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JOINT UNIVERSALITY OF DIRICHLET L-FUNCTIONS AND HURWITZ ZETA-FUNCTIONS

Kęstutis Janulis and Antanas Laurinčikas
(Vilnius, Lithuania)

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Abstract. In the paper, we prove a joint universality theorem of Voronin type for collections of Dirichlet L-functions and Hurwitz zeta-functions. From this theorem, we derive the universality for some class of functions from the above collections. For example, this shows that a product of Dirichlet L-functions and Hurwitz zeta-functions is universal.

1. Introduction

In 1975, S.M. Voronin discovered [29] a remarkable approximation property of the Riemann zeta-function $\zeta(s), s = \sigma + it$, which now is called universality. He proved that the shifts $\zeta(s + i\tau), \tau \in \mathbb{R}$, approximate with a given accuracy any analytic function uniformly on closed discs lying in the right-hand side of the critical strip. Denote by $\mathcal{K}$ the set of compact subsets of the strip

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$D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ with connected complements, and by $H_0(K), K \in \mathcal{K}$, the set of non-vanishing continuous functions on K which are analytic in the interior of K. Moreover, let $meas\{A\}$ stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then a modern version of the Voronin theorem is of the form, see, for example, [7].

Theorem 1.1. *Suppose that $K \in \mathcal{K}$ and $f \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon\} > 0.$$

An analogical universality property is also due to the Hurwitz zeta-function $\zeta(s,\alpha), 0 < \alpha \leq 1$, which is given, for $\sigma > 1$, by the series

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},$$

and by analytic continuation elsewhere, except for a simple pole at $s = 1$ with residue 1. Properties of $\zeta(s,\alpha)$ also depend on arithmetics of the parameter α. For the function $\zeta(s,\alpha)$, the following analogue of Theorem 1.1 is known. Denote by $H(K), K \in \mathcal{K}$, the class of continuous functions on K which are analytic in the interior of K.

Theorem 1.2. *Suppose that α is transcendental or rational $\neq 1, \frac{1}{2}$. Let $K \in \mathcal{K}$ and $f \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon\} > 0.$$

Since $H_0(K) \subset H(K), K \in \mathcal{K}$, we have that the shifts of $\zeta(s,\alpha)$ with transcendental or rational $\neq 1, \ \frac{1}{2}$ approximate a wider class of analytic functions than the shifts $\zeta(s+i\tau)$. Clearly, $\zeta(s,1) = \zeta(s)$,

$$\zeta\Big(s, \frac{1}{2}\Big) = (2^s - 1)\zeta(s),$$

and thus, the functions $\zeta(s,1)$ and $\zeta(s,\frac{1}{2})$ are also universal with approximation property with respect to the set $H_0(K)$.

The universality of $\zeta(s,\alpha)$ with algebraic irrational α remains an open problem.

Theorem 1.2 by different methods was obtained in [3], [1] and [27].

Universality of zeta-functions has an important generalization, the so called joint universality, when a collection of analytic functions is simultaneously approximated by shifts of zeta-functions. The first result in the field also belongs to Voronin who proved [28] the joint universality for Dirichlet L-functions. We also state a modern version of the joint Voronin theorem.

Theorem 1.3. *Suppose that $\chi_1, \ldots, \chi_r$ are pairwise non-equivalent Dirichlet characters, and $L(s, \chi_1), \ldots, L(s, \chi_r)$ are the corresponding Dirichlet L-functions. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}$ and $f_j \in H_0(K_j)$. Then, for every $\varepsilon > 0$.*

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{1\le j\le r} \sup_{s\in K_j} |L(s+i\tau, \chi_j) - f_j(s)| < \varepsilon\} > 0.$$

Proof of Theroem 1.3 is given in [16].

