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CLOSED FORM SOLUTIONS OF MEASURES OF SYSTEMIC RISK

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Dedicated to Professor Karl-Heinz Indlekofer on his seventieth anniversary

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Abstract. This paper derives – considering a Gaussian setting – closed form solutions of the statistics that have been suggested as measures of systemic risk to be attached to individual banks. The statistics equal the product of statistic specific regression coefficients with the mean corrected Value at Risk. Hence, in a Gaussian setting the measures of systemic risks are closely related to well known concepts of financial statistics. A further benefit of the analysis is that it is revealed how the concepts are related to each other.

1. Introduction

Value at Risk (VaR) is an established measure of the riskiness of e.g. financial institutions. However, this statistic measures the riskiness of an institution in "isolation". The financial crisis taught everybody that it is not a good idea to measure the riskiness of financial institutions as if these institutions could be analyzed in isolation.

This paper is based on two influential recent contributions: Adrian and Brunnermeier [3], [4] and Acharya et al. [1], [2]. Adrian and Brunnermeier – like many others – argue very convincingly that incentives of banks would be distorted if regulation were based on Value at Risk not taking into account the systemic risk linked to a bank. They have stressed the need to analyze systemic

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risk and have done so in a very practical manner. They suggest a statistic that could play a practical role when stipulating regulatory capital.

Adrian and Brunnermeier [4] consider the Value of Risk – i.e. the $\alpha-$quantile of some risk related statistic – of a group A of financial institutions (the others) given that a specific institution i has hit its VaR^i. To measure the systemic risk to be attached to bank i, they suggest

$$\begin{aligned}\Delta\mathrm{CoVaR}^{Ai} &= \mathrm{CoVaR}^{Ai} - \mathrm{CoVaRm}^{Ai} = \\ &= \mathrm{VaR}(X_A|X_i = \mathrm{VaR}(X_i)) - \mathrm{VaR}(X_A|X_i = \mathrm{median}(X_i)).\end{aligned}$$

Without loss of generality the statistic X is chosen such that low values relate to bad results. Hence, the risky tail is the left tail. Obviously, CoVaR depends on the dependencies that the other institutions have with the state of the institution i, especially the dependencies at the left tail.

In this paper the approach of Adrian and Brunnermeier is studied in a simple stochastic Gaussian setting. Within this framework the intuition of ΔCoVaR can easily be grasped and a closed form solution can be derived. The approach is very transparent as it can easily be linked to standard concepts of risk management viz. β-coefficient and Value at Risk. The approach is as follows: We consider a system of financial institutions $S = \{i\} \cup A$, where we single out one of these institutions (viz. the institution i). We derive the very simple closed form representation of the systemic risk attached to bank i (Delta Collateral Value at Risk):

$$\begin{aligned}\Delta\mathrm{CollVaR}^{Ai} &= -\Phi^{-1}(\alpha)\frac{\Sigma_{Ai}}{\sqrt{\Sigma_i}} = \beta_{Ai}\,\mathrm{VaR}^{\mathrm{mean}}(X_i), \\ \Delta\mathrm{CollVaR}^{Ai} &= \mathrm{CoVaR}^{Ai} - \mathrm{VaR}^{A}(X_A|X_i = \mathbf{E}(X_i)), \\ \beta_{Ai} &= \frac{\mathrm{cov}(X_i, X_A)}{\mathrm{var}(X_i)}, \mathrm{VaR}^{\mathrm{mean}}(X_i) = \mathrm{VaR}(X_i) - \mathbf{E}(X_i),\end{aligned}$$

where α is the VaR-threshold, Φ is the standard normal distribution function and Σ_{Ai} is the covariance of X_i and X_A. For clarification, the parameter α is close to 1 (say 0.999) – and hence $\varphi = \Phi^{-1}(\alpha)$ is a positive real number ($\varphi = 3.09$ if $\alpha = 0.999$). We derive a series of closed form solutions of statistics related to systemic risk. A table in section 3 provides a summary. Note, the expected value is used instead of the median to define the condition of the conditional Value at Risk. In a Gaussian framework, however, median and mean are the same.

