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ON THE NUMBER OF DIVISORS IN ARITHMETICAL SEMIGROUPS

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Dedicated to Professor Karl-Heinz Indlekofer on his 70th birthday

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Abstract. The generalized divisor function, defined on an arithmetical semigroup, is considered. The asymptotic formula for its mean value is obtained.

1. Introduction

Let $\mathbb{G}$ be a commutative multiplicative semigroup with identity element a_0 and generated by a countable subset $\mathfrak{P}$ of prime elements. We assume that k, l, m, n are non-negative integers, $a, b, d \in \mathbb{G}$, $p \in \mathfrak{P}$ and a completely additive degree function $\partial : \mathbb{G} \to \mathbb{N} \cup \{0\}$ is defined so that $\partial(p) \geq 1$ for each prime p. Moreover, we suppose that the semigroup $\mathbb{G}$ satisfies (see [8], [7]) the following

Axiom A*. *There exist constants $A > 0,\ q > 1$ and $0 \leq \nu < 1$ such that*

$$\mathbb{G}(n) := \#\{a \in \mathbb{G} : \partial(a) = n\} = Aq^n + \mathrm{O}(q^{\nu n}).$$

In this case the generating function

$$Z(z) := \sum_{n \geq 0} \mathbb{G}(n) \left(\frac{z}{q}\right)^n, \quad |z| < 1,$$

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is known to be continuable into the disc $|z| < q^{1-\nu}$ and $Z(z) \neq 0$ for $|z| \leq 1$ with the possible exception at the point $z = -1$. The analogue of the prime number theorem (see [6]) yields

$$\pi(n) := \#\{p \in \mathfrak{P} : \partial(p) = n\} = \frac{q^n}{n}(1 - (-1)^n I(\mathbb{G})) + \mathrm{O}(q^{\mu n}) \tag{1.1}$$

with some $\max(1/2, \nu) < \mu < 1$. Here $I(\mathbb{G}) = 1$, if $Z(-1) = 0$, and $I(\mathbb{G}) = 0$ otherwise.

For a multiplicative function $f : \mathbb{G} \to [0, \infty)$ let

$$T(a, v) := \sum_{d|a,\, \partial(d) \leq v}^{*} f(d), \quad a \in \mathbb{G},\ v \geq 0.$$

Here and thereafter the starred sum or product symbols mean that these operations are used over corresponding elements of the semigroup $\mathbb{G}$. For any $a \in \mathbb{G}$, set

$$X(a, t) := \frac{T(a, \partial(a)t)}{T(a)}, \quad t \in [0, 1],$$

where the multiplicative function $T(a)$ is defined by $T(a) := T(a, \partial(a))$. To evaluate the mean value of the ratio $X(a, t)$ we will consider the sequence

$$F_n(t) := \frac{q-1}{Aq^{n+1}} \sum_{\partial(a) \leq n}^{*} X(a, t), \quad t \in [0, 1], \quad n \in \mathbb{N}.$$

The asymptotic behaviour of $F_n(t)$, as $n \to \infty$, was considered by the first author on the polynomial semigroup [1]. For the multiplicative functions, defined on the set of natural numbers, similar problem was investigated in the series of papers, see for example [4, 3, 2] . The aim of our paper is to improve and generalize the main result in [1] and correct the mistake which was made in this paper by estimating mean value of the shifted multiplicative function.

We consider a non-negative multiplicative function $f(a)$, defined on the semigroup with axiom A^*, provided the associated "divisor" function $T(a)$ satisfies some analytic condition.

Definition 1.1. Let $g : \mathbb{G} \to [0, \infty)$ be a multiplicative function such that $g(p^m) \leq C$ for $m \in \mathbb{N}$, any prime p and some $C > 0$. We say that g belongs to the class $\mathbb{M}(\varkappa, C, c)$, $\varkappa \geq 0$, if the function defined by

$$L_g(z, \varkappa) := \sum_{m \geq 1} \left(\frac{z}{q}\right)^m \sum_{\partial(p)=m}^{*} (g(p) - \varkappa), \quad |z| < 1,$$

has an analytic continuation into the disc $|z| < 1 + c$ for some $c > 0$.