Some other zeta and L-functions are also jointly universal in the above sense. Among them are Hurwitz [10], [26], Lerch zeta-functions [14], [17], [19], [22], [26], zeta-functions of certain cusp forms [20] and their twists [21], periodic [18] and periodic Hurwirz zeta-functions [4], [8], [9], [11], [12], [15], [23], [24]. It is clear that, in the case of the joint universality, the involved zeta-functions must be independent in a certain sense. For Dirichlet L-functions, this independence is implied by the non-equivalence of Dirichlet characters. In the case of Hurwitz zeta-functions, the linear independence over the field of rational numbers $\mathbb{Q}$ of some set is required. For $j = 1, \ldots, r$, let $0 < \alpha_j \le 1$, and

$$L(\alpha_1, \ldots, \alpha_r) = \{\log(m+\alpha_j) : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, j = 1, \ldots, r\}.$$

Theorem 1.4. [10] *Suppose that the set $L(\alpha_1, \ldots, \alpha_r)$ is linearly independent over $\mathbb{Q}$. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}$ and $f_j \in H(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{1\le j\le r} \sup_{s\in K_j} |\zeta(s+i\tau, \alpha_j) - f_j(s)| < \varepsilon\} > 0.$$

Theorem 1.4 with a stronger hypothesis that the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$ has been proved in [26]. An analogical result is also true for periodic Hurwitz zeta-functions [24].

H. Mishou in [25] obtained an interesting theorem on the joint universality of $\zeta(s)$ and $\zeta(s, \alpha)$.

Theorem 1.5. [25] *Suppose that α is a transcendental number, $K_1, K_2 \in \mathcal{K}$ and $f_1 \in H_0(K_1), f_2 \in H(K_2)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{s\in K_1} |\zeta(s+i\tau) - f_1(s)| < \varepsilon,$$

$$\sup_{s\in K_2} |\zeta(s+i\tau,\alpha) - f_2(s)| < \varepsilon\} > 0.$$

Theorem 1.5 is the first example of so called mixed joint universality when the functions from the sets $H_0(K)$ and $H(K)$ are approximated by shifts $\zeta(s+ +i\tau)$ and $\zeta(s+i\tau,\alpha)$ of zeta-functions having and having no the Euler product over primes, respectively.

A generalization of Theorem 1.5 for a periodic zeta-function with multiplicative coefficients and periodic Hurwitz zeta-function is given in [5]. A mixed joint universality theorem for collections of periodic zeta-functions with multiplicative coefficients and of periodic Hurwitz zeta-functions has been obtained in [13]. In the latter theorem, a rank hypothesis on coefficients of periodic zeta-functions is used. The aim of this note is to obtain a mixed universality theorem for collections of Dirichlet L-functions and Hurwitz zeta-functions. In this case, any rank hypothesis is not needed.

Theorem 1.6. *Suppose that $\chi_1, \dots, \chi_{r_1}$ are pairwise non-equivalent Dirichlet characters, and that the numbers $\alpha_1, \dots, \alpha_{r_2}$ are algebraically independent over $\mathbb{Q}$. For $j = 1, \dots, r_1$, let $K_j \in \mathcal{K}$ and $f_j \in H_0(K_j)$, while, for $j = 1, \dots, r_2$, let $\widehat{K}_j \in \mathcal{K}$ and $\widehat{f}_j \in H(\widehat{K})$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{j\le j\le r_1} \sup_{s\in K_j} |L(s+i\tau,\chi_j) - f_j(s)| < \varepsilon,$$

$$\sup_{j\le j\le r_2} \sup_{s\in \widehat{K}_j} |\zeta(s+i\tau,\alpha_j) - \widehat{f}_j(s)| < \varepsilon\} > 0.$$

We see that Theorem 1.6 is a connection of Theorems 1.3 and 1.4. Unfortunately, the linear independence over $\mathbb{Q}$ of the set $L(\alpha_1, \dots, \alpha_{r_2})$ is not sufficient for the proof of Theorem 1.6 because we need the linear independence over $\mathbb{Q}$ of the set $\{\{\log p : p \text{ is prime}\}, L(\alpha_1, \dots, \alpha_{r_2})\}$, and therefore, we require the algebraic independence over $\mathbb{Q}$ for the numbers $\alpha_1, \dots, \alpha_{r_2}$.

Theorem 1.6 implies a more complicated statement for universality of Dirichlet L-functions and Hurwitz zeta-functions. Denote by $H(D)$ the space of analytic functions on D equipped with the topology of uniform convergence on compacta. Let $S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$.