The purpose of this paper is threefold: Improve intuition, offer closed form solutions in a specific stochastic framework and reveal how the concepts are related to each other. The structure of the paper is as follows. In section

2 the Gaussian setting is introduced and the closed form solutions for several statistics relevant for systemic risk are derived. It is shown that all statistics are closely related to well-known concepts of finance viz. β-coefficients and VaR. Also, the relationships between the statistics is derived. Section 3 summarizes.

2. Systemic Risk in a Gaussian setting

2.1. The stochastic framework and the statistics analyzed

We focus on the following problem: There is a system of banks, we single out one of these banks and we aim to measure the systemic risk to be attached to the bank singled out. Such a perspective is quite natural for a supervisor, who actually will study the risk of bank on its own first and augment this narrow risk analysis by a systemic risk analysis. The aim of the paper is to derive simple formulae of quantitative measures of systemic risk to be attached to the bank focused. We denote the bank focused by i. A refers to the complete banking system without the bank i, i.e. $S = \{i\} \cup A$ is the system under investigation.

In the following a vector $(X_i, X_A)'$ of a bank-i respectively group-A related statistic will be considered. Deliberately, we leave indefinite what statistic is considered; one may think of Profit/Loss, Return, Assets, Equity, etc.

Maintained Assumption: *We assume that X_i, X_A are jointly Gaussian with expected values μ_i, μ_A and variance-covariance matrix*

$$\Sigma = \begin{pmatrix} \Sigma_i & \Sigma_{Ai} \\ \Sigma_{Ai} & \Sigma_A \end{pmatrix}.$$

In the following and without further mentioning, we are going to assume that this assumption holds. Of course it is this assumption that allows us to derive closed form solutions. It is well known that empirically many financial statistics are not Gaussian. Hence, the closed form solutions derived later should be used with care. But the benefits of transparency and intuition that result from a closed form solution justify to study the Gaussian case.

The following lemma provides a closed form solution of the conditional expected value and of the conditional variance.

Lemma 2.1 (McNeil et al. [8], p. 68)**.** *The conditional expected value of X_A given that $X_i = x$ is*

$$\mathbf{E}(X_A|X_i = x) = \mu_A + \frac{\Sigma_{Ai}}{\Sigma_i}(x - \mu_i)$$

and the conditional variance equals

$$\mathbf{VAR}(X_A|X_i = x) = \Sigma_A - \frac{\Sigma_{Ai}^2}{\Sigma_i}.$$

Note, that the expected value of X_A depends linearly on realization x of X_i, whereas the conditional variance does not depend on x. Note, that the variance of the conditioned random variable is smaller than its unconditional variance if X_i an X_A are positively correlated (let us call this effect "variance reduction by conditioning").

The second lemma of great use for our purpose endows us with a closed form solution of the Value at Risk of a normally distributed variable.

Lemma 2.2 (McNeil et al. [8], p. 39)**.** *The Value at Risk of a normally distributed variable $X \sim \mathrm{N}(\mu, \Sigma)$ is given by*

$$\mathrm{VaR}(X) = \mu - \varphi \cdot \sqrt{\Sigma}\,,\ \varphi = \Phi^{-1}(\alpha),$$

where Φ^{-1} is the inverse of the standard normal distribution and α is the probability threshold used to define the Value at Risk.

List of Statistics: The aim of this paper is to study statistics that measure systemic risk to be attached to a bank. We will derive closed form solutions of three such statistics and also study the relationships between these statistics:

$$\begin{aligned}
\Delta\mathrm{CollVaR}^{Ai} &= \mathrm{VaR}(X_A|X_i = \mathrm{VaR}(X_i)) - \mathrm{VaR}(X_A|X_i = \mathbf{E}(X_i)),\\
\Delta\mathrm{CondVaR}^{Si} &= \mathrm{VaR}(X_S|X_i = \mathrm{VaR}(X_i)) - \mathrm{VaR}(X_S|X_i = \mathbf{E}(X_i)),\\
\Delta\mathrm{ContrVaR}^{iS} &= \mathrm{VaR}(X_i|X_S = \mathrm{VaR}(X_S)) - \mathrm{VaR}(X_i|X_S = \mathbf{E}(X_S)).
\end{aligned}$$

In each case a stressed situation is compared with an unstressed situation. The stressed situation is modeled by using the VaR in the condition of the conditional Value of Risk, whereas the unstressed situation is modeled by using the expected value to fix the condition of the conditional VaR. Note, that by studying the difference of conditioned VaRs the variance reduction effect mentioned earlier cancels. This is indeed very important, as otherwise misleading statistics would be generated.

