ON STATISTICAL CONVERGENCE AND STATISTICAL MONOTONICITY

E. Kaya (Mersin, Turkey)
M. Kucukaslan (Mersin, Turkey)
R. Wagner (Paderborn, Germany)

Dedicated to the 70th birthday of
Professor Karl-Heinz Indlekofer

Communicated by Imre Kátai
(Received December 28, 2012; accepted January 07, 2013)

Abstract. The main aim of this paper is to investigate properties of statistically convergent sequences. Also, the definition of statistical monotonicity and upper (or lower) peak points of real valued sequences will be introduced. The interplay between the statistical convergence and these concepts are also studied. Finally, the statistically monotonicity is generalized by using a matrix transformation.

1. Introduction

The concept of statistical convergence for real or complex valued sequence was introduced in the journal "Colloq. Math." by H. Fast in [5] and H. Steinhaus in [17] independently in the same year 1951. The idea of this concept is based on the notation of asymptotic density of a set $K \subset \mathbb{N}$ (see for example [11], [12]).

Key words and phrases: Statistical convergence, Lower-Upper peak point, statistical monotone.
2010 Mathematics Subject Classification: 40A05, 40C05.

Statistical convergence has many applications in different fields of mathematics such as: approximation theory [9], measure theory [15], probability theory [8], trigonometric series [18], number theory [4], etc.

Let K be a subset of $\mathbb{N}$ and

$$K(n) := \{k : k \leq n,\ k \in K\}.$$

Then, the asymptotic density of K, denoted by $\delta(K)$, is defined by

$$\delta(K) := \lim_{n\to\infty} \frac{1}{n} |K(n)|, \tag{1.1}$$

if the limit exists. In (1.1), the vertical bars indicate the cardinality of the enclosed set.

A real or complex valued sequence $x = (x_n)$ is said to be *statistically convergent to the number* L, if for every $\varepsilon > 0$, the set

$$K(n,\varepsilon) := \{k : k \leq n,\ |x_k - L| \geq \varepsilon\},$$

has asymptotic density zero, i.e.

$$\lim_{n\to\infty} \frac{|K(n,\varepsilon)|}{n} = 0,$$

and it is denoted by $x_n \to L(S)$.

Deeply connected with this definition is the concept of *strongly Cesàro summability and uniform summability* (see Indlekofer [13]).

A sequence $x = (x_n)$ is called *strongly*-[C,1,α] *summable* ($\alpha > 0$) *to the mean* L in case

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k\leq n} |x_k - L|^{\alpha} = 0 \tag{1.2}$$

and is written

$$x_n \to L[C,1,\alpha].$$

By w_α we denote the space of strongly Cesàro summable sequences.

A sequence $x = (x_n)$ is called *uniformly summable* in case

$$\lim_{K\to\infty} \limsup_{n\to\infty} \frac{1}{n} \sum_{\substack{k\leq n \\ |x_k|\geq K}} |x_k| = 0. \tag{1.3}$$

The space of uniformly summable sequences is denoted by $\mathcal{L}^*$.

Remark 1.1. The space of all complex valued sequences $x = (x_n)$ will be denoted by $\mathbb{C}^{\mathbb{N}}$. In many circumstances we refer to $\mathbb{C}^{\mathbb{N}}$ as the space of arithmetical functions $f : \mathbb{N} \to \mathbb{C}$, especially, when f reflects the multiplicative structure of $\mathbb{N}$. This is the case for additive and multiplicative functions.

We recall the definition. An arithmetical function f is called *additive* or *multiplicative*, if for every pair m, n of positive coprime integers the relation $f(mn) = f(m) + f(n)$ or $f(mn) = f(m)f(n)$, respectively, is satisfied.