Theorem 1.7. *Suppose that* $\chi_1, \ldots, \chi_{r_1}$ *are pairwise non-equivalent Dirichlet characters, the numbers* $\alpha_1, \ldots, \alpha_{r_2}$ *are algebraically independent over* $\mathbb{Q}$*, and* $F : H^{r_1+r_2}(D) \to H(D)$ *is a continuous function such that, for every polynomial* $p = p(s)$*, the set* $(F^{-1}\{p\}) \cap (S^{r_1} \times H^{r_2}(D))$ *is non-empty. Let* $K \in \mathcal{K}$ *and* $f \in H(K)$*. Then, for every* $\varepsilon > 0$*,*

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{s\in K} |F(L(s+i\tau,\chi_1),\ldots,L(s+i\tau,\chi_{r_1}),$$

$$\zeta(s+i\tau,\alpha_1),\ldots,\zeta(s+i\tau,\alpha_{r_2})) - f(s)| < \varepsilon\} > 0.$$

Theorem 1.7 implies the universality for a product of Dirichlet L-functions and Hurwitz zeta-functions.

Corollary 1.1. *Suppose that* $\chi_1, \ldots, \chi_{r_1}$ *and* $\alpha_1, \ldots, \alpha_{r_2}$ *satisfy the hypothesis of Theorem 1.7. Let* $\{j_1, \ldots, j_r\} \neq \emptyset$ *be arbitrary subset of* $\{1, \ldots, r_1\}$*, and* $\{l_1, \ldots, l_k\} \neq \emptyset$ *be arbitrary subset of* $\{1, \ldots, r_2\}$*. Let* $K \in \mathcal{K}$ *and* $f \in H(K)$*. Then, for every* $\varepsilon > 0$*,*

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{s\in K} |L(s+i\tau,\chi_{j_1})\ldots L(s+i\tau,\chi_{j_r})\times$$

$$\times\,\zeta(s+i\tau,\alpha_{l_1})\ldots\zeta(s+i\tau,\alpha_{l_k}) - f(s)| < \varepsilon\} > 0.$$

In our opinion, the most convenient method for proving universality theorems for zeta-functions is that based on probabilistic limit theorems in the space of analytic functions. Thus, we start with limit theorems.

2. Limit theorems

Denote by γ the unit circle on the complex plane, and define two tori

$$\widehat{\Omega} = \prod_p \gamma_p$$

and

$$\Omega = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_p = \gamma$ for all primes p, and $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. The tori $\widehat{\Omega}$ and Ω with the product topology and pointwise multipication are compact topological

Abelian groups. Let

$$\Omega^\kappa = \widehat{\Omega} \times \Omega_1 \times \ldots \times \Omega_{r_2},$$

where $\Omega_j = \Omega$ for $j = 1, \ldots, r_2$, and $\kappa = 1 + r_2$. Then again Ω^κ is a compact topological Abelian group. Therefore, denoting by $\mathcal{B}(S)$ the σ- field of Borel sets of the space S, we have that, on $(\Omega^\kappa, \mathcal{B}(\Omega^\kappa))$, the probability Haar measure m_H^κ can be defined. This gives the probability space $(\Omega^\kappa, \mathcal{B}(\Omega^\kappa), m_H^\kappa)$. We note that the measure m_H^κ is the product of the Haar measures $\widehat{m}_H$ and $m_{1H}, \ldots, m_{r_2 H}$ on $(\widehat{\Omega}, \mathcal{B}(\widehat{\Omega}))$ and $(\Omega_1, \mathcal{B}(\Omega_1)), \ldots, (\Omega_{r_2}, \mathcal{B}(\Omega_{r_2}))$, respectively. For elements of Ω^κ, we use the notation $\underline{\omega} = (\widehat{\omega}, \omega_1, \ldots, \omega_{r_2})$, where $\widehat{\omega} \in \widehat{\Omega}$ and $\omega_j \in \Omega_j$, $j = 1, \ldots, r_2$. Moreover, let $\widehat{\omega}(p)$ be the projection of $\widehat{\omega} \in \widehat{\Omega}$ to γ_p, and let $\omega_j(m)$ denote the projection of $\omega_j \in \Omega_j$ to γ_m. For brevity, we put $r = r_1 + r_2$. Now, on the probability space $(\Omega^\kappa, \mathcal{B}(\Omega^\kappa), m_H^\kappa)$, define the $H^r(D)$ - valued random element $\Xi(s, \underline{\omega})$ by the formula

$$\Xi(s, \underline{\omega}) = (L(s, \widehat{\omega}, \chi_1), \ldots, L(s, \widehat{\omega}, \chi_{r_1}), \zeta(s, \alpha_1, \omega_1), \ldots, \zeta(s, \alpha_{r_2}, \omega_{r_2})),$$

where

$$L(s, \widehat{\omega}, \chi_j) = \prod_p (1 - \frac{\chi_j(p)\widehat{\omega}(p)}{p^s})^{-1}, \quad j = 1, \ldots, r_1,$$

and

$$\zeta(s, \alpha_j, \omega_j) = \sum_{m=0}^\infty \frac{\omega_j(m)}{(m + \alpha_j)^s}, \quad j = 1, \ldots, r_2.$$

