

**A REMARK ON THE ISOMORPHIC
CLASSIFICATION OF WEIGHTED SPACES
OF HOLOMORPHIC FUNCTIONS
ON THE UPPER HALF PLANE**

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Abstract. We complete the isomorphic classification for weighted spaces $H_v(\mathbb{G})$ of holomorphic functions on the upper half plane \mathbb{G} with respect to standard weights v which are of at most moderate growth. We show that there are only two isomorphism classes for the corresponding Banach spaces $H_v(\mathbb{G})$, namely l_∞ and H_∞ . We prove that $H_v(\mathbb{G})$ is isomorphic to H_∞ if and only if v grows slowly. In particular $H_v(\mathbb{G})$ is isomorphic to H_∞ if v is bounded.

1. Introduction

Let $O \subset \mathbb{C}$ be an open subset and $v : O \rightarrow [0, \infty[$ a given function. Then we consider, for $f : O \rightarrow \mathbb{C}$, the weighted sup-norm

$$\|f\|_v = \sup_{z \in O} |f(z)|v(z)$$

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and the space

$$H_v(O) = \{f : O \rightarrow \mathbb{C} \text{ holomorphic} : \|f\|_v < \infty\}$$

In our paper we are concerned with the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and the upper half plane

$$\mathbb{G} = \{w \in \mathbb{C} : \text{Im } w > 0\}.$$

We want to investigate how far the spaces $H_v(\mathbb{D})$ and $H_v(\mathbb{G})$ are related to classical Banach spaces.

Actually the isomorphic classification of $H_v(\mathbb{D})$ is well-known for weights v on \mathbb{D} which satisfy $v(z) = v(|z|)$, $z \in \mathbb{D}$, $v(t) \leq v(s)$ if $0 \leq s \leq t < 1$ and $\lim_{t \rightarrow 1} v(t) = 0$. Then, depending on v , $H_v(\mathbb{D})$ is either isomorphic to l_∞ or to H_∞ , the space of all bounded holomorphic functions on \mathbb{D} endowed with the sup-norm ([7]).

In particular, if v is of moderate decay, i.e. if

$$\sup_{n \in \mathbb{N}} \frac{v(1 - 2^{-n})}{v(1 - 2^{-n-1})} < \infty$$

then $H_v(\mathbb{D})$ is isomorphic to l_∞ if and only if v is ‘normal’, i.e. if

$$\inf_{m \in \mathbb{N}} \sup_{n \in \mathbb{N}} \frac{v(1 - 2^{-n-m})}{v(1 - 2^{-n})} < 1 \quad ([5, 6]).$$

On \mathbb{G} we study now the following class of weights.

Definition 1.1. (i) Let $v : \mathbb{G} \rightarrow]0, \infty[$ be continuous such that $v(w) = v(i \text{Im } w)$, $w \in \mathbb{G}$, $v(is) \leq v(it)$ if $0 < s \leq t$ and $\lim_{t \rightarrow 0} v(it) = 0$. Then v is called a *standard weight*.

(ii) A standard weight v on \mathbb{G} satisfies *condition* (\star) if

$$\sup_{k \in \mathbb{Z}} \frac{v(2^{k+1}i)}{v(2^k i)} < \infty.$$

(iii) A standard weight v satisfies *condition* $(\star\star)$ if

$$\inf_{n \in \mathbb{N}} \sup_{k \in \mathbb{Z}} \frac{v(2^k i)}{v(2^{k+n} i)} < 1.$$

Examples. $v_1(it) = t^\alpha$, $t > 0$, for some $\alpha > 0$, $v_2(it) = \min(t^\alpha, 1)$, $t > 0$,

$$v_3(it) = \begin{cases} (1 - \ln(t))^{-\alpha}, & 0 < t \leq 1, \\ t, & t > 1, \end{cases}$$

define standard weights on \mathbb{G} which satisfy (\star) . Only v_1 satisfies $(\star\star)$.

Condition (\star) means that $v(it)$ is at most moderately growing while a weight with $(\star\star)$ grows at least moderately. The following lemma is easily seen ([1]).