2. Some results about statistical convergence

Define the function $d : \mathbb{C}^{\mathbb{N}} \times \mathbb{C}^{\mathbb{N}} \to [0, \infty)$ for all $x, y \in \mathbb{C}^{\mathbb{N}}$ as follows,

$$d(x, y) := \limsup_{n\to\infty} \frac{1}{n} \sum_{k\le n} \varphi(|x_k - y_k|)$$

where $\varphi : [0, \infty] \to [0, \infty)$

$$\varphi(t) = \begin{cases} t, & \text{if } t \le 1; \\ 1, & \text{otherwise.} \end{cases}$$

It is clear that d is a semi-metric on $\mathbb{C}^{\mathbb{N}}$. With these notations we have

Theorem 2.1. *The sequence $x = (x_n)$ is statistically convergent to L if and only if $d(x, y) = 0$ where $y = (y_n)$ and $y_n = L$ for all $n \in \mathbb{N}$.*

Proof. Let us assume $d(x, y) = 0$ where $y_n = L$ for all $n \in \mathbb{N}$. Then, if $\varepsilon > 0$,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{\substack{k\le n \\ |x_k - L|\ge\varepsilon}} 1 \le \max\left\{1, \frac{1}{\varepsilon}\right\} \limsup_{n\to\infty} \frac{1}{n} \sum_{k\le n} \varphi(|x_k - L|) =$$

$$= \max\left\{1, \frac{1}{\varepsilon}\right\} d(x, y) = 0$$

and $x_n \to L(S)$.

Now, assume that x is statistically convergent to L. Then, for any $\varepsilon > 0$,

$$\frac{1}{n} \sum_{k\le n} \varphi(|x_k - L|) = \frac{1}{n} \sum_{\substack{k\le n \\ |x_k - L|<\varepsilon}} \varphi(|x_k - L|) + \frac{1}{n} \sum_{\substack{k\le n \\ |x_k - L|\ge\varepsilon}} \varphi(|x_k - L|) \le$$

$$\le \varepsilon + \frac{1}{n} \sum_{\substack{k\le n \\ |x_k - L|\ge\varepsilon}} 1$$

which implies immediately

$$d(x,y) \leq \varepsilon \quad \text{for any} \quad \varepsilon > 0$$

where $y = (y_n)$ and $y_n = L \ \ (n \in \mathbb{N})$. This ends the proof of Theorem 2.1. ■

By the first part of this proof we conclude

Corollary 2.1. *If x is strongly Cesàro summable to L then x is statistically convergent to L.*

Remark 2.1. The inverse of Corollary 2.1 is not true in general. To see this, it is enough to consider the sequence $x : \mathbb{N} \to \mathbb{C}$ as

$$x_n := \begin{cases} \sqrt{n}, & n = m^2, \quad m = 1, 2, \ldots; \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand the following holds.

Corollary 2.2. *If $x = (x_n)$ is a bounded sequence and statistically convergent to L, then x is strongly Cesàro summable to L.*

The next result is well-known (see J.A. Fridy [7]). But for the sake of completeness we give a proof of which is different from that of [7].

Theorem 2.2. *A sequence x is statistically convergent to L if and only if there exists $H \subset \mathbb{N}$ with $\delta(H) = 1$ such that x is convergent to L in H,i.e.*

$$\lim_{\substack{n\to\infty \\ n\in H}} x_n = L.$$

Proof. Assume that x is statistically convergent to L. There is $m_j \in \mathbb{N}$ such that

$$\frac{1}{n} \sum_{\substack{k\leq n \\ \frac{1}{2^j} < |x_k - L| < \frac{1}{2^{j-1}}}} 1 \leq \frac{1}{2^j}$$

is satisfied for all $n \geq m_j$. Denote the set

$$H_j := \Big\{ k \in \mathbb{N} : \frac{1}{2^j} \leq |x_k - L| < \frac{1}{2^j} \ \text{ and } \ k < m_j \Big\}.$$