In Lemma 2.1 we obtain the asymptotic formula for the mean value of the shifted multiplicative functions from the class $\mathbb{M}(\varkappa, C, c)$. This enables us to prove the main result contained in following

Theorem 1.1. *Suppose that* $f : \mathbb{G} \to [0, \infty)$ *is a multiplicative function such, that* $1/T \in \mathbb{M}(\alpha, 1, c_1)$ *with some constants* $0 < \alpha < 1$ *and* $c_1 > 0$. *Then for all* $n \in \mathbb{N}$ *and* $0 \le u \le t \le 1$

$$F_n(t) - F_n(u) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_u^t \frac{\mathrm{d}x}{x^\alpha (1-x)^\beta} + \mathrm{O}\left(\rho_n(u, t; \alpha, \beta)\right),$$

where $\beta := 1 - \alpha$ *and*

$$\rho_n(u, t; \gamma, \delta) := n^{-\gamma-\delta} \left((n^{-1} + u)^{-\gamma} + (n^{-1} + 1 - t)^{-\delta} \right).$$

This theorem implies the uniform estimate

$$F_n(t) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \frac{\mathrm{d}x}{x^\alpha (1-x)^\beta} + \mathrm{O}\left(n^{-\alpha} + n^{-\beta}\right), \tag{1.2}$$

for all $n \in \mathbb{N}$ and $0 \le t \le 1$.

When $f \equiv 1$, the value $T(a)$ means the number of divisors of the element a. In this case $\alpha = \beta = 1/2$.

2. Preliminaries

We will need an estimate for the mean value of shifted positive multiplicative functions defined on $\mathbb{G}$

$$M_n(g, d) := \frac{1}{Aq^n} \sum_{\partial(a)=n}^{*} g(ad).$$

The following lemma yields the result of this type. In some cases it intersects with the corresponding results in the papers [9], [10].

Lemma 2.1. *Let* $g : \mathbb{G} \to [0, \infty)$ *be a multiplicative function such that* $g \in \mathbb{M}(\varkappa, C, c)$ *with some positive constants* $\varkappa$, C *and* c. *Then, uniformly for all* $d \in \mathbb{G}$ *and* $n \ge 0$,

$$M_n(g, d) = (A(n+1))^{\varkappa - 1} \left(\frac{L(\varkappa)\tilde{g}(d)}{\Gamma(\varkappa)} + \mathrm{O}\left(\frac{\hat{g}(d)}{n+1} \right) \right),$$

where $L(\varkappa)$ and the multiplicative functions $\tilde{g}$ and $\hat{g}$ are defined by

$$L(\varkappa) := \prod_p{}^{*} \left(1 - \frac{1}{q^{\partial(p)}}\right)^{\varkappa} \sum_{k\ge 0} \frac{g(p^k)}{q^{k\partial(p)}},$$

$$\tilde{g}(p^m) := \left(\sum_{k\ge 0} \frac{g(p^k)}{q^{k\partial(p)}}\right)^{-1} \sum_{k\ge 0} \frac{g(p^{k+m})}{q^{k\partial(p)}}, \tag{2.1}$$

$$\hat{g}(p^m) := \left(1 + \frac{c_1}{q^{2\partial(p)/3}}\right) \sum_{k\ge 0} \frac{g(p^{k+m})}{q^{2k\partial(p)/3}}.$$

Here $c_1 \ge 0$ is a constant, depending on $\varkappa$ and C.

