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CONVERGENCE RATES IN PRECISE ASYMPTOTICS II

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Dedicated to Karl-Heinz Indlekofer on the occasion of his 70th birthday

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Abstract. Let $X_1, X_2, \dots$ be independent, identically distributed (i.i.d.) random variables with partial sums S_n, $n \geq 1$. The now classical Baum–Katz problem concerns finding necessary and sufficient moment conditions for the convergence of $\sum_{n=1}^{\infty} n^{(r/p)-2} P(|S_n| \geq \varepsilon n^{1/p})$ for *fixed* $\varepsilon > 0$. A now equally classical paper by Heyde in 1975 initiated what has later been called precise asymptotics, namely asymptotics for the same sum (for the case $r = 2$ and $p = 1$) when, instead, $\varepsilon \searrow 0$. In a predecessor of this paper we extended a result due to Klesov (1994), in which he determined the convergence rate in Heyde's theorem, to the case $r \geq 2$, $0 < p < 2$. The present companion paper is devoted to the case when the summands belong to the normal domain of attraction of a stable distribution with index $\alpha \in (1, 2]$, in particular to the analog related to Spitzer's 1956-theorem.

1. Introduction

The point of departure of this note is the following part of the main result in Baum and Katz [1].

Key words and phrases: Law of large numbers, Baum–Katz, precise asymptotics, convergence rates.

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Theorem 1.1. *Let $r > 0$, $0 < p < 2$ and $r \geq p$. Suppose that $X, X_1, X_2, \ldots$ are i.i.d. random variables with $E|X|^r < \infty$ and, if $r \geq 1$, $EX = 0$, and set $S_n = \sum_{k=1}^n X_k$, $n \geq 1$. Then*

$$\sum_{n=1}^{\infty} n^{(r/p)-2} P(|S_n| \geq \varepsilon n^{1/p}) < \infty \quad \text{for all} \quad \varepsilon > 0. \tag{1.1}$$

Conversely, if the sum is finite for some $\varepsilon > 0$, then $E|X|^r < \infty$ and, if $r \geq 1$, $EX = 0$. In particular, the conclusion then holds for all $\varepsilon > 0$.

One problem of interest is to examine the rate at which the above probabilities tend to one as $\varepsilon \searrow 0$, more precisely, to find some normalizing function of ε for which the sum in Theorem 1.1, premultiplied by this very function, has a nondegenerate limit as $\varepsilon \searrow 0$. Toward that end, Heyde [9] proved that

$$\lim_{\varepsilon \searrow 0} \varepsilon^2 \sum_{n=1}^{\infty} P(|S_n| \geq \varepsilon n) = EX^2, \tag{1.2}$$

whenever $EX = 0$ and $EX^2 < \infty$. Remaining values of r and p have later been taken care of in [2, 11, 4], and have been coined under the heading "Precise asymptotics for ...".

The following theorem, due du Klesov [10], provides information about the *rate of convergence* in Heyde's result (1.2).

Theorem 1.2. *Let $X, X_1, X_2, \ldots$ be i.i.d. random variables, and set $S_n = \sum_{k=1}^n X_k$, $n \geq 1$.*

(a) *If X is normal with mean 0 and variance $\sigma^2 > 0$, then*

$$\lim_{\varepsilon \searrow 0} \Big(\sum_{n=1}^{\infty} P(|S_n| \geq \varepsilon n) - \frac{\sigma^2}{\varepsilon^2} \Big) = -\frac{1}{2}.$$

(b) *If $EX = 0$, $EX^2 = \sigma^2 > 0$, and $E|X|^3 < \infty$, then*

$$\lim_{\varepsilon \searrow 0} \varepsilon^{3/2} \Big(\sum_{n=1}^{\infty} P(|S_n| \geq \varepsilon n) - \frac{\sigma^2}{\varepsilon^2} \Big) = 0.$$

In [5] we extended Klesov's theorem to the case $r \geq 2$, $0 < p < 2$ as follows.

Theorem 1.3. *Let $r \geq 2$ and $0 < p < 2$. Suppose that $X, X_1, X_2, \ldots$ are i.i.d. random variables, and set $S_n = \sum_{k=1}^n X_k$, $n \geq 1$. Let Y be normal with mean 0 and variance $\sigma^2 > 0$.*

(a) *If* $E\,X = 0$, $E\,X^2 = \sigma^2 > 0$, *and* $E|X|^q < \infty$ *for some* $r < q \le 3$, *then*

$$\lim_{\varepsilon\searrow 0} \varepsilon^{q(r-p)/(q-p)} \Big(\sum_{n=1}^{\infty} n^{(r/p)-2} P(|S_n| \ge \varepsilon n^{1/p}) - \\ - \frac{p}{r-p} \varepsilon^{-2(r-p)/(2-p)} E|Y|^{2(r-p)/(2-p)} \Big) = 0.$$

(b) *If* $E\,X = 0$, $E\,X^2 = \sigma^2 > 0$, *and* $E|X|^q < \infty$ *for some* $q \ge 3$ *with* $q > (2r-3p)/(2-p)$, *then*

$$\lim_{\varepsilon\searrow 0} \varepsilon^{2q(r-p)/(p+q(2-p))} \Big(\sum_{n=1}^{\infty} n^{(r/p)-2} P(|S_n| \ge \varepsilon n^{1/p}) - \\ - \frac{p}{r-p} \varepsilon^{-2(r-p)/(2-p)} E|Y|^{2(r-p)/(2-p)} \Big) = 0.$$

The rate results so far all assume finite variance, that is, it remains to investigate the case when the variance is not necessarily finite. The following two results are the analogs of Heyde's result (1.2) for the case when the summands belong to the normal domain of attraction of a stable distribution with index $\alpha \in (1, 2]$; cf. [4, 11]. The first one is, in particular, related to Spitzer's theorem [12], in which he treats the case $p = 1$.