Lemma 1.2. *Let v be a standard weight on \mathbb{G} . Then*

(i) *v satisfies (\star) if and only if there are $c > 0$ and $\beta > 0$ with*

$$\frac{v(it)}{v(is)} \leq c \left(\frac{t}{s}\right)^\beta \quad \text{whenever } 0 < s \leq t.$$

In this case we can take $c = a^2$ and $\beta = \ln a / \ln 2$ where

$$a = \sup_{k \in \mathbb{Z}} v(2^{k+1}i) / v(2^k i).$$

(ii) *v satisfies $(\star\star)$ if and only if there are $d > 0$ and $\gamma > 0$ with*

$$\frac{v(it)}{v(is)} \geq d \left(\frac{t}{s}\right)^\gamma \quad \text{whenever } 0 < s \leq t.$$

The aim of this note is to complete the isomorphic classification of $H_v(\mathbb{G})$ for standard weights satisfying (\star) . We show that there are only two isomorphism classes for $H_v(\mathbb{G})$. We obtain

Theorem 1.3. *Let v be a standard weight on \mathbb{G} with (\star) . Then*

- (i) *$H_v(\mathbb{G})$ is isomorphic to l_∞ if and only if v satisfies $(\star\star)$.*
- (ii) *$H_v(\mathbb{G})$ is isomorphic to H_∞ if and only if v does not satisfy $(\star\star)$.*

(i) was shown in [1]. We prove (ii) in the following sections.

Corollary 1.4. *Let v be a bounded standard weight on \mathbb{G} with (\star) . Then $H_v(\mathbb{G})$ is isomorphic to H_∞ .*

Proof. If v is bounded then it cannot satisfy $(\star\star)$. Hence 1.4. follows from Theorem 1.3. ■

The preceding theorem for standard weights v on \mathbb{G} cannot be inferred directly from the corresponding result for radial weights on \mathbb{D} . If v is a standard weight on \mathbb{G} and we consider a conformal map $\alpha : \mathbb{D} \rightarrow \mathbb{G}$ then $u(z) = v(\alpha(z))$ is a weight on \mathbb{D} . Moreover $H_u(\mathbb{D})$ is isometrically isomorphic to $H_v(\mathbb{G})$. But $v \circ \alpha$ is not radial and does not satisfy the other requirements for u . For weights on \mathbb{D} of the form $v \circ \alpha$, v a standard weight on \mathbb{G} , nothing is known about $H_{(v \circ \alpha)}(\mathbb{D})$.

2. $H_v(\mathbb{G})$ is complemented in H_∞

Theorem 2.1. *Let v be a standard weight on \mathbb{G} satisfying (\star) . Then $H_v(\mathbb{G})$ is isomorphic to a complemented subspace of H_∞ .*

To prove Theorem 2.1. we need to recall some facts from [1]. At first, consider a holomorphic function f on \mathbb{D} , say $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$. Put

$$(2.1) \quad (R_n f)(z) = \sum_{k=0}^{2^n} \alpha_k z^k + \sum_{k=2^{n+1}}^{2^{n+1}} \frac{2^{n+1} - k}{2^n} \alpha_k z^k.$$

Moreover, for $r > 0$, put $M_\infty(f, r) = \sup_{|z|=r} |f(z)|$. It is well-known (see e.g. [6]) that

$$(2.2) \quad M_\infty(R_n f, r) \leq 3M_\infty(f, r) \quad \text{for all } r \text{ and } n.$$

Clearly, we have

$$(2.3) \quad R_n R_m = R_{\min(m, n)} \quad \text{if } m \neq n.$$

Consider a radial weight u on \mathbb{D} , i.e. u satisfies $u(z) = u(|z|)$, $z \in \mathbb{D}$. Moreover, assume $u(s) \geq u(t)$ if $0 \leq s \leq t < 1$ and $\lim_{t \rightarrow 1} u(t) = 0$. Finally suppose

$$(2.4) \quad a := \sup_{n \in \mathbb{N}} \frac{u(1 - 2^{-n})}{u(1 - 2^{-n-1})} < \infty.$$