Proof. Our proof is similar to that in [2] and based on the properties of the generating function

$$F(z,d) := A \sum_{n\ge 0} M_n(g,d) z^n = \sum_a{}^{*} g(ad) \left(\frac{z}{q}\right)^{\partial(a)}.$$

By the Euler identity, for $|z| < 1$, we have

$$F(z,d) = \prod_p{}^{*} \psi(z,p,d).$$

Here

$$\psi(z,p,d) := \sum_{k\ge 0} g(p^{k+\gamma_p(d)}) \left(\frac{z}{q}\right)^{k\partial(p)}$$

and $\gamma_p(d)$ is defined by $p^{\gamma_p(d)} || d$. Since

$$\sum_{k\ge 1} \frac{1-(-1)^k I(\mathbb{G})}{k} z^k = \ln \frac{1 + zI(\mathbb{G})}{1-z},$$

for $|z| < 1$, we have the representation

$$F(z,d) = G(z,d,\varkappa) W^{\varkappa}(z,\mathbb{G}) \left(\frac{1+zI(\mathbb{G})}{1-z}\right)^{\varkappa}, \tag{2.2}$$

where

$$G(z,d,\varkappa) := \prod_p{}^{*} \psi(z,p,d) \exp\left\{-\varkappa \left(\frac{z}{q}\right)^{\partial(p)}\right\},$$

$$W(z,\mathbb{G}) := \exp\left\{\sum_{k\ge 1} \left(\frac{\pi(k)}{q^k} - \frac{1-(-1)^k I(\mathbb{G})}{k}\right) z^k\right\}.$$

Let us consider the function $G(z,d,\varkappa)$ when $|z| \le r := \min(1+c/2, \sqrt[3]{q})$. Set $k_0 := 1 + [1.5 \log_q(1+2C)]$ and

$$\delta(z,p) := \begin{cases} \exp\left\{-\varkappa\left(\frac{z}{q}\right)^{\partial(p)}\right\}, & \text{if } \partial(p) < k_0\,, \\ (\psi(z,p,a_0))^{-1}, & \text{if } \partial(p) \ge k_0\,. \end{cases}$$

In the disc $|z| \le r$ we have that

$$|\psi(z,p,a_0) - 1| < 1/2, \tag{2.3}$$

when $\partial(p) \ge k_0$. Moreover, there exists a constant $c_1 = c_1(C,\varkappa)$ such that

$$|\delta(z,p)| \le 1 + \frac{c_1}{q^{2\partial(p)/3}}, \tag{2.4}$$

for all $p \in \mathfrak{P}$. Further, let $\mathfrak{P}'$ be the subset of prime elements

$$\mathfrak{P}' := \mathfrak{P} \setminus \{p \in \mathfrak{P} :\ p|d,\ \partial(p) < k_0\}.$$

Then the function $G(z,d,\varkappa)$ can be written in such form

$$\begin{aligned} G(z,d,\varkappa) &= \prod_{p\in\mathfrak{P}'}^{*} \psi(z,p,a_0) \exp\left\{-\varkappa\left(\frac{z}{q}\right)^{\partial(p)}\right\} \cdot \prod_{p|d}^{*} \delta(z,p)\psi(z,p,d), \\ &=: G_1(z,d,\varkappa)\cdot G_2(z,d,\varkappa). \end{aligned}$$

Inequality (2.4) implies that, for $|z| \le r$, the multiplicative functions $G_2(z,d,\varkappa)$ and $\hat{g}(d)$ are related by the inequality

$$|G_2(z,d,\varkappa)| \le \hat{g}(d).$$

Taking exponent and logarithm, which is allowed by (2.3), in the routine way we obtain

$$G_1(z,d,\varkappa) = \mathrm{e}^{L_g(z,\varkappa)} G_3(z,d,\varkappa).$$

Here $G_3(z,d,\varkappa)$ is analytic and bounded for $|z| \le r$. Moreover, the assumptions of lemma allow us to assert that the function $L_g(z,\varkappa)$ has an analytic continuation and is bounded in this domain. Thus we have that $G(z,d,\varkappa)$ is analytic in the disc $|z| \le r$, satisfies there the inequality

$$|G(z,d,\varkappa)| \ll \hat{g}(d), \tag{2.5}$$

and

$$G(1,d,\varkappa) = \tilde{g}(d) \prod_{p}^{*} \psi(1,p,a_0) \exp\left\{-\frac{\varkappa}{q^{\partial(p)}}\right\}. \tag{2.6}$$