Theorem 1.4. *Suppose that* $X, X_1, X_2, \ldots$ *are i.i.d. random variables with mean* 0 *that belong to the normal domain of attraction of a nondegenerate stable law* G *with index* $\alpha \in (1, 2]$, *and set* $S_n = \sum_{k=1}^{n} X_k$, $n \ge 1$. *Then, for* $1 \le p < \alpha \le 2$,

$$\lim_{\varepsilon\searrow 0} \frac{1}{-\log\varepsilon} \sum_{n=1}^{\infty} \frac{1}{n} P(|S_n| \ge \varepsilon n^{1/p}) = \frac{\alpha p}{\alpha - p}. \tag{1.3}$$

In particular, if $\operatorname{Var} X = \sigma^2 < \infty$, *the limit exists and equals* $2p/(2-p)$.

Theorem 1.5. *Suppose that* $X, X_1, X_2, \ldots$ *are i.i.d. random variables with mean* 0 *that belong to the normal domain of attraction of a nondegenerate stable law* G *with index* $\alpha \in (1, 2]$, *and set* $S_n = \sum_{k=1}^{n} X_k$, $n \ge 1$. *Then, for* $1 \le p < r < \alpha$,

$$\lim_{\varepsilon\searrow 0} \varepsilon^{\alpha(r-p)/(\alpha-p)} \sum_{n=1}^{\infty} n^{(r/p)-2} P(|S_n| \ge \varepsilon n^{1/p}) = \\ = \frac{p}{r-p} \cdot E|Y|^{\alpha(r-p)/(\alpha-p)}, \tag{1.4}$$

where Y *is a random variable with distribution* G.

In the present, as the title suggests, companion of [5] we provide Klesov type results related to Theorems 1.4 and 1.5.

We remark that there exist an abundance of results on precise asymptotics *without rates* of different levels of generalization with proofs whose skeletons are very much the same; see [6] for more on this. In the same vein one can, of course, extend our present results and those of [5] to more general cases.

2. Results

The first aim of the present paper is to prove a convergence rate result with respect to Theorem 1.4. For $p = 1$ this corresponds to the Spitzer case; [12].

Theorem 2.1. *Let $1 \le p < \alpha \le 2$, suppose that X, X_1, X_2, ... are i.i.d. random variables with mean 0 and distribution function F belonging the normal domain of attraction of a nondegenerate stable law G with index $\alpha \in (1, 2]$, and set $S_n = \sum_{k=1}^{n} X_k$, $n \ge 1$.*

(a) *If $\alpha = 2$, $\mathrm{Var}\,(X) = \sigma^2$, and $EX^2 \log(1 + |X|) < \infty$, then*

$$\begin{aligned}\lim_{\varepsilon \searrow 0} \Big(\sum_{n=1}^{\infty} \frac{1}{n} P(|S_n| \ge \varepsilon n^{1/p}) + \frac{2p}{2-p} \log \varepsilon \Big) &= \\ = \frac{2(1-p)}{2-p}\gamma + \frac{p}{2-p} \log\Big(\frac{\sigma^2}{2}\Big) - \varrho,&\end{aligned} \tag{2.1}$$

where γ is Euler's constant $(= 0.57721\dots)$ and $\varrho = \sum_{n=1}^{\infty} n^{-1} P(S_n = 0)$.

(b) *If $1 < \alpha < 2$ and $\int_{-\infty}^{\infty} |x|^{\alpha-1} |F(x) - G(x)|\,dx < \infty$, then*

$$\begin{aligned}\lim_{\varepsilon \searrow 0} \Big(\sum_{n=1}^{\infty} \frac{1}{n} P(|S_n| \ge \varepsilon n^{1/p}) + \frac{\alpha p}{\alpha - p} \log \varepsilon \Big) &= \\ = \frac{\alpha p}{\alpha - p} \cdot E \log|Y| + \gamma - \varrho,&\end{aligned} \tag{2.2}$$

where γ and ϱ are as defined above and Y is Stable(α)-distributed with mean 0.

Remark 2.1. (i) It is well-known that ϱ as given in Theorem 2.1 is finite, since the first moment exists (and $EX = 0$); cf. Spitzer [13], Corollary 3.3.

(ii) Note also that $E \log|Y| < \infty$ in (2.2), since $E|Y|^\beta < \infty$ for all $0 < \beta < \alpha$, the density of Y is bounded on $[0, 1]$, and $\int_0^1 \log y\,dy$ is finite.