Then we use induction to find integers $m_0 = 0 < m_1 < m_2 < \dots$ such that

$$(2.5) \quad \frac{1}{2a} \leq \frac{u(1 - 2^{-m_{k+1}})}{u(1 - 2^{-m_k})} \leq \frac{1}{2}.$$

We have (see [1])

Proposition 2.2. *Put $\|f\| = \sup_k M_\infty((R_{m_k} - R_{m_{k-1}})f, 1)u(1 - 2^{-m_k})$. Then there is a universal constant $b > 0$ depending only on a such that*

$$\frac{1}{96} \|f\| \leq \|f\|_u \leq b \|f\| \quad \text{for all } f \in H_u(\mathbb{D}).$$

(Actually, $b = 32a + 4a \sum_{j=1}^{\infty} 2^j a^j \exp(-2^{j-1})$.)

Corollary 2.3. *Let $f : \mathbb{D} \rightarrow \mathbb{C}$ be holomorphic such that $\|f\| < \infty$. Then $f \in H_u(\mathbb{D})$.*

Proof. Fix n . Then (2.2) and (2.3) imply $\sup_n \|R_n f\| \leq 3\|f\|$. Since $R_n f$ is a polynomial it is an element of $H_u(\mathbb{D})$. Hence (2.2) yields

$$\sup_n \|R_n f\|_u \leq 3b\|f\|.$$

On the other hand, $(R_n f)$ converges to f pointwise on \mathbb{D} . Hence we have $\|f\|_u \leq \sup_n \|R_n f\|_u < \infty$. This implies $f \in H_u(\mathbb{D})$. ■

Let, for $n \in \mathbb{N}$, A_n be the space of all complex polynomials of degree $\leq n$ endowed with the norm $M_\infty(\cdot, 1)$. It is well-known that H_∞ is isomorphic to $(\sum_{n=1}^\infty \oplus A_n)_{(\infty)}$ (see [8]).

Proposition 2.4. *There is an isomorphism*

$$T : H_u(\mathbb{D}) \rightarrow \left(\sum_{k=1}^\infty \oplus A_{2^{m_k+2}}\right)_{(\infty)}$$

and a projection

$$P : \left(\sum_{k=1}^\infty \oplus A_{2^{m_k+2}}\right)_{(\infty)} \rightarrow TH_u(\mathbb{D})$$

with $\|P\| \leq 2 \cdot 10^4 ab$ and $\|T\| \cdot \|T^{-1}\| \leq 96b$.

Proof. This is essentially the argument of the proof of Lemma 3.3. of [6].

Let $B_k = A_{2^{m_k+2}}$ be endowed with the norm $M_\infty(\cdot, 1)u(1 - 2^{-m_k})$. Then $Y := (\sum_{k=1}^\infty \oplus B_k)_{(\infty)}$ is isometrically isomorphic to $(\sum_{k=1}^\infty \oplus A_{2^{m_k+2}})_{(\infty)}$. We work with Y instead of $(\sum_{k=1}^\infty \oplus A_{2^{m_k+2}})_{(\infty)}$. Define $T : H_u(\mathbb{D}) \rightarrow Y$ by $Tf = (R_{m_k} - R_{m_{k-1}})f$. Then, by Proposition 2.2., $\|T\| \cdot \|T^{-1}\| \leq 96b$. Moreover, define $S : Y \rightarrow H_u(\mathbb{D})$ by $S(g_k) = \sum_{j=1}^\infty (R_{m_k+1} - R_{m_{k-1}-1})g_k$ where $g_k \in B_k$ for all k . (Put $R_{-1} = 0$). S makes sense, $S(g_k)$ is the Taylor series of a holomorphic function on \mathbb{D} . Indeed, we have

$$(R_{m_k+1} - R_{m_{k-1}-1})g_k = \sum_{j=2^{m_{k-1}-1}}^{2^{m_k+2}} \alpha_j z^j$$

where

$$|\alpha_j| \leq \frac{\|f\|}{u(1 - 2^{-m_k})} \leq u(0)2^k \|f\|.$$

Hence this Taylor series converges pointwise on \mathbb{D} . We obtain, with (2.3),