2.2. The closed form solution

We first apply the two lemmata to derive a closed form solution of the VaR of A given that bank i has hit its VaR. We obtain

$$\begin{aligned}
\mathrm{CoVaR}^{Ai} &= \mu_A + \frac{\Sigma_{Ai}}{\Sigma_i}\left(\mu_i - \sqrt{\Sigma_i}\cdot\varphi - \mu_i\right) - \varphi\sqrt{\Sigma_A - \frac{\Sigma_{Ai}^2}{\Sigma_i}} =\\
&= \mu_A - \varphi\frac{\Sigma_{Ai}}{\sqrt{\Sigma_i}} - \varphi\sqrt{\Sigma_A - \frac{\Sigma_{Ai}^2}{\Sigma_i}}\,.
\end{aligned}$$

The sum of the first two terms is the conditional expected value and the third is the conditional standard deviation. Lemma 1 is used to calculate the conditional VaR of X_A. Lemma 2 is used twice. First when calculating conditional VaR of X_A and second when calculating the VaR of X_i. We are now in the position to calculate the closed form solution of

$$\Delta\mathrm{CollVaR}^{Ai} = \mathrm{CoVaR}^{Ai} - \mathrm{CoVaRe}^{Ai},$$

where

$$\mathrm{CoVaRe}^{Ai} = \mathrm{VaR}(X_A|X_i = \mathbf{E}(X_i)) = \mu_A - \varphi\sqrt{\Sigma_A - \frac{\Sigma_{Ai}^2}{\Sigma_i}}.$$

We observe that most terms – viz. the conditional variance and the unconditional expected value – cancel out if we calculate the difference $\mathrm{CoVaR}^{Ai} - -\mathrm{CoVaRe}^{Ai}$, so that we obtain

$$\Delta\mathrm{CollVaR}^{Ai} = -\varphi\frac{\Sigma_{Ai}}{\sqrt{\Sigma_i}} = -\Phi^{-1}(\alpha)\frac{\Sigma_{Ai}}{\sqrt{\Sigma_i}}.$$

This verifies

Proposition 2.1. *Delta Collateral Value of Risk equals*

$$\Delta\mathrm{CollVaR}^{Ai} = \left(\frac{\Sigma_{Ai}}{\Sigma_i}\right)\cdot(-\varphi\sqrt{\Sigma_i}) = \beta_{Ai}\cdot\mathrm{VaR}^{\mathrm{mean}}(X_i).$$

The interpretation is as follows: Delta Collateral Value at Risk equals the regression coefficient $\beta_{Ai} = \Sigma_{Ai}/\Sigma_i$ times the impulse $\mathrm{VaR}^{\mathrm{mean}}(X_i)$. In other words: The calamities of bank i measured by its own mean-corrected Value at Risk $\mathrm{VaR}^{\mathrm{mean}}(X_i)$ are translated into systemic calamities via the regression coefficient β_{Ai}.

