOMENTS IN THE HSU–ROBBINS–ERDŐS THEOREM

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Abstract. We consider the so-called empirical version of the Hsu–Robbins series and find conditions for the existence of its moments.

1. Introduction

Let X_k, $k \geq 1$, be a sequence of independent identically distributed random variables and let S_n, $n \geq 1$, be the sequence of their partial sums. According to Hsu and Robbins [5], the sequence $\{S_n/n\}$ is said to converge completely to a constant μ if

$$\sum_{n=1}^{\infty} \mathsf{P}\left(\left|\frac{S_n}{n} - \mu\right| \geq \varepsilon\right) < \infty \qquad \text{for all } \varepsilon > 0.$$

In a more convenient form, the latter condition is written as

$$\sum_{n=1}^{\infty} \mathsf{P}(|S_n - n\mu| \geq n\varepsilon) < \infty \qquad \text{for all } \varepsilon > 0.$$

Hsu and Robbins [5] found the sufficient condition for the complete convergence of $\{S_n/n\}$ to μ, namely

$$\mathsf{E}[X_1] = \mu, \qquad \mathsf{E}[X_1^2] < \infty.$$

Later, Erdős [2] proved the converse.

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Motivated by their results we assume throughout that the first moment exists and is zero and we introduce the random variable

$$\xi \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \mathbb{I}_{\{|S_n| \ge \varepsilon n\}},$$

where $\mathbb{I}_A$ is the indicator of a random event A. Note that the right hand side depends on ε but since our results do not depend on this quantity we suppress this variable in the variable ξ. Then the Hsu–Robbins–Erdős theorem is stated as follows

$$\mathsf{E}[\xi] < \infty \qquad \text{for all } \varepsilon > 0 \qquad \Longleftrightarrow \qquad \mathsf{E}[X_1] = 0, \qquad \mathsf{E}[X_1^2] < \infty.$$

The aim of this note is to find necessary and sufficient conditions for

$$\mathsf{E}[\xi^r] < \infty \qquad \text{for all } \varepsilon > 0 \tag{1}$$

if $r > 0$. It turns out that this question is related to a Baum–Katz [1] result extending the Hsu–Robbins–Erdős theorem. Below is a particular case of the Baum–Katz result.

Theorem (Baum and Katz [1]). *If $r > 0$, then*

$$\sum_{n=1}^{\infty} n^{r-1} \mathsf{P}(|S_n| \ge \varepsilon n) < \infty \qquad \textit{for all } \varepsilon > 0$$

if and only if

$$\mathsf{E}[X_1] = 0 \quad \textit{and} \quad \mathsf{E}[|X_1|^{r+1}] < \infty. \tag{2}$$

2. Main result

Theorem 1. *The following implications hold:*

(a) *if $r \ge 1$, then* (2) *implies* (1)*;*

(b) *if $0 < r \le 1$, then* (1) *implies* (2)*.*

Remark 1. In the case $r = 1$, we obtain that (1) is equivalent to (2). This result is, in fact, the Hsu–Robbins–Erdős theorem.

Remark 2. The case (a) is easy to treat for integer r. We show this for a particular case of $r = 2$. Then

$$\xi^2 = \sum_{i,j\ge 1} \mathbb{I}_{\{|S_i|\ge \varepsilon i\}} \mathbb{I}_{\{|S_j|\ge \varepsilon j\}} =$$
$$= \sum_{i=1}^{\infty} \sum_{j\le i} \mathbb{I}_{\{|S_i|\ge \varepsilon i\}} \mathbb{I}_{\{|S_j|\ge \varepsilon j\}} + \sum_{i=1}^{\infty} \sum_{j> i} \mathbb{I}_{\{|S_i|\ge \varepsilon i\}} \mathbb{I}_{\{|S_j|\ge \varepsilon j\}}.$$