$$\begin{aligned}
\|S(g_k)\| &\leq \sup_k M_\infty((R_{m_k} - R_{m_{k-1}})(R_{m_{k+1}} - R_{m_{k-1}-1})g_k + \\
&\quad + (R_{m_{k-1}+1} - R_{m_{k-2}-1})g_{k-1} + (R_{m_{k+1}+1} - R_{m_{k-1}})g_{k+1}, 1)u(1 - 2^{-m_k}) \\
&\leq 36 \sup_k (M_\infty(g_{k-1}, 1) + M_\infty(g_k, 1) + M_\infty(g_{k+1}, 1))u(1 - 2^{-m_k}) \leq \\
&\leq 36 \sup_k (M_\infty(g_{k-1}, 1)u(1 - 2^{-m_k+1}) + \\
&\quad + M_\infty(g_k, 1)u(1 - 2^{-m_k}) + aM_\infty(g_{k+1}, 1)u(1 - 2^{-m_{k+1}})) \leq \\
&\leq 108a\|(g_k)\|.
\end{aligned}$$

Hence, in view of Corollary 2.3., $S(g_k) \in H_u(\mathbb{D})$. With (2.3) we see that $STf = f$. Then $P = TS$ is a projection with $\|P\| \leq 108a \cdot 96b \leq 2 \cdot 10^4 ab$. ■

2.5. Proof of Theorem 2.1. Let

$$u_n(z) = v\left(n \frac{1 - |z|}{1 + |z|} i\right), \quad z \in \mathbb{D}, \quad n \in \mathbb{N}.$$

Then $H_v(\mathbb{G})$ is isometrically isomorphic to a complemented subspace of $Z := (\sum_{n=1}^\infty \oplus H_{u_n}(\mathbb{D}))_{(\infty)}$ ([1], Corollary 1.5.). Put

$$(2.6) \quad a_n = \sup_{j \in \mathbb{N}} \frac{u_n(1 - 2^{-j})}{u_n(1 - 2^{-j-1})}$$

and find integers $0 = m_{n,0} < m_{n,1} < m_{n,2} < \dots$ with

$$(2.7) \quad \frac{1}{2a_n} \leq \frac{u_n(1 - 2^{-m_{n,k+1}})}{u_n(1 - 2^{-m_{n,k}})} \leq \frac{1}{2}.$$

If v satisfies (\star) then $\sup_n a_n < \infty$ ([1], Lemma 4.1.). According to Proposition 2.4., there are isomorphisms $T_n : H_{u_n}(\mathbb{D}) \rightarrow (\sum_k \oplus A_{2^{m_{n,k}+2}})_{(\infty)} =: Y_n$ and projections $P_n : Y_n \rightarrow T_n H_{u_n}(\mathbb{D})$ where $\|T_n\| \cdot \|T_n^{-1}\|$ and $\|P_n\|$ depend only on a_n . Since $\sup_n a_n < \infty$ we see that there is an isomorphism $\tilde{T} : Z := (\sum_n \oplus H_{u_n}(\mathbb{D}))_{(\infty)} \rightarrow (\sum_n \oplus Y_n)_{(\infty)} =: Y$ and a bounded projection $\tilde{P} : Y \rightarrow Z$. Hence there is an isomorphism $T : H_v(\mathbb{G}) \rightarrow Y$ and a bounded projection $P : Y \rightarrow TH_v(\mathbb{G})$. We have $Y = (\sum_{n,k} \oplus A_{2^{m_{n,k}+2}})_{(\infty)}$. According to [6], Lemma 2.2., Y is isomorphic to $(\sum_{n=1}^\infty \oplus A_n)_{(\infty)}$ and hence to H_∞ . ■

3. H_∞ is complemented in $H_v(\mathbb{G})$

Theorem 3.1. *Let v be a standard weight on \mathbb{G} satisfying (\star) but not $(\star\star)$. Then $H_v(\mathbb{G})$ contains a complemented subspace which is isomorphic to H_∞ .*

Theorem 3.1. together with Theorem 2.1. and Pelczynski's decomposition method prove that $H_v(\mathbb{G})$ is isomorphic to H_∞ if v satisfies (\star) but not $(\star\star)$ since H_∞ is isomorphic to $(H_\infty \oplus H_\infty \oplus \dots)_{(\infty)}$ ([8]). The converse of Theorem 1.3.(ii) follows from Theorem 1.3.(i).