The procedure in the present paper is the same as that of [5], that is, the proof of Theorem 2.1 is based on the following propositions concerning the Gaussian and the stable laws, respectively, and a Berry–Esseen remainder term type argument.

Proposition 2.1. *Let $0 < p < 2$, suppose that $Y, X_1, X_2, \ldots$ are i.i.d. standard normal random variables, and set $S_n = \sum_{k=1}^n X_k$, $n \geq 1$. Then*

$$\lim_{\varepsilon \searrow 0} \Big(\sum_{n=1}^{\infty} \frac{1}{n} P(|S_n| \geq \varepsilon n^{1/p}) + \frac{2p}{2-p} \log \varepsilon \Big) = \frac{2(1-p)}{2-p} \gamma - \frac{p}{2-p} \log 2.$$

Proposition 2.2. *Let $1 \leq p < \alpha \leq 2$, suppose that $Y, X_1, X_2, \ldots$ are i.i.d. Stable(α)-distributed random variables with mean 0, and set $S_n = \sum_{k=1}^n X_k$, $n \geq 1$. Then*

$$\lim_{\varepsilon \searrow 0} \Big(\sum_{n=1}^{\infty} \frac{1}{n} P(|S_n| \geq \varepsilon n^{1/p}) + \frac{\alpha p}{\alpha - p} \log \varepsilon \Big) = \frac{\alpha p}{\alpha - p} \cdot E \log |Y| + \gamma.$$

For the next statement, which is related to Theorem 1.5, we need the following kind of Eulerian constant, the existence of which we shall return to ahead; recall the ordinary Euler constant $\gamma = \sum_{k=1}^n \frac{1}{n} - \log n = 0.57721\ldots$ above.

Definition 2.1. Let $-1 < \theta < 0$. The constant γ_θ is defined via the relation

$$\gamma_\theta = \lim_{n \to \infty} \Big(\frac{n^{\theta+1}}{\theta+1} - \sum_{j=1}^{n} j^\theta \Big).$$

Then, with arguments similar to those of the proof of Theorem 2.1, we also obtain the following convergence rate result.

Theorem 2.2. *Let $1 \leq p < r < \alpha$, suppose that $X, X_1, X_2, \ldots$ are i.i.d. random variables with mean 0 and distribution function F belonging to the normal domain of attraction of a nondegenerate stable distribution (function) G with index $\alpha \in (1, 2]$, and set $S_n = \sum_{k=1}^n X_k$, $n \geq 1$.*

(a) *If $\alpha = 2$, $\mathrm{Var}\,(X) = \sigma^2$, and $E|X|^{2r/p} < \infty$, then*

$$\begin{aligned} &\lim_{\varepsilon \searrow 0} \Big(\sum_{n=1}^{\infty} n^{(r/p)-2} P(|S_n| \geq \varepsilon n^{1/p}) - \\ &- \frac{p}{r-p} \cdot \varepsilon^{-2(r-p)/(2-p)} E|Y|^{2(r-p)/(2-p)} \Big) = \gamma_{(r/p)-2} - \varrho_{r,p}\,, \end{aligned} \tag{2.3}$$

where Y is $N(0, \sigma^2)$-distributed and $\varrho_{r,p} = \sum_{n=1}^{\infty} n^{(r/p)-2} P(S_n = 0)$.

(b) *If* $1 < \alpha < 2$ *and* $\int_{-\infty}^{\infty} |x|^{(\alpha r/p)-1} |F(x) - G(x)|\, dx < \infty$, *then*

$$\lim_{\varepsilon \searrow 0} \Big(\sum_{n=1}^{\infty} n^{(r/p)-2} P(|S_n| \geq \varepsilon n^{1/p}) - \tag{2.4}$$
$$- \frac{p}{r-p} \cdot \varepsilon^{-\alpha(r-p)/(\alpha-p)} E|Y|^{\alpha(r-p)/(\alpha-p)} \Big) = \gamma_{(r/p)-2} - \varrho_{r,p}\,,$$

where $\varrho_{r,p}$ *is as defined above and* Y *is Stable*(α)*-distributed with mean* 0.

Remark 2.2. It can easily be verified that $\varrho_{r,p}$ in Theorem 2.2 is finite under the given assumptions.

Namely, set

$$\Delta_n = \sup_y \big| P(|S_n| \geq n^{1/\alpha} y) - P(|Y| \geq y) \big|$$

and note that, if $\alpha = 2$ and $E|X|^{2r/p} < \infty$, then $\sum_{n=1}^{\infty} n^{(r/p)-2}\, \Delta_n < \infty$ (cf. Heyde [8], Theorem, p. 12, with $\delta = (2r/p) - 2$). Moreover, if $1 < \alpha < 2$ and $\int_{-\infty}^{\infty} |x|^{(\alpha r/p)-1} |F(x) - G(x)|\, dx < \infty$, the latter sum also converges (see Hall [7], pp. 351-352, with $\beta = \alpha r/p$).