Remark 2.1. If i is in a stressed situation A is also pulled down (if i and A are positively correlated). The statistic $\Delta\mathrm{CollVaR}^{Ai}$ captures this effect actually through the change in expected value:

$$\Delta\mathrm{CollVaR}^{Ai} = \mathbf{E}(X_A|X_i = \mathrm{VaR}(X_i)) - \mu_A.$$

Also note that

$$\begin{aligned}\Delta\mathrm{CollVaR}^{Ai} &= \beta_{Ai}\cdot\mathrm{VaR}^{\mathrm{mean}}(X_i) = \frac{\mathrm{cov}(X_i, X_A)}{\mathrm{var}(X_i)}\cdot\mathrm{VaR}^{\mathrm{mean}}(X_i) = \\ &= -\frac{\rho\cdot\mathrm{std}(X_i)\cdot\mathrm{std}(X_A)}{\mathrm{std}(X_i)^2}\cdot\varphi\cdot\mathrm{std}(X_i) = -\varphi\cdot\rho\cdot\mathrm{std}(X_A),\end{aligned}$$

where $\rho = \mathrm{corr}(X_i, X_A)$.

Remark 2.2. Note, that the size of the institution i has no direct effect on $\Delta\text{CollVaR}^{Ai} = -\varphi \cdot \rho \cdot \text{std}(X_A)$. $\Delta\text{CollVaR}^{Ai}$ is affected by the size of A and the correlation with i. Hence, a small i may be systemically relevant if A is highly correlated with i.

Note that in the process of supervising a bank, the statistic $\text{VaR}^{i,\text{mean}}$ is calculated anyway. All that is additionally needed to implement the formulae derived in this paper is the covariance $\text{cov}(X_i, X_S)$ (or β_{Si} or β_{iS}). These statistics are relatively well understood and relatively easy to communicate to banks and to the market.

So far we have considered the system $S = \{i\} \cup A$. We have singled out the bank i and studied the CoVaR attached to this bank. In this context CoVaR is best translated as Collateral Value at Risk: We consider the state of A – the others – if i is under stress. We thereby focus on the spillover (or the externality) attached to bank i exerted on the group A. Obviously, it is also of interest to study the shape of the complete system S if i is under stress. In other words, we are as much interested to study the stochastic vector $(X_i, X_i + X_A)'$ as we are interested to analyze the vector $(X_i, X_A)'$. It is easy to derive variance-covariance matrix of $(X_i, X_i + X_A)' = (X_i, X_S)'$:

$$\Sigma^{iS} = \begin{pmatrix} \Sigma_i & \Sigma_{iA} + \Sigma_i \\ \Sigma_{iA} + \Sigma_i & \Sigma_i + 2\Sigma_{iA} + \Sigma_A \end{pmatrix}.$$

Using this we find that the conditional Value at Risk of the system given that X_i has hit the value x is

$$\begin{aligned} \text{VaR}(X_S|X_i = x) = \mu_S + \frac{\Sigma_{iA} + \Sigma_i}{\Sigma_i}(x - \mu_i) - \\ -\varphi\sqrt{\Sigma_i + 2\Sigma_{iA} + \Sigma_A - \frac{(\Sigma_{iA} + \Sigma_i)^2}{\Sigma_i}} \end{aligned}$$

and consequently we have the

Proposition 2.2. *Delta Conditional Value at Risk equals*

$$\begin{aligned} \Delta\text{CondVaR}^{Si} &= \text{VaR}(X_S|X_i = \text{VaR}(X_i)) - \text{VaR}(X_S|X_i = \mathbf{E}(X_i)) = \\ &= -\varphi\frac{\Sigma_{iA} + \Sigma_i}{\sqrt{\Sigma_i}} = \\ &= \beta_{Ai}\text{VaR}^{\text{mean}}(X_i) + \text{VaR}^{\text{mean}}(X_i) = \beta_{Si}\text{VaR}^{\text{mean}}(X_i). \end{aligned}$$

Again, this is intuitive. It means that the overall risk attached to bank i consists of the bank's own risk plus its collateral risk. If bank i hits its VaR the system is directly pulled down as i is a member of the group: consequently we have the term $-\varphi\sqrt{\Sigma_i} = \mathrm{VaR}^{\mathrm{mean}}(X_i)$ (which is the mean corrected VaR). In addition to this we have an indirect negative effect (at least in the more likely case that i and A are positively correlated) as the other banks are simultaneously drawn down: $-\varphi\Sigma_{Ai}/\Sigma_i = \beta_{Ai}\mathrm{VaR}^{\mathrm{mean}}(X_i)$.