Denoting the terms on the right hand side by ξ_1^2 and ξ_2^2, respectively, we have

$$\xi_1^2 = \sum_{i=1}^{\infty} \mathbb{I}_{\{|S_i|\ge \varepsilon i\}} \left[\sum_{j\le i} \mathbb{I}_{\{|S_j|\ge \varepsilon j\}} \right] \le \sum_{i=1}^{\infty} i \mathbb{I}_{\{|S_i|\ge \varepsilon i\}}.$$

Passing to the expectations

$$\mathsf{E}[\xi_1^2] \le \sum_{i=1}^{\infty} i \mathsf{P}(|S_i| \ge \varepsilon i).$$

By the Baum–Katz theorem with $r = 2$, the latter series is finite if and only if $\mathsf{E}[X_1] = 0$ and $\mathsf{E}[|X_1|^3] < \infty$. The same holds for ξ_2^2 and thus case (a) follows.

3. Proof of the main result

We start with the following elementary lemma.

Lemma. *Let $a_n \in \{0, 1\}$ for each n.*

(i) Let $r \ge 1$. *Then, for all $n \ge 1$,*

$$\left(\sum_{k=1}^{n} a_k \right)^r \le r \sum_{k=1}^{n} k^{r-1} a_k.$$

(ii) *Let $0 < r \le 1$. Then, for all $n \ge 1$,*

$$\left(\sum_{k=1}^{n} a_k \right)^r \ge \frac{r}{r+1} \sum_{k=1}^{n} k^{r-1} a_k.$$

Proof of Lemma. (i) It is clear that for $r \ge 1$

$$(k-1)^{r-1} \le \int_{k-1}^{k} x^{r-1}\, dx \le k^{r-1}, \qquad k \ge 1,$$

whence

$$r \int_0^n x^{r-1}\, dx = n^r \le r \sum_{k=1}^n k^{r-1} \qquad \text{for all } n \ge 1\,.$$

Next, for $0 < r \le 1$

$$r \int_0^n x^{r-1}\, dx = n^r \ge r \left(\sum_{k=1}^n (k-1)^{r-1} + n^{r-1} - n^{r-1} \right) \ge r \left(\sum_{k=1}^n k^{r-1} \right) - rn^r$$

which in turn implies

$$n^r \ge \frac{r}{r+1} \sum_{k=1}^n k^{r-1}\,.$$

Now fix n and let

$$I_n = \{k \le n : a_k = 1\}, \qquad m = m_n = \operatorname{card}(I_n).$$

Then

$$I_n = \{i_1, \dots, i_m\}, \qquad 1 \le i_1 < \cdots < i_m \le n.$$

It is clear that

$$i_1 \ge 1, \quad \dots, \quad i_m \ge m.$$

Therefore

$$\begin{aligned}(a_1 + \cdots + a_n)^r = m^r &\le r(1^{r-1} + \cdots + m^{r-1}) \le \\ &\le r(i_1^{r-1} + \cdots + i_m^{r-1}) = \\ &= r(i_1^{r-1} a_{i_1} + \cdots + i_m^{r-1} a_{i_m}) = \\ &= r \sum_{k=1}^n k^{r-1} a_k\end{aligned}$$

which proves the case (i).

(ii) We use the same notation I_n, m, and $i_1, \dots, i_m$ as in the proof of case (i). Since $0 < r \le 1$ we find

$$\begin{aligned}(a_1 + \cdots + a_n)^r = m^r &\ge \\ &\ge \frac{r}{r+1} \sum_{k=1}^m k^{r-1} \ge \\ &\ge \frac{r}{r+1} (i_1^{r-1} a_{i_1} + \cdots + i_m^{r-1} a_{i_m}) \ge \\ &\ge \frac{r}{r+1} \sum_{k=1}^n k^{r-1} a_k\end{aligned}$$

which proves the case (ii). ■

Proof of Theorem 1. Let $a_n = \mathbb{I}_{\{|S_n| \geq \varepsilon n\}}$. For (a), we apply case (i) of the above Lemma:

$$\xi^r \leq r \sum_{k=1}^{\infty} k^{r-1} \mathbb{I}_{\{|S_k| \geq \varepsilon k\}}.$$

Passing to the expectations

$$\mathsf{E}[\xi^r] \leq r \sum_{k=1}^{\infty} k^{r-1} \mathsf{P}(|S_k| \geq \varepsilon k).$$

By the Baum–Katz theorem, the right hand side is finite if $\mathsf{E}[|X|^{r+1}] < \infty$.