Let P_Ξ be the distribution of the random element $\Xi(s, \underline{\omega})$, i.e.,

$$P_\Xi(A) = m_H^\kappa(\underline{\omega} \in \Omega^\kappa : \Xi(s, \underline{\omega}) \in A), A \in \mathcal{B}(H^r(D)).$$

Moreover, we will use the notation

$$\Xi(s) = (L(s, \chi_1), \ldots, L(s, \chi_{r_1}), \zeta(s, \alpha_1), \ldots, \zeta(s, \alpha_{r_2})).$$

Let, for $A \in \mathcal{B}(H^r(D))$,

$$P_T(A) \stackrel{\text{def}}{=} \frac{1}{T} meas\{\tau \in [0; T] : \Xi(s + i\tau) \in A\}.$$

Then we have the following limit theorem.

Theorem 2.1. *Suppose that the numbers $\alpha_1, \ldots, \alpha_{r_2}$ are algebraically independent over $\mathbb{Q}$. Then P_T converges weakly to P_Ξ as $T \to \infty$.*

The theorem is contained in Theorem 2 of [13].

Theorem 2.2. *Suppose that the numbers $\alpha_1, \dots, \alpha_{r_2}$ are algebraically independent over $\mathbb{Q}$, and that $F : H^\kappa(D) \to H(D)$ is a continuous function. Then*

$$P_{T,F}(A) \stackrel{def}{=} \frac{1}{T} meas\{\tau \in [0;T] : F(\Xi(s+i\tau)) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

converges weakly to $P_\Xi F^{-1}$ as $T \to \infty$.

Proof. We have that $P_{T,F} = P_T F^{-1}$, where $P_T F^{-1}(A) = P_T(F^{-1}A)$, $A \in \mathcal{B}(H(D))$. Therefore, the theorem is a consequence of Theorem 2.1, Theorem 5.1 of [2], and of continuity of the function F. ■

3. Supports

Let S be a separable metric space, and P be a probability measure on $(S, \mathcal{B}(S))$. We remind that a minimal closed set $S_P \in \mathcal{B}(S)$ is called the support of the measure P if $P(S_P) = 1$. The set S_P consists of all elements x such that, for every open neighbourhood G of x, the inquality $P(G) > 0$ is satisfied.

Theorem 3.1. *Suppose that $\chi_1, \dots, \chi_{r_1}$ are pairwise non-equivalent Dirichlet characters, and the numbers $\alpha_1, \dots, \alpha_{r_2}$ are algebraically independent over $\mathbb{Q}$. Then the support of the measure P_Ξ is the set $S^{r_1} \times H^{r_2}(D)$.*

The theorem is deduced from two following lemmas. Let $\underline{\chi} = (\chi_1, \dots, \chi_{r_1})$, $\underline{\alpha} = (\alpha_1, \dots, \alpha_{r_2})$, $\underline{\underline{\omega}} = (\omega_1, \dots, \omega_{r_2})$,

$$\underline{L}(s, \widehat{\omega}, \underline{\chi}) = (L(s, \widehat{\omega}, \chi_1), \dots, L(s, \widehat{\omega}, \chi_{r_1}))$$

and

$$\underline{\zeta}(s, \underline{\alpha}, \underline{\underline{\omega}}) = (\zeta(s, \alpha_1, \omega_1), \dots, \zeta(s, \alpha_{r_2}, \omega_{r_2})).$$

Moreover, let $P_{\underline{L}}$ and $P_{\underline{\zeta}}$ denote the distributions of the random elements $\underline{L}(s, \widehat{\omega}, \underline{\chi})$ and $\underline{\zeta}(s, \underline{\alpha}, \underline{\underline{\omega}})$, respectively.

Lemma 3.1. *Suppose that $\chi_1, \dots, \chi_{r_1}$ are pairwise non-equivalent Dirichlet characters. Then the support of the measure $P_{\underline{L}}$ is the set S^{r_1}.*

Proof of the lemma is given in [13].