Now, let $x > 0$ be fixed and $\kappa > 0$ to be chosen below. Then

$$\begin{aligned}
\varrho_{r,p} &= \sum_{n=1}^{\infty} n^{(r/p)-2} P(S_n = 0) \;\leq\; \sum_{n=1}^{\infty} n^{(r/p)-2} P(|S_n| < x n^{(1/\alpha)-\kappa}) \leq \\
&\leq \sum_{n=1}^{\infty} n^{(r/p)-2} \Big| P(|S_n|/n^{1/\alpha} < x/n^{\kappa}) - P(|Y| < x/n^{\kappa}) \Big| + \\
&\qquad + \sum_{n=1}^{\infty} n^{(r/p)-2} P(|Y| < x/n^{\kappa}) \leq \\
&\leq \sum_{n=1}^{\infty} n^{(r/p)-2}\, \Delta_n + C \sum_{n=1}^{\infty} n^{(r/p)-2-\kappa},
\end{aligned}$$

where C is a positive constant. Then the first sum is finite (in view of Hall [7] and Heyde [8]), and the second sum is finite if $\kappa > (r/p) - 1$, which proves the finiteness of $\varrho_{r,p}$.

The proof of Theorem 2.2 is based on the following propositions, again concerning the Gaussian and the stable laws, respectively.

Proposition 2.3. *Let* $1 \leq p < r < 2$, *and suppose that* Y, X_1, X_2, $\dots$ *are i.i.d.* $N(0, \sigma^2)$*-distributed random variables with* $\sigma^2 > 0$, *and set*

$$S_n = \sum_{k=1}^{n} X_k, \quad n \geq 1.$$

Then

$$\lim_{\varepsilon\searrow 0}\Big(\sum_{n=1}^{\infty} n^{(r/p)-2}P(|S_n|\ge \varepsilon n^{1/p})-$$
$$-\frac{p}{r-p}\cdot \varepsilon^{-2(r-p)/(2-p)}E|Y|^{2(r-p)/(2-p)}\Big)=\gamma_{(r/p)-2}\,.$$

Proposition 2.4. *Let* $1\le p<r<\alpha<2$*, suppose that* $Y, X_1, X_2, \ldots$ *are i.i.d. Stable*(α)*-distributed random variables with mean* 0*, and set* $S_n = \sum_{k=1}^{n} X_k$*,* $n\ge 1$*. Then*

$$\lim_{\varepsilon\searrow 0}\Big(\sum_{n=1}^{\infty} n^{(r/p)-2}P(|S_n|\ge \varepsilon n^{1/p})-$$
$$-\frac{p}{r-p}\cdot \varepsilon^{-\alpha(r-p)/(\alpha-p)}E|Y|^{\alpha(r-p)/(\alpha-p)}\Big)=\gamma_{(r/p)-2}\,.$$

3. Proofs

Proof of Proposition 2.1. Let $p\in(0,2)$ and Y be a standard normal random variable. Then,

$$\lambda_p(\varepsilon)=\sum_{n=1}^{\infty}\frac{1}{n}P(|S_n|\ge\varepsilon n^{1/p})=\sum_{n=1}^{\infty}\frac{1}{n}P(|Y|\ge\varepsilon n^{(2-p)/(2p)})=$$
$$=\sum_{j=1}^{\infty}\frac{1}{j}\sqrt{\frac{2}{\pi}}\int_{\varepsilon j^{(2-p)/(2p)}}^{\infty}\exp\{-y^2/2\}\,dy=$$
$$=\sqrt{\frac{2}{\pi}}\sum_{j=1}^{\infty}\frac{1}{j}\sum_{n=j}^{\infty}\int_{\varepsilon n^{(2-p)/(2p)}}^{\varepsilon(n+1)^{(2-p)/(2p)}}\exp\{-y^2/2\}\,dy=$$
$$=\sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}\Big(\sum_{j=1}^{n}\frac{1}{j}\Big)\int_{\varepsilon n^{(2-p)/(2p)}}^{\varepsilon(n+1)^{(2-p)/(2p)}}\exp\{-y^2/2\}\,dy=$$
$$=\sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}(\log n+\gamma_n)\int_{\varepsilon n^{(2-p)/(2p)}}^{\varepsilon(n+1)^{(2-p)/2p}}\exp\{-y^2/2\}\,dy=$$

$$
\begin{aligned}
&= \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} (\log n + \gamma) \int\limits_{\varepsilon n^{(2-p)/(2p)}}^{\varepsilon (n+1)^{(2-p)/(2p)}} \exp\{-y^2/2\}\, dy = \\
&\qquad + \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} (\gamma_n - \gamma) \int\limits_{\varepsilon n^{(2-p)/(2p)}}^{\varepsilon (n+1)^{(2-p)/(2p)}} \exp\{-y^2/2\}\, dy = \\
&= \Sigma_p(\varepsilon) + R_p(\varepsilon)\,.
\end{aligned}
\tag{3.1}
$$

Now, let $\delta \in (0,1)$ be small and choose n_0 such that $|\gamma_n - \gamma| < \delta$ for $n > n_0$. By splitting the sum at n_0 into two parts and noticing that $0 < \gamma_n, \gamma \le 1$, we first observe that

$$
\begin{aligned}
\limsup_{\varepsilon \searrow 0} |R_p(\varepsilon)| \le \limsup_{\varepsilon \searrow 0} \Big(& P(\varepsilon \le |Y| < \varepsilon (n_0+1)^{(\alpha-p)/(\alpha p)}) + \\
& + \delta P(|Y| \ge \varepsilon (n_0+1)^{(\alpha-p)/(\alpha p)}) \Big) \le \delta.
\end{aligned}
\tag{3.2}
$$