Then

$$\frac{1}{n} \sum_{\substack{k\leq n \\ \frac{1}{2^j} < |x_k - L| < \frac{1}{2^{j-1}} \\ \text{and } k\in\mathbb{N}\setminus H_j}} 1 < \frac{1}{2^j}$$

holds for all $n \in \mathbb{N}$. If we consider the set $H := \bigcup_{j=1}^{\infty} H_j \cup \{k : x_k = L\}$ then $|x_n - L| \geq \varepsilon$ holds only to finitely many $n \in H$. This means that x is convergent to L in the usual case. Now, let us show that $\delta(\mathbb{N} \setminus H) = 0$. Let $\varepsilon > 0$ be given and choose an arbitrary $r \in \mathbb{N}$ such that

$$\sum_{j=r+1}^{\infty} \frac{1}{2^j} < \frac{\varepsilon}{2}$$

holds. For this r, there exists a $l_r \in \mathbb{N}$ such that

$$\frac{1}{n} \sum_{\substack{k \leq n \\ \frac{1}{2^j} < |x_k - L| < \frac{1}{2^{j-1}} \\ \text{and } k \in \mathbb{N}\setminus H}} 1 < \frac{1}{r+1} . \frac{\varepsilon}{7} \qquad \text{and} \qquad \frac{1}{n} \sum_{\substack{k \leq n \\ |x_k - L| \geq 1 \\ \text{and } k \in \mathbb{N}\setminus H}} 1 < \frac{1}{r+1} . \frac{\varepsilon}{4}$$

for all $n > l_r$ and $j \in \{1, 2, \ldots, r\}$. Therefore,

$$\frac{1}{n} \sum_{\substack{k \leq n \\ k \in \mathbb{N}\setminus H}} 1 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

holds for all $n \geq l_r$.

The inverse of theorem is obviously obtained. ■

3. Some results for multiplicative and additive functions

In this section, we will give some results for multiplicative and additive functions. With $M(f)$ we denote the mean-value of the arithmetical function f, if the limit

$$M(f) := \lim_{n \to \infty} \frac{1}{n} \sum_{k \leq n} f(k)$$

exists.

Theorem 3.1. *Assume that $f : \mathbb{N} \to \mathbb{C}$ is bounded and statistically convergent to L and $H \subset \mathbb{N}$ is an arbitrary set which possesses an asymptotic density $\delta(H)$. Then, $M(1_H \cdot f)$ exists and equals $L \cdot \delta(H)$.*

Proof. The proof is obtained obviously by the following inequality

$$\left|\frac{1}{n}\sum_{\substack{k\le n\\ k\in H}} f(k) - \frac{1}{n}\sum_{\substack{k\le n\\ k\in H}} L\right| \le$$

$$\le \frac{1}{n}\sum_{\substack{k\le n\\ k\in H,\ |f(k)-L|<\varepsilon}} |f(k)-L| + \frac{1}{n}\sum_{\substack{k\le n\\ k\in H,\ |f(k)-L|\ge\varepsilon}} |f(k)-L|. \qquad \blacksquare$$

Theorem 3.2. *If a multiplicative function f is bounded and statistically convergent to $L \neq 0$, then $f \equiv 1$.*

Proof. Let $p_0 \in P$, P is the set of primes, $k_0 \in \mathbb{N}$ and let

$$H := \{n \in \mathbb{N} : p_0^{k_0} || n\},$$

be the set of all elements of $\mathbb{N}$ divisible exactly by $p_0^{k_0}$, i.e. n can be written in the form $n = p_0^{k_0} m$ where $p_0 \nmid m$. It is clear from Theorem 3.1 that

$$M(1_H \cdot f) = L\delta(H) = L\frac{1}{p_0^{k_0}}(1 - \frac{1}{p_0})$$

holds. Since f is multiplicative, we have

$$f(k) = f(p_0^{k_0}) \cdot f\left(\frac{k}{p_0^{k_0}}\right) \quad \text{for} \quad k \in H.$$

Therefore,

$$M(1_H \cdot f) = \lim_{n\to\infty} \frac{1}{n}\sum_{\substack{k\le n\\ k\in H}} f(k) = \lim_{n\to\infty} \frac{1}{n} f(p_0^{k_0}) \sum_{\substack{m \le \frac{n}{p_0^{k_0}}\\ p_0 \neq m}} f(m) =$$

$$= \lim_{n\to\infty} f(p_0^{k_0}) \frac{1}{p_0^{k_0}} \cdot \frac{1}{\frac{n}{p_0^{k_0}}} \cdot \sum_{\substack{m \le \frac{n}{p_0^{k_0}}\\ p_0 \nmid m}} f(m) =$$

$$= f(p_0^{k_0}) \frac{1}{p_0^{k_0}} \cdot L \cdot \left(1 - \frac{1}{p_0}\right).$$