From (1.1) it follows that $W(z,\mathbb{G})$ is analytic in the disc $|z|<q^{1-\mu}$. Moreover, it can be shown (see eg. [9]) that

$$W(1,\mathbb{G}) = \frac{A}{1+I(\mathbb{G})}\prod_p{}^{*}\left(1-q^{-\partial(p)}\right)\exp\left\{q^{-\partial(p)}\right\}.$$

Therefore in virtue of (2.2), (2.5) and (2.6) we obtain

$$F(z,d) = H(z,d,\varkappa)(1-z)^{-\varkappa},$$

where $H(z,d,\varkappa)$ is analytic and satisfies the inequality

$$|H(z,d,\varkappa)| \ll \hat{g}(d),$$

when $|z|\le r_1 := \min(r, q^{(1-\mu)/2})$. Moreover,

$$H(1,d,\varkappa) = A^{\varkappa}L(\varkappa)\tilde{g}(d). \tag{2.7}$$

Thus with the sole exception at point $z=1$ for $|z|\le r_1$ we have

$$F(z,d) = H(1,d,\varkappa)(1-z)^{-\varkappa} + \mathrm{O}\left(\hat{g}(d)|1-z|^{1-\varkappa}\right). \tag{2.8}$$

According to Theorem 1 and Corollary 3 in [5] this estimate implies

$$M_n(g,d)A = H(1,d,\varkappa)\binom{n+\varkappa-1}{n} + \mathrm{O}\left(\hat{g}(d)n^{\varkappa-2}\right).$$

Since $M_0(g,d) = A^{-1}F(0,d) \ll \hat{g}(d)$ and

$$\binom{n+\varkappa-1}{n} = \frac{(n+1)^{\varkappa-1}}{\Gamma(\varkappa)}\left(1+\mathrm{O}\left(\frac{1}{n+1}\right)\right),$$

the desired estimate follows from (2.8) and (2.7). ■

In addition, we provide some asymptotic formulas which will be useful in the sequel.

Lemma 2.2 ([8] p. 86)**.** *Suppose that* $\sigma\in\mathbb{R}$. *Then*

$$\sum_{m=1}^{n} m^{\sigma}q^{m} = \frac{q}{q-1}n^{\sigma}q^{n} + \mathrm{O}\left(n^{\sigma-1}q^{n}\right).$$

Lemma 2.3. *For $0 \le u \le t \le 1$, $n \ge 1$, $\gamma > 0$ and $\delta > 0$, we have*

$$\sum_{nu<k\le nt} \frac{1}{(1+k)^{\gamma}(1+n-k)^{\delta}} = n^{1-\gamma-\delta} I(u,t;\gamma,\delta,n^{-1}) + \mathrm{O}(\rho_n(u,t;\gamma,\delta)),$$

where

$$I(u,t;\gamma,\delta,\eta) := \int\limits_u^t \frac{dv}{(\eta+v)^{\gamma}(\eta+1-v)^{\delta}}\,.$$

Moreover,

$$I(u,t;\gamma+1,\delta,n^{-1}) + I(u,t;\gamma,\delta+1,n^{-1}) \ll n^{\gamma+\delta}\rho_n(u,t;\gamma,\delta) \tag{2.9}$$

and

$$I(u,t;\gamma,\delta,n^{-1}) = I(u,t;\gamma,\delta,0) + \mathrm{O}\left(\rho_n(u,t;\gamma,\delta)\right). \tag{2.10}$$

Proof. The first formula in this lemma follows from Euler-Maclaurin summation formula. The relations (2.9) and (2.10) follow from the definition of the integral $I(u,t;\gamma,\delta,\eta)$ by straightforward estimations (see, eg. in [2]). ■