Next we note that, as $\varepsilon \searrow 0$,

$$
\begin{aligned}
\Sigma_p(\varepsilon) &= \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} \log n \cdot \exp\{-\varepsilon^2 n^{(2-p)/p}/2\} \cdot \\
&\qquad \cdot \big(\varepsilon (n+1)^{(2-p)/(2p)} - \varepsilon n^{(2-p)/(2p)}\big) + \gamma + \mathcal{O}(\varepsilon) = \\
&= \frac{\varepsilon}{\sqrt{2\pi}} \cdot \frac{2-p}{p} \sum_{n=1}^{\infty} \log n \cdot n^{(2-3p)/(2p)} \big(1 + \mathcal{O}(1/n)\big) \cdot \\
&\qquad \cdot \exp\{-\varepsilon^2 n^{(2-p)/p}/2\} + \gamma + \mathcal{O}(\varepsilon)\,,
\end{aligned}
\tag{3.3}
$$

where the constant in the $\mathcal{O}(1/n)$-term is independent of ε. Furthermore, via a change of variable,

$$
\begin{aligned}
&\frac{\varepsilon}{\sqrt{2\pi}} \cdot \frac{2-p}{p} \int\limits_0^{\infty} \log x \cdot x^{(2-3p)/(2p)} \cdot \exp\{-\varepsilon^2 x^{(2-p)/p}/2\}\, dx = \\
&= \frac{1}{\sqrt{\pi}} \cdot \frac{p}{2-p} \int\limits_0^{\infty} \frac{\log y + \log 2 - 2\log\varepsilon}{\sqrt{y}} \cdot e^{-y}\, dy = \\
&= \frac{\log 2 - 2\log\varepsilon}{\sqrt{\pi}} \cdot \frac{p}{2-p} \cdot \Gamma(1/2) + \frac{1}{\sqrt{\pi}} \cdot \frac{p}{2-p} \cdot \int\limits_0^{\infty} \frac{\log y}{\sqrt{y}} \cdot e^{-y}\, dy = \\
&= -\frac{2p}{2-p} \log\varepsilon + \frac{p}{2-p} \log 2 + \frac{1}{\sqrt{\pi}} \cdot \frac{p}{2-p} \cdot \int\limits_0^{\infty} \frac{\log y}{\sqrt{y}} \cdot e^{-y}\, dy,
\end{aligned}
\tag{3.4}
$$

so that, upon recalling that

$$\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{\log y}{\sqrt{y}}\cdot e^{-y}\,dy = \frac{\Gamma'(1/2)}{\Gamma(1/2)} = -\gamma - 2\log 2,$$

and by combining the above numbered relations, we finally obtain that

$$\begin{aligned}\lim_{\varepsilon\searrow 0}\Big(\lambda_p(\varepsilon) + \frac{2p}{2-p}\log\varepsilon\Big) &= \frac{p}{2-p}\Big(\log 2 - 2\log 2\Big) + \gamma\Big(1 - \frac{p}{2-p}\Big) = \\ &= \gamma\frac{2(1-p)}{2-p} - \frac{p}{2-p}\log 2\,,\end{aligned}$$

in view of the arbitrariness of δ.

It remains to justify that the sum in (3.3) can be approximated by the integral in (3.4). Toward that end, note that in the sum we can always replace $\log n$ by $\log(n+1)$ and $n^{(2-3p)/(2p)}$ by $(n+1)^{(2-3p)/(2p)}$ (in view of the $\mathcal{O}(1/n)$-term) and thus get either a lower or an upper bound.

Moreover, the remainder term is negligible, i.e.,

$$\begin{aligned}&\lim_{\varepsilon\searrow 0}\ \varepsilon\sum_{n=1}^\infty \log n\cdot n^{(2-3p)/(2p)}\cdot\frac{1}{n}\cdot\exp\{-\varepsilon^2 n^{(2-p)/p}/2\} = \\ &= \lim_{\varepsilon\searrow 0}\ \sum_{n=1}^\infty \frac{\log n}{n^2}\big(\varepsilon^2 n^{(2-p)/p}\big)^{1/2}\exp\{-\varepsilon^2 n^{(2-p)/p}/2\} = 0,\end{aligned}$$

by dominated convergence. ∎

Proof of Proposition 2.2. The proof follows the basic lines of the previous one.