So far we followed Adrian and Brunnermeier and considered the VaR of the system given that a bank has hit its VaR. As an alternative we may consider the VaR of bank i given that the financial system has hit its VaR, i.e.

$$\mathrm{VaR}(X_i|X_S = \mathrm{VaR}(X_S)).$$

This perspective is very close to the perspective that Acharya et al. [1] and Brownless and Engle [6] have recommended. They suggest to use a statistic that features prominently (e.g. Mc Neil, et. al. [8], p. 258) in risk management viz. the VaR-contribution:

$$\text{VaR-Contribution}_{iS} = \mathbf{E}(X_i|X_S = \mathrm{VaR}(X_S)).$$

The idea of the VaR-Contribution: If the financial system S has hit its VaR what is the contribution of i to this loss. In other words, if we were to recapitalize a system because of a systemic crisis what is the share attributable to a specific institution.

Proposition 2.3. *The difference between the stressed and the normal situation is*

$$\begin{aligned}\Delta\mathrm{ContrVaR}^{iS} &= \mathrm{VaR}(X_i|X_S = \mathrm{VaR}(X_S)) - \mathrm{VaR}(X_i|X_S = \mathrm{E}(X_S)) = \\ &= -\varphi\frac{\Sigma_{iA} + \Sigma_i}{\sqrt{\Sigma_i + 2\Sigma_{iA} + \Sigma_A}} = -\varphi\frac{\Sigma_{iA} + \Sigma_i}{\sqrt{\Sigma_S}} = \\ &= \frac{\Sigma_{iA} + \Sigma_i}{\Sigma_S}(-\varphi)\sqrt{\Sigma_S} = \beta_{iS}\mathrm{VaR}^{\mathrm{mean}}(X_S).\end{aligned}$$

Proof. Observe

$$\begin{aligned}\mathbf{E}(X_i|X_S = x) &= \mu_i + \frac{\Sigma_{iA} + \Sigma_i}{\Sigma_i + 2\Sigma_{iA} + \Sigma_A}(x - \mu_S), \\ \mathbf{VAR}(X_i|X_S = x) &= \Sigma_i - \frac{(\Sigma_{iA} + \Sigma_i)^2}{\Sigma_i + 2\Sigma_{iA} + \Sigma_A}.\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{VaR}(X_i|X_S = \mathrm{VaR}(X_S)) &= \mu_i - \frac{\Sigma_{iA} + \Sigma_i}{\Sigma_i + 2\Sigma_{iA} + \Sigma_A}\,\varphi\,\sqrt{\Sigma_i + 2\Sigma_{iA} + \Sigma_A} - \\ &\quad -\varphi\sqrt{\Sigma_i - \frac{(\Sigma_{iA} + \Sigma_i)^2}{\Sigma_i + 2\Sigma_{iA} + \Sigma_A}} = \\ &= \mu_i - \varphi\frac{\Sigma_{iA} + \Sigma_i}{\sqrt{\Sigma_i + 2\Sigma_{iA} + \Sigma_A}} - \varphi\sqrt{\Sigma_i - \frac{(\Sigma_{iA} + \Sigma_i)^2}{\Sigma_i + 2\Sigma_{iA} + \Sigma_A}}\end{aligned}$$

■

Several remarks are appropriate:

Remark 2.3. A useful feature of $\Delta\text{ContrVaR}^{iS}$ is that the sum of the systemic risks attached to i and A equals aggregate risk. Observe

$$\beta_{iS} + \beta_{AS} = \frac{\Sigma_{iA} + \Sigma_i}{\Sigma_S} + \frac{\Sigma_{iA} + \Sigma_A}{\Sigma_S} = \frac{\Sigma_i + 2\Sigma_{iA} + \Sigma_A}{\Sigma_S} = 1.$$

Hence

$$\Delta\text{ContrVaR}^{iS} + \Delta\text{ContrVaR}^{AS} = \text{VaR}^{\text{mean}}(X_S).$$