To prove Theorem 3.1. we consider at first polynomials on \mathbb{D} .

Lemma 3.2. *Let $0 \leq n_1 < n_2 < n_2 + 2 < n_3 < n_4$ be integers and consider*

$$\begin{aligned} X &= \text{span}\{z^{2^{n_1}+1}, z^{2^{n_1}+2}, \dots, z^{2^{n_2+1}-1}\}, \\ Y &= \text{span}\{z^{2^{n_3}+1}, z^{2^{n_3}+2}, \dots, z^{2^{n_4+1}-1}\}. \end{aligned}$$

For any $r > 0$ and $f \in X$, $g \in Y$ we have

$$(3.1) \quad (R_{n_4} - R_{n_3})f = 0 = (R_{n_2} - R_{n_1})g$$

and

$$\frac{1}{6} \max(M_\infty(f, r), M_\infty(g, r)) \leq M_\infty(f + g, r) \leq 2 \max(M_\infty(f, r), M_\infty(g, r)).$$

Proof. (3.1) follows from the definition of the operators R_n . The lower inequality follows from the fact that $(R_{n_2+1} - R_{n_1-1})(f + g) = f$ and $(R_{n_4+1} - R_{n_3-1})(f + g) = g$. The upper inequality is a consequence of the triangle inequality. ■

We need the following Lemma from [6].

Lemma 3.3. *Let X and Y be as in Lemma 3.2. Fix some constant $c > 0$. Consider the norm $M_\infty(\cdot, 1)c$ on X and on Y . Let*

$$m = \min(2^{n_2-n_1-1}, 2^{n_4-n_3-1}).$$

Then there is an isometry $i : A_m \rightarrow (X \oplus Y)_\infty$ and a projection $Q : (X \oplus Y)_{(\infty)} \rightarrow i(A_m)$ with $\|Q\| \leq 2$ such that

$$(3.2) \quad ((R_{n_2} - R_{n_1})f, (R_{n_4} - R_{n_3})g) = (f, g)$$

whenever $(f, g) \in i(A_m) \subset (X \oplus Y)_\infty$. (We regard $(X \oplus Y)_\infty$ as the space of all pairs (f, g) with $f \in X$ and $g \in Y$).

Then we obtain

Proposition 3.4. *Let v be a standard weight on \mathbb{G} satisfying (\star) but not $(\star\star)$. Then there is a universal constant $d > 0$ such that for every $j > 0$ there exists an integer $m > j$, an isomorphism $T : A_m \rightarrow H_v(\mathbb{G})$ and a projection $P : H_v(\mathbb{G}) \rightarrow TA_m$ with $\|P\| \leq d$ and $\|T\| \cdot \|T^{-1}\| \leq d$.*

Proof. Put $v_n(w) = v(nw)$, $w \in \mathbb{G}$, $n \in \mathbb{N}$. Then $(S_n f)(w) = f(nw)$, $w \in \mathbb{G}$, defines an isometry between $H_v(\mathbb{G})$ and $H_{v_n}(\mathbb{G})$. Moreover, put

$$u_n(z) = v \left(n \frac{1 - |z|}{1 + |z|} i \right), \quad z \in \mathbb{D}.$$

Then u_n is a radial weight on \mathbb{D} . We consider again a_n with (2.6) and $m_{n,k}$ with (2.7). (\star) implies $a := \sup_n a_n < \infty$. Since v does not satisfy $(\star\star)$ we have

$$(3.3) \quad \sup_n \sup_k (m_{n,k} - m_{n,k-1}) = \infty$$

([1], Lemma 4.1.)