Let $Y \in \text{Stable}\,(\alpha)$, set $\Psi(y) = P(|Y| \geq y)$, $y > 0$, and note that $P(|Y|\geq y) = -\int_y^\infty d\Psi(x)$. Now,

$$\begin{aligned}\lambda_{\alpha,p}(\varepsilon) &= \sum_{n=1}^\infty \frac{1}{n}P(|S_n|\geq\varepsilon n^{1/p}) = \sum_{n=1}^\infty\frac{1}{n}P\big(|Y|\geq\varepsilon n^{(\alpha-p)/(\alpha p)}\big) = \\ &= -\sum_{j=1}^\infty\frac{1}{j}\int_{\varepsilon j^{(\alpha-p)/(\alpha p)}}^\infty d\Psi(y) = -\sum_{j=1}^\infty\frac{1}{j}\sum_{n=j}^\infty\int_{\varepsilon n^{(\alpha-p)/(\alpha p)}}^{\varepsilon(n+1)^{(\alpha-p)/(\alpha p)}} d\Psi(y) = \\ &= -\sum_{n=1}^\infty\Big(\sum_{j=1}^n\frac{1}{j}\Big)\int_{\varepsilon n^{(\alpha-p)/(\alpha p)}}^{\varepsilon(n+1)^{(\alpha-p)/(\alpha p)}} d\Psi(y) =\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&= -\sum_{n=1}^{\infty}(\log n + \gamma_n) \int\limits_{\varepsilon n^{(\alpha-p)/(\alpha p)}}^{\varepsilon(n+1)^{(\alpha-p)/(\alpha p)}} d\Psi(y) = \\
&= -\sum_{n=1}^{\infty}(\log n + \gamma) \int\limits_{\varepsilon n^{(\alpha-p)/(\alpha p)}}^{\varepsilon(n+1)^{(\alpha-p)/(\alpha p)}} d\Psi(y) - \\
&\qquad - \sum_{n=1}^{\infty}(\gamma_n - \gamma) \int\limits_{\varepsilon n^{(\alpha-p)/(\alpha p)}}^{\varepsilon(n+1)^{(\alpha-p)/(\alpha p)}} d\Psi(y) = \\
&= -\Sigma_{\alpha,p}(\varepsilon) - R_{\alpha,p}(\varepsilon),
\end{aligned} \tag{3.5}$$

where, again, γ is Euler's constant.

Once again, let $\delta \in (0,1)$ be small and choose n_0 such that $|\gamma_n - \gamma| < \delta$ for $n > n_0$. Then, by arguing as in (3.2), we first observe that

$$\limsup_{\varepsilon \searrow 0} |R_{\alpha,p}(\varepsilon)| \le \delta\,.$$

Next, noticing that

$$\log n = \frac{\alpha p}{\alpha - p}\Big(\log\big(\varepsilon n^{(\alpha-p)/(\alpha p)}\big) - \log\varepsilon\Big),$$

it follows that

$$\begin{aligned}
-\Sigma_{\alpha,p}(\varepsilon) &\le -\frac{\alpha p}{\alpha-p}\int\limits_{\varepsilon}^{\infty} \log y \; d\Psi(y) + \big(\frac{\alpha p}{\alpha - p}\cdot \log\varepsilon - \gamma\big)\cdot \int\limits_{\varepsilon}^{\infty} d\Psi(y) = \\
&= -\frac{\alpha p}{\alpha-p}\big(-E\log|Y| - \int\limits_{0}^{\varepsilon} \log y \; d\Psi(y)\big) + \\
&\qquad + \big(\frac{\alpha p}{\alpha - p}\cdot \log\varepsilon - \gamma\big)\cdot\big(-1 - \int\limits_{0}^{\varepsilon} d\Psi(y)\big) = \\
&= \big(-\frac{\alpha p}{\alpha-p}\cdot\log\varepsilon + \gamma\big)\cdot(1 + \mathcal{O}(\varepsilon)) + \frac{\alpha p}{\alpha-p}\cdot\big(E\log|Y| + \mathcal{O}(\varepsilon)\big) = \\
&= -\frac{\alpha p}{\alpha-p}\cdot\log\varepsilon + \gamma + \frac{\alpha p}{\alpha-p}\cdot E\log|Y| + \mathcal{O}(\varepsilon\log\varepsilon) \quad \text{as} \quad \varepsilon \searrow 0\,,
\end{aligned}$$

which, by combining as before, results in

$$\limsup_{\varepsilon \searrow 0}\Big(\lambda_{\alpha,p} + \frac{\alpha p}{\alpha-p}\cdot\log\varepsilon\Big) \le \frac{\alpha p}{\alpha-p}\cdot E\log|Y| + \gamma\,.$$

Upon observing that $\lambda_{\alpha,p}$ can similarly be represented as

$$\lambda_{\alpha,p} = -\sum_{n=1}^{\infty}\{\log(n+1)+\tilde{\gamma}_n\} \int\limits_{\varepsilon n^{(\alpha-p)/(\alpha p)}}^{\varepsilon(n+1)^{(\alpha-p)/(\alpha p)}} d\Psi(y),$$

where $\tilde{\gamma}_n = \gamma_n + \log n - \log(n+1) \to \gamma$ as $n \to \infty$, and by arguments analogous to those above, we obtain the same lower bound

$$\liminf_{\varepsilon \searrow 0} \Big(\lambda_{\alpha,p} + \frac{\alpha p}{\alpha - p}\cdot \log\varepsilon\Big) \geq \frac{\alpha p}{\alpha - p}\cdot E\log|Y| + \gamma\,,$$

which completes the proof. ■

Proof of Theorem 2.1. (a) Without loss of generality we can and will assume that $\sigma^2 = 1$. Set $\Delta_n = \sup_y \big|P(|S_n| \geq n^{1/2}y) - P(|Y| \geq y)\big|$ as before, and

$$\begin{aligned}\Delta_n(\varepsilon) &= P(|S_n| \geq n^{1/p}\varepsilon) - P(|Y| \geq n^{1/p-1/2}\varepsilon) = \\ &= P(|Y| < n^{1/p-1/2}\varepsilon) - P(|S_n| < n^{1/p}\varepsilon),\end{aligned} \tag{3.6}$$

where Y has a standard normal distribution.