3. Proof of Theorem 1.1

Assumptions of the theorem imply, that the multiplicative functions $1/T(a) \in \mathbb{M}(\alpha,1,c_1)$ and $f(a)/T(a) \in \mathbb{M}(\beta,1,c_2)$ with some positive constants c_1 and c_2. We have

$$F_n(t) = S_n(t) + R_n(t), \tag{3.1}$$

where

$$S_n(t) := \frac{q-1}{Aq^{n+1}} \sum_{0\le m\le n} \sideset{}{^*}\sum_{\partial(a)=m} \frac{T(a,nt)}{T(a)},$$

$R_n(0) = 0$ and

$$R_n(t) \ll q^{-n} \sideset{}{^*}\sum_{\partial(d)\le nt} f(d) \sum_{k\le \partial(d)(1-t)/t} \sideset{}{^*}\sum_{\partial(a)=k} \frac{1}{T(ad)}, \quad t\in(0,1].$$

To evaluate the most inner sum we apply Lemma 2.1 with $g(a) = g_0(a) :=$ $:= 1/T(a)$. We have

$$\sideset{}{^*}\sum_{\partial(a)=k} \frac{1}{T(ad)} = \frac{A^{\alpha}q^k}{(1+k)^{\beta}} \left(\frac{L_0(\alpha)\tilde{g}_0(d)}{\Gamma(\alpha)} + \mathrm{O}\left(\frac{\hat{g}_0(d)}{1+k}\right)\right). \tag{3.2}$$

Here $L_0(\alpha)$ and multiplicative functions $\tilde{g}_0, \hat{g}_0$ are defined in (2.1) by setting $g = g_0$. An easy calculation shows that

$$\begin{aligned}\tilde{g}_0(p^m) &= g_0(p^m)\left(1 + \mathrm{O}\left(q^{-\partial(p)}\right)\right),\\ \hat{g}_0(p^m) &= g_0(p^m)\left(1 + \mathrm{O}\left(q^{-\frac{2\partial(p)}{3}}\right)\right).\end{aligned}$$

Thus

$$R_n(t) \ll q^{-n} \sum_{\partial(d)\le tn}^{*} f(d)\hat{g}_0(d) \sum_{k\le \partial(d)(1-t)/t} \frac{q^k}{(1+k)^\beta}.$$

By the Lemma 2.2

$$R_n(t) \ll \frac{q^{-nt}}{(1+n(1-t))^\beta} \sum_{m\le nt} \sum_{\partial(a)=m}^{*} f(a)\hat{g}_0(a).$$

Since $f(a)\hat{g}_0(a) \in \mathbb{M}(\beta, C_1, c_3)$ with some positive C_1 and c_3, for the inner sum we can apply Lemma 2.1 by setting $g = f \cdot \hat{g}_0$ and $d = a_0$. Then employing Lemma 2.2 again we obtain that

$$R_n(t) \ll (1+n(1-t))^{-\beta}(1+nt)^{-\alpha},$$

for all $0 \le t \le 1$. Setting $S_n(u,t) := S_n(t) - S_n(u)$, from this and (3.1) we deduce

$$F_n(t) - F_n(u) = S_n(u,t) + \mathrm{O}(\rho_n(u,t;\alpha,\beta)). \tag{3.3}$$

It remains to evaluate the sum $S_n(u,t)$. Changing order of summation we have

$$S_n(u,t) = \frac{q-1}{Aq^{n+1}} \sum_{nu<\partial(d)\le nt}^{*} f(d) \sum_{m=0}^{n-\partial(d)} \sum_{\partial(a)=m}^{*} \frac{1}{T(ad)}.$$

Since $\tilde{g}_0(a) \le \hat{g}_0(a)$, applying (3.2) and Lemma 2.2 we get

$$S_n(u,t) = S_1(n;u,t) + \mathrm{O}\left(R_1(n;u,t)\right),$$

where

$$S_1(n;u,t) := \frac{L_0(\alpha)A^{-\beta}}{\Gamma(\alpha)} \sum_{nu<m\le nt} \frac{q^{-m}}{(1+n-m)^\beta} \sum_{\partial(d)=m}^{*} f(d)\tilde{g}_0(d)$$

and

$$R_1(n;u,t) := \sum_{nu<m\le nt} \frac{q^{-m}}{(1+n-m)^{\beta+1}} \sum_{\partial(d)=m}^{*} f(d)\hat{g}_0(d).$$