By [1], Propositions 3.1 and 3.2., there is a universal constant $c > 0$, depending only on a , an (into-)isomorphism $T_n : H_{u_n}(\mathbb{D}) \rightarrow H_{v_n}(\mathbb{G})$ and a projection $P_n : H_{v_n}(\mathbb{G}) \rightarrow T_n H_{u_n}(\mathbb{D})$ with $\|T_n\| \cdot \|T_n^{-1}\| \leq c$ and $\|P_n\| \leq c$.

Now let $j > 0$. By (3.3) we find n and k and integers $m_{n,k-1} < n_1 < n_2 < n_2 + 2 < n_3 < n_4 < m_{n,k}$ such that $m := \min(2^{n_2 - n_1 - 1}, 2^{n_4 - n_3 - 1}) > j$. Using Proposition 2.2. with u_n and Lemmas 3.2. and 3.3. we see that there is an isomorphism $i : A_m \rightarrow H_{u_n}(\mathbb{D})$ and a projection $Q : (R_{n_4} - R_{n_3} + R_{n_2} - R_{n_1})H_{u_n}(\mathbb{D}) \rightarrow i(A_m)$ with $\|Q\| \leq 2$ satisfying (3.2). We have $\|i\| \cdot \|i^{-1}\| \leq 96b$. (3.1) and (3.2) imply that $Q(R_{n_4} - R_{n_3} + R_{n_2} - R_{n_1})$ is a projection from $H_{u_n}(\mathbb{D})$ onto $i(A_m)$. Put $T = S_n^{-1} T_n i$ and $P = S_n^{-1} T_n Q (R_{n_4} - R_{n_3} + R_{n_2} - R_{n_1}) T_n^{-1} P_n S_n$. Then P is a projection from $H_v(\mathbb{G})$ onto TA_m and we have $\|T\| \cdot \|T^{-1}\| \leq 96bc$, $\|P\| \leq 24 \cdot 96bc^2$. ■

Now, for a standard weight v on \mathbb{G} we introduce

$$(H_v)_0(\mathbb{G}) = \{f \in H_v(\mathbb{G}) : |f(w)|v(w) \text{ vanishes at infinity}\}.$$

(Here $|f(w)|v(w)$ vanishes at infinity if for every $\epsilon > 0$ there is a compact subset $K \subset \mathbb{G}$ such that $|f(w)|v(w) \leq \epsilon$ for $w \in \mathbb{G} \setminus K$.)

It is well known ([2, 3]) that $H_v(\mathbb{G})$ is isometrically isomorphic to $(H_v)_0(\mathbb{G})^{**}$ and the canonical embedding of $(H_v)_0(\mathbb{G})$ into $(H_v)_0(\mathbb{G})^{**}$ corresponds to the embedding of $(H_v)_0(\mathbb{G})$ into $H_v(\mathbb{G})$.

Lemma 3.5. *Let $E \subset H_v(\mathbb{G})$ be a finite dimensional subspace and $P : H_v(\mathbb{G}) \rightarrow E$ a projection. Then, for every $\epsilon > 0$, there is an isomorphism $T : E \rightarrow (H_v)_0(\mathbb{G})$ and a projection $Q : (H_v)_0(\mathbb{G}) \rightarrow TE$ with $\|Q\| \leq (1+\epsilon)\|P\|$ and $\|T\| \cdot \|T^{-1}\| \leq 1 + \epsilon$.*

Proof. Put $X = (H_v)_0(\mathbb{G})$ and identify X^{**} with $H_v(\mathbb{G})$. Let $i : X \rightarrow X^{**}$ be the canonical embedding. Then $i^* : X^{***} \rightarrow X^*$ is the map with $(i^*x^{***})(x) = x^{***}(ix)$, $x^{***} \in X^{***}$, $x \in X$. Put $F = i^*P^*X^{***}$. By the principle of local reflexivity ([4], p.53,) we find $T : E \rightarrow X$ with $\|T\| \cdot \|T^{-1}\| \leq 1 + \epsilon$ and $e(f) = f(Te)$ for $e \in E$ and $f \in F$. Put $Q = TPi$. Let $x^* \in X^*$ and $x \in X$. Then we obtain $(i^*P^*T^*)(x^*) \in F$ and hence $x^*(TPiTPix) = x^*(TPix)$. This follows since we have $P^{**}i^{**}P = P$ taking into account that $\dim E < \infty$. Therefore, Q is the desired projection. ■