Now, in view of Proposition 2.1 it suffices to show that

$$\lim_{\varepsilon \searrow 0} \sum_{n=1}^{\infty} \frac{1}{n}\,\Delta_n(\varepsilon) = -\varrho, \tag{3.7}$$

where ϱ is finite (recall Remark 2.1). For doing so, we make use of the fact that, if $EX^2\log(1+|X|) < \infty$, then $\sum_{n=1}^{\infty} n^{-1}\Delta_n < \infty$ (cf. [8], Theorem, p. 12; in fact, the two assertions are equivalent).

But then, by using the second equality in (3.6), relation (3.7) is an immediate consequence of Lebesgue's dominated convergence theorem in combination with the estimation $|\Delta_n(\varepsilon)| \leq \Delta_n$ and the continuity of the distribution of Y.
(b) The arguments for the proof of (2.2) are exactly the same as above. Just note that, under the given conditions, $\sum_{n=1}^{\infty} n^{-1}\Delta_n < \infty$ still is in force (cf. Hall [7], pp. 351-352, with $\beta = \alpha$), and make use of Proposition 2.2 instead of Proposition 2.1. ■

Proof of Proposition 2.3. For the proof we need the following auxiliary result taken from [5].

Lemma 3.1. *We have, as $n \to \infty$,*

$$\sum_{j=1}^{n} j^{\theta} = \begin{cases} \dfrac{n^{\theta+1}}{\theta+1} + \dfrac{n^{\theta}}{2} + \mathcal{O}\big(n^{\theta-1}\big), & \text{for} \quad \theta > 1, \\ \dfrac{n^2}{2} + \dfrac{n}{2}, & \text{for} \quad \theta = 1, \\ \dfrac{n^{\theta+1}}{\theta+1} + \dfrac{n^{\theta}}{2} + \mathcal{O}(1), & \text{for} \quad 0 < \theta < 1, \\ n, & \text{for} \quad \theta = 0, \\ \dfrac{n^{\theta+1}}{\theta+1} - \gamma_{\theta} + \mathcal{O}\big(n^{\theta}\big), & \text{for} \quad -1 < \theta < 0, \end{cases}$$

where, in the last case, $0 < -\frac{\theta}{\theta+1} < \gamma_{\theta} \le \frac{1}{\theta+1}$.

Proof. The even relations are trivial, the first and third ones are a consequence of the Euler–MacLaurin sum formula (cf. [3], p. 124). As for the last one, set

$$D_n = \int_0^n x^{\theta}\, dx - \sum_{j=1}^{n} j^{\theta} = \frac{n^{\theta+1}}{\theta+1} - \sum_{j=1}^{n} j^{\theta}.$$

Then, since $D_{n+1} - D_n > 0$, it follows that $\{D_n,\, n \ge 1\}$ is increasing, and, in particular, that

$$D_n > D_1 = \int_0^1 x^{\theta}\, dx - 1 = -\frac{\theta}{\theta+1} > 0.$$

Moreover, the sequence is bounded, since

$$\begin{aligned} D_n &= \sum_{j=0}^{n-1} \int_j^{j+1} \big(x^{\theta} - (j+1)^{\theta}\big)\, dx = \\ &= -\frac{\theta}{\theta+1} + \sum_{j=1}^{n-1} \Big(\frac{(j+1)^{\theta+1} - j^{\theta+1}}{\theta+1} - (j+1)^{\theta} \Big) \le \\ &\le -\frac{\theta}{\theta+1} + \sum_{j=1}^{n-1} \big(j^{\theta} - (j+1)^{\theta} \big) = \frac{1}{\theta+1} - n^{\theta} < \frac{1}{\theta+1}. \end{aligned}$$

Combining the upper and lower bounds tells us that

$$0 < -\frac{\theta}{\theta+1} < \gamma_{\theta} \le \frac{1}{\theta+1},$$

and the monotonicity shows that the $\lim_{n\to\infty} D_n$ exists. By the same estimation, for $m > n$,

$$D_m - D_n = \sum_{j=n}^{m-1} \int_j^{j+1} (x^\theta - (j+1)^\theta)\, dx < n^\theta - m^\theta,$$

which, by letting $m \to \infty$, gives the desired rate and completes the proof. ■