This implies $f(p_0^{k_0}) = 1$. Since p_0 is a prime, $k_0 \in \mathbb{N}$ have been chosen arbitrarily it follows $f = 1$ (and $L = 1$). $\blacksquare$

Remark 3.1. If f is bounded and statistically convergent to L, then f is Cesáro summable, i.e. (1.2) holds (for every $\alpha > 0$). Then Indlekofer proved in [13] (Theorem 2′).

Proposition 3.3. *Let f be multiplicative and $\alpha > 0$. Then the following assertions are valid:*

(i) *If* (1.2) *holds for $L \neq 0$, then $L = 1$ and $f(n) = 1$ for all $n \in \mathbb{N}$.*

(ii) (1.2) *holds for $L = 0$ if and only if $|f|^\alpha \in \mathcal{L}^*$ and one of the series*

$$\sum_{\substack{p \\ ||f(p)|-1| \leq \frac{1}{2}}} \frac{(|f(p)|-1)^2}{p}, \qquad \sum_{\substack{p \\ ||f(p)|-1| > \frac{1}{2}}} \frac{||f(p)|-1|^\alpha}{p}$$

diverges or

$$\sum_{p \leq x} \frac{|f(p)|-1}{p} \to -\infty \quad as \quad x \to \infty.$$

Assertion (i) corresponds to Theorem 3.2.

Theorem 3.4. *If an additive function $f : \mathbb{N} \to \mathbb{C}$ is statistically convergent, then $f \equiv 0$.*

Proof. It is enough to prove Theorem 3.4 for real-valued additive functions. Let us define the multiplicative function $g : \mathbb{N} \to \mathbb{C}$ by

$$g(n) := e^{i\alpha f(n)}$$

where $\alpha \in \mathbb{R}$. Obviously, g is bounded and also statistically convergent to some L,where $L \neq 0$. So, $g \equiv 1$. Therefore, $\alpha f(n) = 2\pi i k_n$ where $k_n \in \mathbb{Z}$. For an arbitrary $\beta \in \mathbb{R}$ we have also

$$\beta f(n) = 2\pi i \tilde{k}_n$$

where $\tilde{k}_n \in \mathbb{Z}$. If we assume $f(n) \neq 0$, then $\frac{\beta}{\alpha} \in \mathbb{Q}$. This is a contradiction because of β is an arbitrary real numbers. So, $f \equiv 0$. ■

4. Statistical monotonicity and related results

In this section we consider only real-valued sequences and introduce the concept of statistical monotonicity.

Definition 4.1. (Statistical monotone increasing (or decreasing) sequence.) A sequence $x = (x_n)$ is statistical monotone increasing (decreasing) if there exists a subset $H \subset \mathbb{N}$ with $\delta(H) = 1$ such that the sequence $x = (x_n)$ is monotone increasing (or decreasing) on H.

A sequence $x = (x_n)$ is statistical monotone if it is statistical monotone increasing or statistical monotone decreasing.

In the following we list some (obvious) properties of statistical monotone sequences.

(i) If the sequence $x = (x_n)$ is bounded and statistical monotone then it is statistically convergent.

(ii) If $x = (x_n)$ is statistical monotone increasing or statistical monotone decreasing then

$$\lim_{n\to\infty} \tfrac{1}{n}|\{k : k \leq n : x_{k+1} < x_k\}| = 0 \tag{4.1}$$

or

$$\lim_{n\to\infty} \tfrac{1}{n}|\{k : k \leq n : x_{k+1} > x_k\}| = 0 \tag{4.2}$$

respectively. The inverse of these assertions is not correct because of the following example:

Define $x = (x_n)$ by

$$x_n = \begin{cases} 1, & \text{if } 2^k \leq n < 2^{k+1} - 1 \text{ for even } k, \\ 0, & \text{otherwise.} \end{cases}$$

Then the relations (4.1) and (4.2) hold but $x = (x_n)$ is not statistical monotone (and not statistically convergent).