Lemma 3.6. *Let $B_j \subset (H_v)_0(\mathbb{G})$, $j = 1, 2$, be two finite dimensional subspaces and assume that $P_j : (H_v)_0(\mathbb{G}) \rightarrow B_j$, $j = 1, 2$, are bounded and linear and P_2 is a projection. Then, for every $\epsilon > 0$, there is an isometry $T : B_2 \rightarrow (H_v)_0(\mathbb{G})$ and a linear map $Q : (H_v)_0(\mathbb{G}) \rightarrow B_1 + TB_2$ such that*

$$(1-\epsilon) \max(\|f\|_v, \|g\|_v) \leq \|f+Tg\|_v \leq (1+\epsilon) \max(\|f\|_v, \|g\|_v), \quad f \in B_1, g \in B_2,$$

$$\|Q\| \leq (1 + \epsilon) \max(\|P_1\|, \|P_2\|) \text{ and}$$

$$\|(Q-id)|_{(B_1+TB_2)}\| \leq (1+\epsilon)\|(P_1-id)|_{B_1}\| + \epsilon \max(\|P_1\|, \|P_2\|), \quad f \in B_1, g \in B_2.$$

Proof. Fix $0 < \epsilon' < 1$. Since $B_j \subset (H_v)_0(\mathbb{G})$ are finite dimensional we find compact subsets $K_j \subset \mathbb{G}$ such that $|f(w)|v(w) \leq \epsilon' \|f\|_v$ for all $w \in \mathbb{G} \setminus K_j$ and $f \in B_j$, $j = 1, 2$. For any $x \in \mathbb{R}$ and $f \in H_v(\mathbb{G})$ put $(T_x f)(w) = f(x+w)$, $w \in \mathbb{G}$. Since v is a standard weight, T_x is an isometry $(H_v)_0(\mathbb{G}) \rightarrow (H_v)_0(\mathbb{G})$. Let $\varphi \in (H_v)_0(\mathbb{G})^*$. In view of the Riesz representation theorem there is a regular Borel measure μ on \mathbb{G} with $|\mu|(\mathbb{G}) < \infty$ such that

$$\varphi(f) = \int_{\mathbb{G}} f(w)v(w)d\mu(w).$$

Hence $\lim_{x \rightarrow \pm\infty} \varphi(T_x f) = 0$. Since $P_j^*(H_v)_0(\mathbb{G})^*$ are finite dimensional we find $x \in \mathbb{R}$ so large that $K_1 \cap (x + K_2) = \emptyset = K_2 \cap (-x + K_1)$ and $|\varphi_1(T_x g)| \leq \epsilon' \|g\|_v$ for $g \in B_2$, $\varphi_1 \in P_1^*(H_v)_0(\mathbb{G})^*$ and $|\varphi_2(T_{-x} f)| \leq \epsilon' \|f\|_v$ for $f \in B_1$, $\varphi_2 \in P_2^*(H_v)_0(\mathbb{G})^*$. Hence $\|P_1 T_x g\|_v \leq \epsilon' \|g\|_v$ for $g \in B_2$ and $\|P_2 T_{-x} f\|_v \leq \epsilon' \|f\|_v$ for $f \in B_1$.