Lemma 3.2. *Suppose that the numbers $\alpha_1, \dots, \alpha_{r_2}$ are algebraically independent over $\mathbb{Q}$. Then the support of the measure $P_{\underline{\zeta}}$ is the set $H^{r_2}(D)$.*

The proof of the lemma can be found in [10].

Proof of Theorem 3.1. Let $A_1 \in \mathcal{B}(H^{r_1}(D))$ and $A_2 \in \mathcal{B}(H^{r_2}(D))$, and $A = A_1 \times A_2$. Since the spaces $H^{r_1}(D)$ and $H^{r_2}(D)$ are separable, we have [2] that

$$\mathcal{B}(H^{r_1+r_2}(D)) = \mathcal{B}(H^{r_1}(D)) \times \mathcal{B}(H^{r_2}(D)).$$

Let m_H be the Haar measure on $(\Omega_1 \times \ldots \times \Omega_{r_2}, \mathcal{B}(\Omega_1 \times \ldots \times \Omega_{r_2}))$. Then m_H^κ is the product of the measures $\widehat{m}_H$ and m_H. Therefore, by the above remark,

$$P_\Xi(A) = m_H^\kappa(\underline{\omega} \in \Omega^\kappa : \Xi(s, \underline{\omega}) \in A) =$$

$$= \widehat{m}_H(\widehat{\omega} \in \widehat{\Omega} : \underline{L}(s, \widehat{\omega}, \chi) \in A_1) m_H(\underline{\underline{\omega}} \in \Omega^{r_2} : \underline{\zeta}(s, \underline{\alpha}, \underline{\underline{\omega}} \in A_2).$$

From this, and Lemmas 3.1 and 3.2 the theorem follows. ■

Theorem 3.2. *Suppose that $F : H^{r_1+r_2}(D) \to H(D)$ is a continuous function such that, for every polynomial $p = p(s)$, the set $(F^{-1}\{p\}) \bigcap (S^{r_1} \times H^{r_2}(D))$ is non-empty. Then the support of the measure $P_\Xi F^{-1}$ is the whole of $H(D)$.*

Proof. Let g be an arbitrary element of $H(D)$, and G be any open neighbourhood of g. It was noted in [6] that the approximation on $H(D)$ reduces to that on compact subsets $K \subset D$ with connected complements. Moreover, by the Mergelyan theorem on the approximation of analytic functions by polynomials, for every $\varepsilon > 0$, there exists a polynomial $p = p(s)$ such that

$$\sup_{s \in K} |g(s) - p(s)| < \varepsilon.$$

If ε is small enough, we have that the polynomial $p(s)$ belongs to G. Thus, by a hypothesis of the theorem, the set $F^{-1}G$ is open, and, in view of Theorem 3.1, is a neighbourhood of an element of the support of the measure P_Ξ. Therefore,

$$P_\Xi(F^{-1}G) > 0.$$

Hence,

$$P_\Xi F^{-1}(G) = P_\Xi(F^{-1}G) > 0.$$

Since g and G are arbitrary, this proves the theorem. ■

4. Proof of Theorems 1.6 and 1.7

Proof of Theorem 1.6. By the Mergelyan theorem, there exist polynomials $p_1(s), \ldots, p_{r_1}(s)$ and $\widehat{p}_1(s), \ldots, \widehat{p}_{r_2}(s)$ such that

$$\sup_{1 \le j \le r_1} \sup_{s \in K_j} |f_j(s) - e^{p_j(s)}| < \frac{\varepsilon}{2} \tag{4.1}$$

and

$$\sup_{1\le j\le r_2}\sup_{s\in\widehat{K}_j}|\widehat{f}_j(s)-\widehat{p}_j(s)|<\frac{\varepsilon}{2}. \tag{4.2}$$

Define

$$G=\{g_1,\dots,g_{r_1},\widehat{g}_1,\dots,\widehat{g}_{r_2})\in H^{r_1+r_2}(D):\sup_{1\le j\le r_1}\sup_{s\in K_j}|g_j(s)-e^{p_j(s)}|<\tfrac{\varepsilon}{2},$$

$$\sup_{1\le j\le r_2}\sup_{s\in\widehat{K}_j}|\widehat{g}_j(s)-\widehat{p}_j(s)|<\tfrac{\varepsilon}{2}\}.$$