Remark 2.4. The VaR-Contribution – a conditional expected value – equals the unconditional expected value plus ΔContrVaR:

$$\text{VaR} - \text{Contribution}_{iS} = \mu_i + \Delta\text{ContrVaR}^{iS}.$$

Proof Indeed, using Lemma (2.1) we have

$$\begin{aligned}\mathbf{E}(X_i|X_S = \text{VaR}(X_S)) = \mu_i - \frac{\Sigma_{iA} + \Sigma_i}{\Sigma_i + 2\Sigma_{iA} + \Sigma_A}\,\varphi\,\sqrt{\Sigma_i + 2\Sigma_{iA} + \Sigma_A} = \\ = \mu_i + \Delta\text{ContrVaR}^{iS}\end{aligned}$$

or

$$\Delta\text{ContrVaR}^{iS} = \mathbf{E}(X_i|X_S = \text{VaR}(X_S)) - \mu_i. \qquad \blacksquare$$

Remark 2.5. Compare the following statistics

$$\Delta\text{CollVaR}^{Ai} = \mathbf{E}(X_A|X_i = \text{VaR}(X_i)) - \mu_A,$$

$$\Delta\text{ContrVaR}^{iS} = \mathbf{E}(X_i|X_S = \text{VaR}(X_S)) - \mu_i.$$

In both cases a change of an expected value is considered. Whereas in the top-down approach one considers the change of expected value of i given stress in the system, in the bottom up approach the perspective is reversed as one considers the expected value of the system given stress at i.

Remark 2.6. Furthermore, note that

$$\sum_i (\text{VaR} - \text{Contribution})_{iS} = \text{VaR}(X_S),$$

whereas

$$\Delta\text{ContrVaR}^{iS} + \Delta\text{ContrVaR}^{AS} = \text{VaR}^{\text{mean}}(X_S).$$

This is as intuition suggests, as the statistics ContrVaR^{xy} equal conditional expectations corrected with unconditional expectations.

The next result relates the statistics based on Adrian and Brunnermeier [4] to that of Proposition (2.3). Indeed, the formula for $\Delta\text{ContrVaR}^{iS}$ closely resembles that of $\Delta\text{CondVaR}^{Si}$. The only difference is the denominator. Whereas it is $1/\text{std}(X_i)$ in case of $\Delta\text{CondVaR}^{Si}$ it is $1/\text{std}(X_S)$ in the case of $\Delta\text{ContrVaR}^{iS}$.

Proposition 2.4. *The statistics* $\Delta\text{CondVaR}^{Si}$ *and* $\Delta\text{ContrVaR}^{iS}$ *are closely related:*

$$\Delta\text{CondVaR}^{Si} = \left(\frac{\sqrt{\Sigma_S}}{\sqrt{\Sigma_i}}\right) \cdot \Delta\text{ContrVaR}^{iS}$$

or

$$\frac{\Delta\text{CondVaR}^{Si}}{\sqrt{\Sigma_S}} = \frac{\Delta\text{ContrVaR}^{iS}}{\sqrt{\Sigma_i}}.$$

Using Proposition (2.4) we can quickly find an additivity result for the Conditional Value at Risk.

Proposition 2.5. *The weighted sum of Conditional Values at Risk equals aggregated risk:*

$$\begin{aligned} & \left(\frac{\sqrt{\Sigma_i}}{\sqrt{\Sigma_S}}\right) \Delta\text{CondVaR}^{Si} + \left(\frac{\sqrt{\Sigma_A}}{\sqrt{\Sigma_S}}\right) \Delta\text{CondVaR}^{SA} \\ = \quad & \Delta\text{ContrVaR}^{iS} + \Delta\text{ContrVaR}^{AS} = \text{VaR}^{\text{mean}}(X_S). \end{aligned}$$

This result mirrors the observation of remark (2.3). The sum of systemic risks attributed to the several banks equals mean corrected aggregate risk. However in case of the Conditional Value at Risk it is a *weighted* sum, where the weights reflect the relative risk of the institutions. Proposition (2.5) shows how the top-down of Acharya et al. [1] is related to the bottom-up approach of Adrian and Brunnermeier (see Drehmann and Tarashev [7] for a discussion of top-down and bottom-up).