Definition 4.2. The real number sequence $x = (x_n)$ is said to be statistical bounded if there is a number $M > 0$ such that

$$\delta(\{n \in \mathbb{N} \ : \ |x_n| > M\}) = 0.$$

Let $\{n_k\}$ be a strictly increasing sequence of positive natural numbers and $x = (x_n)$, define $x' = (x_{n_k})$ and $K_{x'} := \{n_k : k \in \mathbb{N}\}$. With this notation

Definition 4.3. (Dense Subsequence) The subsequence $x' = (x_{n_k})$ of $x = (x_n)$ is called a dense subsequence, if $\delta(K_{x'}) = 1$.

Two more simple properties are given by

(iii) Every dense subsequence of a statistical monotone sequence is statistical monotone.

(iv) The statistical monotone sequence $x = (x_n)$ is statistically convergent if and only if $x = (x_n)$ is statistical bounded.

Definition 4.4. The sequence $x = (x_n)$ and $y = (y_n)$ are called statistical equivalent if there is a subset M of $\mathbb{N}$ with $\delta(M) = 1$ such that $x_n = y_n$ for each $n \in M$. It is denoted by $x \asymp y$.

With this definition we formulate

(v) Let $x = (x_n)$ and $y = (y_n)$ be statistical equivalent. Then $x = (x_n)$ statistical monotone if and only if $y = (y_n)$ is statistical monotone.

For additive functions we have

Theorem 4.1. *If a real-valued additive function f is statistical monotone, then there exists $c \in \mathbb{R}$ such that $f(n) = c \log n \quad (n \in \mathbb{N})$.*

I. Kátai [14] and B.J. Birch [2] showed independently, that it is enough to assume in Theorem 4.1, that f is monotone on a set having upper density one.

An immediate consequence is given in

Corollary 4.1. *Let $f \geq 1$ be a (real-valued) multiplicative function, which is statistical monotone. Then $f(n) = n^c \quad (n \in \mathbb{N})$ with some $c \geq 0$.*

5. Peak points and related results

In this section, for real-valued sequences upper and lower peak points will be defined and its relation with statistical convergence and statistical monotonicity will be given.

Definition 5.1. (Upper (or Lower) Peak Point) The point x_k is called upper (or lower) peak point of the sequence $x = (x_n)$ if $x_k \geq x_l \quad (\text{or } x_l \geq x_k)$ holds for all $l \geq k$.

Theorem 5.1. *If the index set of peak points of the sequence $x = (x_n)$ has asymptotic density 1, then the sequence is statistical monotone.*

Proof. Let us denote the index set of upper peak points of the sequence $x = (x_n)$ by

$$H := \{k_n : x_{k_n} \text{ upper peak point of } (x_n)\} \subset \mathbb{N}.$$

Since $\delta(H) = 1$, and $x = (x_n)$ is monotone on H, the sequence $x = (x_n)$ is statistical monotone. ■

Remark 5.1. The inverse of Theorem 5.1 is not true.

Consider $x = (x_n)$ where

$$x_n = \begin{cases} \dfrac{1}{m}, & n = m^2,\ m \in \mathbb{N}; \\ n, & n \neq m^2, \end{cases}$$

i.e. $x = (x_n) = (1, 2, 3, \frac{1}{2}, 5, 6, 7, 8, \frac{1}{3}, ...)$.

Since the set $H = \left\{m^2 : m \in \mathbb{N}\right\}$ possesses an asymptotic density $\delta(H) = 0$, the sequence $x = (x_n)$ is statistical monotone increasing. But, it has not any peak points.

Corollary 5.1. *If $x = (x_n)$ is bounded and the index set of upper (or lower) peak points*

$$H = \{k_n : x_{k_n} \text{ upper(lower) peak point of } (x_n)\}$$

possesses an asymptotic density 1, then $x = (x_n)$ is statistically convergent.

Remark 5.2. In Corollary 5.1, it can not be replaced usual convergence by statistical convergence.