Then G is an open set in $H^{r_1+r_2}(D)$. Moreover, by Theorem 3.1, the collection

$$(e^{p_1(s)},\dots,e^{p_{r_1}(s)},\widehat{p}_1(s),\dots,\widehat{p}_{r_2}(s))\in H^{r_1+r_2}(D)$$

is an element of the support of the measure P_Ξ. Therefore, $P_\Xi(G)>0$. From this and Theorem 8, using the equivalent of weak convergence of probability measures in terms of open sets (Theorem 2.1 of [2]), we find that

$$\liminf_{T\to\infty}\frac{1}{T}meas\{\tau\in[0;T]:\Xi(s+i\tau)\in G\}\ge P_\Xi(G)>0,$$

or, by the definition of the set G,

$$\liminf_{T\to\infty}\frac{1}{T}meas\{\tau\in[0;T]:\sup_{1\le j\le r_1}\sup_{s\in K_j}|L(s+i\tau,\chi_j)-e^{p_j(s)}|<\tfrac{\varepsilon}{2},$$

$$\sup_{1\le j\le r_2}\sup_{s\in\widehat{K}_j}|\zeta(s+i\tau,\alpha_j)-\widehat{p}_j(s)|<\tfrac{\varepsilon}{2}\}>0. \tag{4.3}$$

However, in virtue of (4.1) and (4.2),

$$\{\tau\in[0;T]:\sup_{1\le j\le r_1}\sup_{s\in K_j}|L(s+i\tau,\chi_j)-e^{p_j(s)}|<\frac{\varepsilon}{2},$$

$$\sup_{1\le j\le r_2}\sup_{s\in\widehat{K}_j}|\zeta(s+i\tau,\alpha_j)-\widehat{p}_j(s)|<\tfrac{\varepsilon}{2}\}\subset$$

$$\subset\{\tau\in[0;T]:\sup_{1\le j\le r_1}\sup_{s\in K_j}|L(s+i\tau,\chi_j)-f_j(s)|<\varepsilon,$$

$$\sup_{1\le j\le r_2}\sup_{s\in\widehat{K}_j}|\zeta(s+i\tau,\alpha_j)-f_j(s)|<\varepsilon\}$$

Therefore, taking into account (4.3), we obtain that

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{1\le j\le r_1} \sup_{s\in K_j} |L(s+i\tau,\chi_j) - f_j(s)| < \varepsilon, \\ \sup_{1\le j\le r_2} \sup_{s\in \widehat{K}_j} |\zeta(s+i\tau,\alpha_j) - \widehat{f}_j(s)| < \varepsilon\} > 0.$$

■

Proof of Theorem 1.7. We apply similar arguments as in the proof of Theorem 1.6. The Mergelyan theorem implies the existence of a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{4.4}$$

Define

$$G = \{g \in H(D) : \sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}\}.$$

Since the set G is open, and, by Theorem 3.2, the polynomial $p(s)$ is an element of the support of the measure $P_\Xi F^{-1}$, we have that

$$P_\Xi F^{-1}(G) > 0.$$

This and Theorem 2.2 show that

$$\liminf_{T\to\infty} \frac{1}{T} meas\{\tau \in [0;T] : \sup_{s\in K} |F(\Xi(s+i\tau)) - p(s)| < \frac{\varepsilon}{2}\} > 0.$$

Combining this with (4.4) proves the theorem. ■

Proof of Corollary 1.1. We take a function $F : H^{r_1+r_2}(D) \to H(D)$ given by

$$F(g_1, \dots, g_{r_1}; \widehat{g}_1, \dots, \widehat{g}_{r_2}) = g_{j_1} \dots g_{j_r} \widehat{g}_{l_1} \dots \widehat{g}_{l_k}.$$

Then the function F is continuous. Moreover, for every polynomial $p = p(s)$, we have that

$$F(1, \dots, 1; 1, \dots, 1, \widehat{g}_{l_1}, 1, \dots, 1) = p$$

and

$$(1, \dots, 1; 1, \dots, 1, \widehat{g}_{l_1}, 1, \dots, 1) \in S^{r_1} \times H^{r_2}(D)$$

with $\widehat{g}_{l_1} = p$. Thus, the hypotheses of Theorem 1.7 are satisfied, and we have the assertion of the corollary. ■

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K. Janulis and A. Laurinčkas
Faculty of Mathematics and Informatics, Naugarduko St. 24
LT-03225 Vilnius
Lithuania
kestutis.janulis@gmail.com
antanas.laurincikas@mif.vu.lt