3. Conclusion

We have derived a battery of closed form solutions of statistics of systemic risk calculated in a Gaussian setting. The formulae allow us to relate the systemic risk statistics to well known concepts of financial statistics viz. VaR and β-coefficients. We also derive a collection of result that reveal how the different statistics of systemic risk are related to each other. For sake of transparency a summary of the formulae is provided.

Collecting Results:

$$\begin{aligned}
\Delta\text{CollVaR}^{Ai} &= \text{VaR}(X_A|X_i = \text{VaR}(X_i)) - \text{VaR}(X_A|X_i = \mathbf{E}(X_i)) = \\
&= \beta_{Ai}\text{VaR}^{\text{mean}}(X_i) = \rho \cdot \varphi \cdot \text{std}(X_A) = \\
&= \mathbf{E}(X_A|X_i = \text{VaR}(X_i)) - \mu_A,
\end{aligned}$$

$$\begin{aligned}
\Delta\text{CondVaR}^{Si} &= \text{VaR}(X_S|X_i = \text{VaR}(X_i)) - \text{VaR}(X_S|X_i = \mathbf{E}(X_i)) = \\
&= \beta_{Ai}\text{VaR}^{\text{mean}}(X_i) + \text{VaR}^{\text{mean}}(X_i) = \beta_{Si}\text{VaR}^{\text{mean}}(X_i),
\end{aligned}$$

$$\begin{aligned}
\Delta\text{ContrVaR}^{iS} &= \text{VaR}(X_i|X_S = \text{VaR}(X_S)) - \text{VaR}(X_i|X_S = \mathbf{E}(X_S)) = \\
&= \beta_{iS}\text{VaR}^{\text{mean}}(X_i) = \mathbf{E}(X_i|X_S = \text{VaR}(X_S)) - \mu_i,
\end{aligned}$$

$$\Delta\text{CondVaR}^{Si} = \left(\frac{\sqrt{\Sigma_S}}{\sqrt{\Sigma_i}}\right) \cdot \Delta\text{ContrVaR}^{iS},$$

$$\text{VaR}^{\text{mean}} = \Delta\text{ContrVaR}^{iS} + \Delta\text{ContrVaR}^{AS},$$

$$\text{VaR}^{\text{mean}} = \left(\frac{\sqrt{\Sigma_i}}{\sqrt{\Sigma_S}}\right)\Delta\text{CondVaR}^{Si} + \left(\frac{\sqrt{\Sigma_A}}{\sqrt{\Sigma_S}}\right)\Delta\text{CondVaR}^{SA}.$$

References

[1] **Acharya, V., L. Pedersen, Z. Philippon and M. Richardson,** Regulating Systemic Risk, in: Acharya, V. , Richardson, M. (Eds.), *Restoring Financial Stability,* Wiley, Hoboken, 2009, 283–303.

[2] **Acharya, V., L. Pedersen, T. Philippon and M. Richardson,** Measuring Systemic Risk,
`http://pages.stern.nyu.edu/~sternfin/vacharya/public_html/MeasuringSystemicRisk_final.pdf`

[3] **Adrian, T. and M. Brunnermeier,** CoVaR, FRB of New York Staff Report No. 348,
`http://www.newyorkfed.org/research/staff_reports/sr348.pdf`

[4] **Adrian, T. and M. Brunnermeier,** CoVaR,
`http://www.princeton.edu/~markus/research/papers/CoVaR`

[5] **Brunnermeier, M. and L. Pedersen,** Market Liquidity and Funding Liquidity, *Review of Financial Studies*, **22(6)** (2009), 2201–2238.

[6] **Brownless, C. and R. Engle,** Volatility, Correlation and Tails for Systemic Risk Measurement, `http://ssrn.com/abstract=1611229`

[7] **Drehmann, M. and N. Tarashev,** Systemic importance: some simple indicators. *BIS Quarterly Review March*, 2011, 25–37.

[8] **McNeil, A., R. Frey and P. Embrechts,** *Quantitative Risk Management,* Princeton University Press, Princeton, 2005.

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