It is easy to see, that $f(a)\tilde{g}_0(a) \in \mathbb{M}(\beta, C_2, c_4)$ with some positive C_2 and c_4. Therefore the inner sums in the expressions of $S_1(n; u, t)$ and $R_1(n; u, t)$ we can evaluate by means of Lemma 2.1. So we have

$$S_n(u,t) = \frac{L_0(\alpha)L_1(\beta)}{\Gamma(\alpha)\Gamma(\beta)} S(\alpha,\beta) + \mathrm{O}\left(S(\alpha+1,\beta) + S(\alpha,\beta+1)\right),$$

where

$$S(\gamma,\delta) := \sum_{nu \le j \le nt} \frac{1}{(1+n-j)^{\delta}(1+n)^{\gamma}}$$

for short. Now Lemma 2.3 implies

$$S_n(u,t) = \frac{L_0(\alpha)L_1(\beta)}{\Gamma(\alpha)\Gamma(\beta)} I(u,t;\alpha,\beta,0) + \mathrm{O}\left(\rho_n(u,t;\alpha,\beta)\right). \tag{3.4}$$

Here $L_0(\alpha)$ and $L_1(\beta)$ are defined in (2.1) by setting $g = g_0$ and $g = f \cdot \tilde{g}_0$ respectively. The routine calculation yields that $L_0(\alpha) \cdot L_1(\beta) = 1$ (see, eg. [1, 2]). Finally the proof of the theorem follows from (3.4) and (3.3). ■

The estimate (1.2) is an easy consequence of Theorem 1.1 with $u = 0$, since $F_n(0) \ll n^{-\beta}$ by (3.2) and Lemma 2.2.

References

[1] **Bareikis, G.,** Beta distribution in the polynomial semigroup, *Annales Univ. Sci. Budapest., Sect. Comp.*, **22** (2003), 35–48.

[2] **Bareikis, G. and A. Mačiulis,** Asymptotic expectation of a sequence of arithmetical processes, *Anal. Probab. Methods in Number Theory,* Kubilius Memorial Volume, TEV, Vilnius, 2012, 35–48.

[3] **Bareikis, G. and E. Manstavičius,** On the DDT theorem, *Acta Arith.*, **126**(2007), 155–168.

[4] **Deshouillers, J.M., F. Dress and G. Tenenbaum,** Lois de répartition des diviseurs, *Acta Arith.*, **34** (1979), 7–19.

[5] **Flajolet, P. and A. Odlyzko,** Singularity analysis of generating functions, *SIAM J. Discrete Math.*, **3(2)** (1990), 216–240.

[6] **Indlekofer, K.-H., E. Manstavičius and R. Warlimont,** On a certain class of infinite products with an application to arithmetical semigroups, *Archiv der Math.*, **56** (1991), 446–453.

[7] **Indlekofer, K.-H. and E. Manstavičius,** Additive and multiplicative functions on arithmetical semigroups, *Publ. Math.*, **45** (1994), 1–17.

[8] **Knopfmacher, J.,** *Analytic Arithmetic of Algebraic Function Fields,* Marcel Dekker, New York, 1979.

[9] **Mačiulis, A.,** The mean values of multiplicative functions defined on a semigroup, in: New Trends in Probab. and Statist., Vol. 2, *Anal. Probab. Methods in Number Theory*, F. Schweiger and E. Manstavičius (Eds), TEV, Vilnius, Lithuania/VSP, Utrecht, The Netherlands (1992), 121–133.

[10] **Manstavičius, E. and R. Skrabutėnas,** Summation of values of multiplicative functions on semigroups, *Lith. Math. J.*, **33**(3) (1993), 330–340.

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