Now, let $1 \le p < r < 2$ and Y be a standard normal random variable. Then, by arguments similar to those in the proof of Proposition 2.1, together with Lemma 3.1, we obtain

$$\begin{aligned}
\lambda_{r,p}(\varepsilon) &= \sum_{n=1}^{\infty} n^{(r/p)-2} P(|S_n| \ge \varepsilon n^{1/p}) = \\
&= \sum_{n=1}^{\infty} n^{(r/p)-2} P(|Y| \ge \varepsilon n^{(2-p)/(2p)}) = \\
&= \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} \Big(\sum_{j=1}^{n} j^{(r/p)-2} \Big) \int_{\varepsilon n^{(2-p)/(2p)}}^{\varepsilon (n+1)^{(2-p)/(2p)}} \exp\{-y^2/2\}\, dy = \\
&= \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} \Big(\frac{p}{r-p} \cdot n^{(r-p)/p} - \gamma_{(r/p)-2} + \delta_n \Big) \cdot \\
&\quad \cdot \int_{\varepsilon n^{(2-p)/(2p)}}^{\varepsilon (n+1)^{(2-p)/(2p)}} \exp\{-y^2/2\}\, dy = \\
&= I + II + III,
\end{aligned} \tag{3.8}$$

where $\lim_{n\to\infty} \delta_n = 0$. Now, by straightforward computations,

$$\begin{aligned}
I &= \frac{p}{r-p}\, \varepsilon^{-\frac{2(r-p)}{2-p}} \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} \big(\varepsilon n^{\frac{2-p}{2p}}\big)^{\frac{2(r-p)}{2-p}} \int_{\varepsilon n^{(2-p)/(2p)}}^{\varepsilon (n+1)^{(2-p)/(2p)}} \exp\{-y^2/2\}\, dy = \\
&= \frac{p}{r-p}\, \varepsilon^{-\frac{2(r-p)}{2-p}} \sqrt{\frac{2}{\pi}} \int_0^{\infty} y^{\frac{2(r-p)}{2-p}} \exp\{-y^2/2\}\, dy + \mathcal{O}(\varepsilon) = \\
&= \frac{p}{r-p}\, \varepsilon^{-\frac{2(r-p)}{2-p}}\, E|Y|^{\frac{2(r-p)}{2-p}} + \mathcal{O}(\varepsilon) \quad \text{as} \quad \varepsilon \searrow 0.
\end{aligned} \tag{3.9}$$

Next,

$$
\begin{aligned}
II &= \gamma_{(r/p)-2}\sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}\int\limits_{\varepsilon n^{(2-p)/(2p)}}^{\varepsilon(n+1)^{(2-p)/(2p)}}\exp\{-y^2/2\}\,dy = \\
&= \gamma_{(r/p)-2}\sqrt{\frac{2}{\pi}}\int\limits_{\varepsilon}^{\infty}\exp\{-y^2/2\}\,dy \to \\
&\to \gamma_{(r/p)-2} \quad \text{as} \quad \varepsilon \searrow 0,
\end{aligned}
\tag{3.10}
$$

and, similar to (3.2),

$$
III \quad = \quad o(1) \quad \text{as} \quad \varepsilon \searrow 0. \tag{3.11}
$$

A combination of (3.8)–(3.11) finishes the proof. ■

Proof of Proposition 2.4. The proof follows the usual pattern. Let $1 \le p < r < \alpha < 2$, $Y \in$ Stable(α) with mean 0, set $\Psi(y) = P(|Y| \ge y)$, $y > 0$, and recall that $P(|Y| \ge y) = -\int_y^\infty d\Psi(x)$. Then,

$$
\begin{aligned}
\lambda_{\alpha,r,p}(\varepsilon) &= \sum_{n=1}^{\infty} n^{(r/p)-2}P(|S_n| \ge \varepsilon n^{1/p}) = \\
&= \sum_{n=1}^{\infty} n^{(r/p)-2}P(|Y| \ge \varepsilon n^{(\alpha-p)/(\alpha p)}) = \\
&= -\sum_{n=1}^{\infty}\Big(\sum_{j=1}^{n} j^{(r/p)-2}\Big)\int\limits_{\varepsilon n^{(\alpha-p)/(\alpha p)}}^{\varepsilon(n+1)^{(\alpha-p)/(\alpha p)}} d\Psi(y) = \\
&= -\sum_{n=1}^{\infty}\Big(\frac{p}{r-p}\,n^{(r-p)/p} + \gamma_{(r/p)-2} + \delta_n\Big)\int\limits_{\varepsilon n^{(\alpha-p)/(\alpha p)}}^{\varepsilon(n+1)^{(\alpha-p)/(\alpha p)}} d\Psi(y) = \\
&= I + II + III,
\end{aligned}
\tag{3.12}
$$

where $\lim_{n\to\infty}\delta_n = 0$, after which the three contributions are taken care of, and then combined, exactly as in the proof of Proposition 2.3. ■

Proof of Theorem 2.2. (a) The arguments for (2.3) are similar to those for (2.1). By Remark 2.2, $\sum_{n=1}^{\infty} n^{(r/p)-2}\Delta_n < \infty$, so that we only have to replace ϱ by $\varrho_{r,p} = \sum_{n=1}^{\infty} n^{(r/p)-2} P(S_n = 0)$, which is finite, and make use of Proposition 2.3.

(b) Since again $\sum_{n=1}^{\infty} n^{(r/p)-2}\Delta_n < \infty$, the arguments for the proof of (2.4) are the same as above with Proposition 2.4 replacing Proposition 2.3. ■

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