For (b), we apply case (ii) of the above Lemma:

$$\xi^r \geq \frac{r}{r+1} \sum_{k=1}^{\infty} k^{r-1} \mathbb{I}_{\{|S_k| \geq \varepsilon k\}}$$

hence, (1) implies that the expectation of the right hand side is finite which implies $\mathsf{E}[|X|^{r+1}] < \infty$ by the Baum–Katz theorem. Now standard arguments imply that random variable X has expectation zero. ■

Remark 3. We do not know whether the implications (1) $\Longrightarrow$ (2) for $0 < r < 1$ and (2) $\Longrightarrow$ (1) for $r \geq 1$ hold but we conjecture that the two statements are equivalent for all $r > 0$.

4. Extensions

Here we will consider the multiindex case. Therefore, let $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}_+^d\}$, X be i.i.d. random variables, that is, we discuss a random field with index set $\mathbb{Z}_+^d$, $d \geq 2$, denoting the positive integer d-dimensional lattice with coordinate-wise partial ordering $\leq$. As before we discuss partial sums $S_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}}$, $\mathbf{n} \in \mathbb{Z}_+^d$. Finally, let $|\mathbf{n}| = n_1 \cdot \dots \cdot n_d$.

In the following we assume again that the expectation of X exists and is zero. Now we investigate the random variable

$$\xi_d \stackrel{\text{def}}{=} \sum_{\mathbf{k} \in \mathbb{Z}_+^d} \mathbb{I}_{\{|S_{\mathbf{k}}| \geq \varepsilon\, |\mathbf{k}|\}}.$$

Similarly to the case of $d = 1$, we are interested in moment conditions implying the existence of the r-th moment of ξ_d, i.e.,

$$\mathsf{E}[\xi_d^r] < \infty \qquad \text{for all } \varepsilon > 0. \tag{3}$$

Typically in these kind of results one needs a somewhat stronger moment condition on X in case $d > 1$ as compared to $d = 1$, namely

$$\mathsf{E}[|X|^{r+1}(\log_+ |X|)^{(d-1)\,r}] < \infty, \tag{4}$$

here $\log_+ x = \log(1 + x)$ for $x \geq 0$. Now, we are ready to formulate our second result.

Theorem 2. *The following implications hold:*

(a) *if* $r \geq 1$*, then* (4) *implies* (3)*;*

(b) *if* $0 < r \leq 1$*, then* (3) *implies* (4)*.*

Proof. The proof follows similar arguments as the one for Theorem 1. Let $a_{\mathbf{k}} = \mathbb{1}_{\{|S_{\mathbf{k}}| \geq \varepsilon\, |\mathbf{k}|\}}$, $\mathbf{k} \in \mathbb{Z}_+^d$, and for any positive integer n let

$$I_n = \{\mathbf{k} : |\mathbf{k}| \leq n \text{ and } a_{\mathbf{k}} = 1\}, \qquad m = m_n = \operatorname{card}(I_n)\,.$$