Let us consider the sequence $x = (x_n)$ where

$$x_n := \begin{cases} -1, & n = m^2, m \in \mathbb{N}, \\ \frac{1}{k}, & n \neq m^2. \end{cases}$$

The index set of upper peak points of the sequence $x = (x_n)$ is $H = \left\{k : k \neq m^2, m \in \mathbb{N}\right\}$. It is clear that $\delta(H) = 1$ and $x = (x_n)$ is bounded. So, the hypothesis of Corollary 5.1 is fulfilled. The subsequence

$$(x_{k_n}) = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{10}, \ldots\right)$$

is convergence to zero. Also, $x = (x_n)$ is statistical convergence to zero but it is not convergence to zero.

6. *A*-generalization of statistical monotonicity

Statistical monotonicity can be generalized by using A-density of a subset K of $\mathbb{N}$ for a regular nonnegative summability matrix $A = (a_{nk})$.

Recall A-density of a subset K of $\mathbb{N}$, if

$$\delta_A(K) := \lim_{n\to\infty} \sum_{k\in K} a_{nk} = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} 1_K(k) = \\ = \lim_{n\to\infty} (A \cdot 1_K)(n)$$

exists and is finite.

The sequence $x = (x_n)$ is A-statistically convergent to l, if for every $\varepsilon > 0$ the set $K_\varepsilon = \{k \in \mathbb{N} : |x_k - l| \geq \varepsilon\}$ possesses A-density zero (see for details [3]).

Definition 6.1. A sequence $x = (x_n)$ is called A-statistical monotone, if there exists a subset H of $\mathbb{N}$ with $\delta_A(H) = 1$ such that the sequence $x = (x_n)$ is monotone on H.

$A = (a_{nk})$ and $B = (b_{nk})$ will denote nonnegative regular matrices.

Theorem 6.1. *If the condition*

$$\limsup_{n\to\infty} \sum_{k=1}^{\infty} |a_{nk} - b_{nk}| = 0 \tag{6.1}$$

holds. Then, $x = (x_n)$ is A-statistical monotone if and only if $x = (x_n)$ is B-statistical monotone.

Proof. For an arbitrary $H \subset \mathbb{N}$ the inequality

$$0 \leq |(A \cdot 1_H)(n) - (B \cdot 1_H)(n)| = \Big|\sum_{k\in H} a_{nk} - \sum_{k\in H} b_{nk}\Big| \leq \\ \leq \sum_{k\in H} |a_{nk} - b_{nk}| \leq \\ \leq \sum_{k=1}^{\infty} |a_{nk} - b_{nk}|,$$

holds. Under the condition (6.1) $\delta_A(H)$ exists if and only if $\delta_B(H)$ exists, and in this case $\delta_A(H) = \delta_B(H)$. Therefore, A-statistical monotonicity of $x = (x_n)$ implies B-statistical monotonicity vice versa. ■

Let us consider strictly increasing and nonnegative sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ and $E = \{\lambda_n\}_{n=0}^{\infty}$. If $A = (a_{nk})$ is a summability matrix, then $A_\lambda := (a_{\lambda(n),k})$

is the submatrix of $A = (a_{nk})$. Thus, the A_λ transformation of a sequence $x = (x_n)$ as

$$(A_\lambda x)_n = \sum_{k=0}^{\infty} a_{\lambda(n),k}\, x_k.$$

Since, A_λ is a row submatrix of A, it is regular whenever A is a regular summability matrix (See [6],[10]). The Cesáro submethod's has been defined by D.H. Armitage and I.J. Maddox in [1], and later studied by J.A. Osikiewicz in [16].