Put $T = T_x|_{B_2}$ and $Q = P_1 + T_x P_2 T_{-x}$. Then we obtain

$$(1 - \epsilon') \max(\|f\|_v, \|g\|_v) \leq \|f + Tg\|_v \leq (1 + \epsilon') \max(\|f\|_v, \|g\|_v)$$

for $f \in B_1$ and $g \in B_2$. Using this we see that

$$\|Q\| \leq \left(\frac{1 + \epsilon'}{1 - \epsilon'} \right) \max(\|P_1\|, \|P_2\|).$$

Finally, we have, for $f \in B_1$ and $g \in B_2$,

$$\begin{aligned} \|Q(f + T_x g) - (f + T_x g)\|_v &= \|P_1 f - f + P_1 T_x g + T_x P_2 T_{-x} f\|_v \leq \\ &\leq (\|(P_1 - id)|_{B_1}\| + 2\epsilon' \max(\|P_1\|, \|P_2\|)) \cdot \\ &\quad \cdot \max(\|f\|_v, \|g\|_v) \leq \\ &\leq (\|(P_1 - id)|_{B_1}\| + 2\epsilon' \max(\|P_1\|, \|P_2\|)) \cdot \\ &\quad \cdot \frac{\|f + T_x g\|_v}{1 - \epsilon'}. \end{aligned}$$

If ϵ' is small enough we obtain the estimates of the assertion of Lemma 3.6. ■

Proof of Theorem 3.1. Use Proposition 3.4. and Lemma 3.5. to find integers $0 < k_1 < k_2 < \dots$, isomorphisms $T_n : A_{k_n} \rightarrow (H_v)_0(\mathbb{G})$ and projections $P_n : (H_v)_0(\mathbb{G}) \rightarrow T_n A_{k_n}$ with $\|T_n\| \cdot \|T_n^{-1}\| \leq d$ and $\|P_n\| \leq d$ where $d > 0$ is a universal constant.

Then use Lemma 3.6. and induction to find an isomorphic copy $X \subset (H_v)_0(\mathbb{G})$ of $(\sum_{n=1}^{\infty} \oplus A_{k_n})_{(0)}$ and a linear bounded map $Q : (H_v)_0(\mathbb{G}) \rightarrow X$ with $\|(Q - id)|_X\| < 1$. (Apply Lemma 3.6. successively with ϵ_n small enough such that in particular $0 < \prod_{n=1}^{\infty} (1 - \epsilon_n) < \prod_{n=1}^{\infty} (1 + \epsilon_n) < \infty$.)

Hence $S := (Q|_X)^{-1}$ exists and is bounded. Put $P = SQ$. Then P is a bounded projection from $(H_v)_0(\mathbb{G})$ onto X . With biduality we see that $P^{**} : H_v(\mathbb{G}) \rightarrow X^{**}$ is a bounded projection from $H_v(\mathbb{G})$ onto an isomorphic copy of $(\sum_{n=1}^{\infty} \oplus A_{k_n})_{(\infty)}$. This space is isomorphic to $(\sum_{n=1}^{\infty} \oplus A_n)_{(\infty)}$ and hence to H_{∞} ([6]). ■

References

- [1] **Ardalani, M.A. and W. Lusky**, Weighted spaces of holomorphic functions on the upper half-plane, *Math. Scand.*, **111** (2012), 244–260.
- [2] **Bierstedt, K.D. and W.H. Summers**, Biduals of weighted Banach spaces of analytic functions, *J. Austral. Math. Soc. Sec A*, **34** (1993), 70–79.
- [3] **Holtmanns, S.**, *Operator Representation and Biduals of Weighted Function Spaces*, PhD thesis, Inst. for Math., Univ. of Paderborn, 2000.
- [4] **Johnson, W.B. and J. Lindenstrauss**, Basic concepts in the geometry of Banach spaces, in: *Handbook of the Geometry of Banach Spaces*, Elsevier, 2001, 1–84.

- [5] **Lusky, W.**, On weighted spaces of harmonic and holomorphic functions, *J. Lond. Math. Soc.*, (2) **51** (1995), 309–320.
- [6] **Lusky, W.**, On the isomorphic classification of weighted spaces of holomorphic functions, *Acta Universitatis Carolinae, Mat. et Phys.*, **41** (2000), 51–60.
- [7] **Lusky, W.**, On the isomorphism classes of weighted spaces of harmonic and holomorphic functions, *Studia Math.*, **175** (2006), 19–45.
- [8] **Wojtaszczyk, P.**, On projections in spaces of bounded analytic functions with applications, *Studia Math.*, **65** (1979), 147–173.

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