Then

$$I_n = \{\mathbf{k}_1, \ldots, \mathbf{k}_m\}$$

where the indices are ordered along the hyperbolas $|\mathbf{k}| = \ell$, $\ell = 1, 2, \ldots$, and therein in lexicographic order. In general we do not have any more that $|\mathbf{k}_\nu| \geq \nu$ since several indices may be incident to the same "hyperbola" $|\mathbf{k}| = \nu$. We write $d(\ell) = \#\{\mathbf{k} \in \mathbb{Z}_+^d \text{ with } |\mathbf{k}| = \ell\}$. The terms $d(\ell)$ themselves do not have a nice asymptotic behavior, but their partial sums $M(n) = \sum_{\ell=1}^n d(\ell)$ have the asymptotic

$$M(n) \sim \frac{1}{(d-1)!}\, n\, (\log n)^{d-1}, \qquad n \to \infty,$$

see [7]. This asymptotic is well known in the generalized Dirichlet divisor problem. In fact, we do not need the precise asymptotic for the proof below, while the asymptotic relation

$$M(n) \asymp n(\log n)^{d-1}$$

is sufficient for our purposes. The latter relation can be proved by comparing $M(n)$ and the volume of the domain

$$x_1 \geq 1, \ldots, x_d \geq 1, \qquad x_1 \ldots x_d \leq n$$

in the space $\mathbb{R}^d$.

Now we conclude that if $|\mathbf{k}_\nu| = \ell$, then

$$\nu \leq M(\ell) = M(|\mathbf{k}_\nu|) \leq c|\mathbf{k}_\nu|(\log |\mathbf{k}_\nu|)^{d-1} \tag{5}$$

with some positive $c > 0$. As above, we conclude that for $r \geq 1$ and any positive integer n

$$\begin{aligned}\Big(\sum_{|\mathbf{k}|\leq n} a_{\mathbf{k}}\Big)^r &= m^r \leq \\ &\leq r\left(1^{r-1}+\cdots+m^{r-1}\right) \leq \\ &\leq c\,r\Big(|\mathbf{k}_1|^{r-1}(\log|\mathbf{k}_1|)^{(d-1)(r-1)}+\cdots+ \\ &\qquad\qquad + |\mathbf{k}_m|^{r-1}(\log|\mathbf{k}_m|)^{(d-1)(r-1)}\Big) = \\ &= c\,r\Big(|\mathbf{k}_1|^{r-1}(\log|\mathbf{k}_1|)^{(d-1)(r-1)}a_{\mathbf{k}_1}+\cdots+ \\ &\qquad\qquad + |\mathbf{k}_m|^{r-1}(\log|\mathbf{k}_m|)^{(d-1)(r-1)}a_{\mathbf{k}_m}\Big) = \\ &= c\,r\sum_{|\mathbf{k}|\leq n}|\mathbf{k}|^{r-1}(\log|\mathbf{k}|)^{(d-1)(r-1)}a_{\mathbf{k}}\,.\end{aligned}$$

Hence

$$\begin{aligned}\mathsf{E}[\xi_d^r] &\leq cr\sum_{|\mathbf{k}|\geq 1}|\mathbf{k}|^{r-1}(\log|\mathbf{k}|)^{(d-1)(r-1)}\mathsf{P}(|S_{\mathbf{k}}|>|\mathbf{k}|\varepsilon) = \\ &= cr\sum_{\ell\geq 1}d(\ell)\,\ell^{r-1}(\log\ell)^{(d-1)(r-1)}\mathsf{P}(|S_\ell|>\ell\,\varepsilon)\,.\end{aligned}$$

In sums with otherwise smooth summands we may replace $d(\ell)$ by the asymptotic derivative of $M(\ell)$, i.e., $(\log\ell)^{d-1}$. Hence, the last sum above is finite if and only if (4) holds, as it can formally be seen using arguments similar to those in the proof Lemma 3.1 in [3] or on page 2448 in [6]. The converse implication follows as in the proof of Theorem 1 (b) and the arguments in the proof of Lemma 3.1 in [3] or [6] again (the case of an arbitrary slowly varying function is treated in [4] for $d = 1$). In doing so we use the inequality

$$\nu^{r-1} \geq M(\ell)^{r-1} \geq c(|\mathbf{k}_\nu|(\log|\mathbf{k}_\nu|+1)^{d-1})^{r-1}, \qquad 0 < r \leq 1,$$

that follows from (5). ■

Remark 4. Again we conjecture that the moment condition (4) is equivalent to the moment condition (3).

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