Theorem 6.2. *Let A be a summability matrix and let $E = \{\lambda_n\}$ and $F = \{\rho_n\}$ be an infinite subset of $\mathbb{N}$. If $F \setminus E$ is finite, then A_λ-statistical monotonicity implies A_ρ-statistical monotonicity.*

Proof. Assume that $F \setminus E$ is finite, and $x = (x_n)$ is A_λ-statistical monotone sequence. From the assumption there exists a $n_0 \in \mathbb{N}$ such that

$$\{\rho(n) : n \geq n_0\} \subseteq E.$$

It means that there is a monotone increasing sequence $j(n)$ such that $\rho(n) := := \lambda_j(n)$. So, the A_ρ asymptotic density of the set $K := \{n \in \mathbb{N} : \rho(n) = \lambda_j(n)\}$ is

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{\rho(n),k} 1_K(k) = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{\lambda_{j(n)},k} 1_K(k) = = 1.$$

This give us, $x = (x_n)$ is A_ρ-statistical monotone sequence. ■

By the Theorem 6.2 we have following corollaries:

Corollary 6.1. *A-statistical monotone sequence is A_λ-statistical monotone.*

Corollary 6.2. *Under the condition of Theorem* 6.2*, if $E \Delta F$ is finite, then the sequence $x = (x_n)$, A_λ- statistical monotone if and only if A_ρ-statistical monotone.*

References

[1] **Armitage D.H. and I.J. Maddox,** A new type of Cesaro mean, *Analysis*, **9(1-2)** (1989), 195–206.

[2] **Birch, B.J.,** Multiplicative functions with non-decraasing normal order, *J. London Math Soc.*, **42** (1967), 149–151.

[3] **Buck, R.C.,** Generalized asymptotic density, *Amer. J. Math.*, **75** (1953), 335–346.

[4] **Erdős, P. and G. Tenenbaum,** Sur les densities de certaines suites déntiers, *Proc. London Math. Soc.*, **59** (1989), 417–438.

[5] **Fast, H.,** Sur la convergence statistique, *Colloq. Math.*, **2** (1951), 241–244.

[6] **Fridy, J.A.,** Submatrices of summability matrices, *Int. J. Math. Sci.*, **1(4)** (1978), 519–524.

[7] **Fridy, J.A.,** On statistical convergence,, *Analysis*, **5** (1985), 301–313.

[8] **Fridy, J.A. and M.K. Khan,** Tauberian theorems via statistical convergence, *J. Math. Anal. Appl.*, **228** (1998), 73–95.

[9] **Gadjiev, A.D. and C. Orhan,** Some approximation theorems via statistical convergence, *Rocky Mountain J. Math.*, **32** (2002), 129–138.

[10] **Goffman, C. and G.M. Peterson,** Submethods of regular matrix summability methods, *Canad. J. Math.*, **8** (1956), 40–46.

[11] **Halberstam, H. and K.F. Roth,** *Sequences*, Clarendon Press, Oxford, 1966.

[12] **Indlekofer, K.-H.,** New approach to probabilistic number theory-compactifications and integration, *Adv. Studies in Pure Mathematics*, **49** (2005), 133–170.

[13] **Indlekofer, K.-H.,** Cesàro means of additive functions, *Analysis*, **6** (1986), 1–24.

[14] **Kátai, I.,** A remark on number-theoretical functions, *Acta Arith.*, **14** (1968), 409–415.

[15] **Miller, H.I.,** A measure theoretical subsequence characterization of statistical convergence, *Trans. Amer. Math. Soc.*, **347** (1995), 1881–1819.

[16] **Osikiewicz, J.A.,** Equivalence results for Cesaro submethods, *Analysis*, **20(1)** (2000), 35–43.

[17] **Steinhaus, H.,** Sur la convergence ordinaire et la convergence asymtotique, *Colloq. Math.*, **2** (1951), 73–74.

[18] **Zygmund, A.,** *Trigonometric Series*, Second ed., Cambridge Univ. Press, Cambridge, 1979.

E. Kaya
Tarsus Vocational School
University of Mersin
Takbaş Köyü Mevkii
33480 Mersin, Turkey
kayaerdener@mersin.edu.tr

M. Kucukaslan
Faculty of Science,
Department of Mathematics
University of Mersin
33343 Mersin, Turkey
mkucukaslan@mersin.edu.tr

R. Wagner
Faculty of Computer Science,
Electrical Engineering and Mathematics
University of Paderborn
Warburger Straße 100
D-33098 Paderborn, Germany
Robert.Wagner43@gmx.de