ON THE NUMBER OF DIVISORS IN ARITHMETICAL SEMIGROUPS

Gintautas Bareikis and Algirdas Mačiulis
(Vilnius, Lithuania)

Dedicated to Professor Karl-Heinz Indlekofer
on his 70th birthday

Communicated by Imre Kátai
(Received December 19, 2012; accepted January 08, 2013)

Abstract. The generalized divisor function, defined on an arithmetical semigroup, is considered. The asymptotic formula for its mean value is obtained.

1. Introduction

Let $G$ be a commutative multiplicative semigroup with identity element $a_0$ and generated by a countable subset $\mathcal{P}$ of prime elements. We assume that $k, l, m, n$ are non-negative integers, $a, b, d \in G$, $p \in \mathcal{P}$ and a completely additive degree function $\partial : G \to \mathbb{N} \cup \{0\}$ is defined so that $\partial(p) \geq 1$ for each prime $p$. Moreover, we suppose that the semigroup $G$ satisfies (see [8], [7]) the following Axiom $A^\ast$.

There exist constants $A > 0$, $q > 1$ and $0 \leq \nu < 1$ such that

$$G(n) := \# \{a \in G : \partial(a) = n\} = Aq^n + O(q^\nu).$$

In this case the generating function

$$Z(z) := \sum_{n \geq 0} G(n) \left(\frac{z}{q}\right)^n, \quad |z| < 1,$$

Key words and phrases: Arithmetical semigroup, multiplicative function, divisor function.

2010 Mathematics Subject Classification: 11N45, 11N80, 11K65.
is known to be continuable into the disc $|z| < q^{1-\nu}$ and $Z(z) \neq 0$ for $|z| \leq 1$ with the possible exception at the point $z = -1$. The analogue of the prime number theorem (see [6]) yields

\begin{equation}
\pi(n) := \# \{ p \in \mathbb{P} : \partial(p) = n \} = \frac{q^n}{n} (1 - (-1)^n I(\mathbb{G})) + O(q^{\mu n})
\end{equation}

with some $\max(1/2, \nu) < \mu < 1$. Here $I(\mathbb{G}) = 1$, if $Z(-1) = 0$, and $I(\mathbb{G}) = 0$ otherwise.

For a multiplicative function $f : \mathbb{G} \to [0, \infty)$ let

$$ T(a, v) := \sum_{d(a, \partial(a)) \leq v}^* f(d), \quad a \in \mathbb{G}, \ v \geq 0. $$

Here and thereafter the starred sum or product symbols mean that these operations are used over corresponding elements of the semigroup $\mathbb{G}$. For any $a \in \mathbb{G}$, set

$$ X(a, t) := \frac{T(a, \partial(a)t)}{T(a)}, \quad t \in [0, 1], $$

where the multiplicative function $T(a)$ is defined by $T(a) := T(a, \partial(a))$. To evaluate the mean value of the ratio $X(a, t)$ we will consider the sequence

$$ F_n(t) := \frac{q - 1}{Aq^{n+1}} \sum_{\partial(a) \leq n}^* X(a, t), \quad t \in [0, 1], \ n \in \mathbb{N}. $$

The asymptotic behaviour of $F_n(t)$, as $n \to \infty$, was considered by the first author on the polynomial semigroup [1]. For the multiplicative functions, defined on the set of natural numbers, similar problem was investigated in the series of papers, see for example [4, 3, 2]. The aim of our paper is to improve and generalize the main result in [1] and correct the mistake which was made in this paper by estimating mean value of the shifted multiplicative function.

We consider a non-negative multiplicative function $f(a)$, defined on the semigroup with axiom $A^*$, provided the associated "divisor" function $T(a)$ satisfies some analytic condition.

**Definition 1.1.** Let $g : \mathbb{G} \to [0, \infty)$ be a multiplicative function such that $g(p^m) \leq C$ for $m \in \mathbb{N}$, any prime $p$ and some $C > 0$. We say that $g$ belongs to the class $\mathcal{M}(\kappa, C, c)$, $\kappa \geq 0$, if the function defined by

$$ L_g(z, \kappa) := \sum_{m \geq 1} \left(\frac{z}{q}\right)^m \sum_{\partial(p) = m}^* (g(p) - \kappa), \quad |z| < 1, $$

has an analytic continuation into the disc $|z| < 1 + c$ for some $c > 0$. 
In Lemma 2.1 we obtain the asymptotic formula for the mean value of the shifted multiplicative functions from the class $M(\kappa, C, c)$. This enables us to prove the main result contained in following

**Theorem 1.1.** Suppose that $f : G \rightarrow [0, \infty)$ is a multiplicative function such, that $1/T \in M(\alpha, 1, c_1)$ with some constants $0 < \alpha < 1$ and $c_1 > 0$. Then for all $n \in \mathbb{N}$ and $0 \leq u \leq t \leq 1$

\[ F_n(t) - F_n(u) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_u^t \frac{dx}{x^\alpha(1-x)^\beta} + O \left( \rho_n(u, t; \alpha, \beta) \right), \]

where $\beta := 1 - \alpha$ and

\[ \rho_n(u, t; \gamma, \delta) := n^{-\gamma-\delta} \left( (n^{-1} + u)^{-\gamma} + (n^{-1} + 1 - t)^{-\delta} \right). \]

This theorem implies the uniform estimate

\[ (1.2) \quad F_n(t) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \frac{dx}{x^\alpha(1-x)^\beta} + O \left( n^{-\alpha} + n^{-\beta} \right), \]

for all $n \in \mathbb{N}$ and $0 \leq t \leq 1$.

When $f \equiv 1$, the value $T(a)$ means the number of divisors of the element $a$. In this case $\alpha = \beta = 1/2$.

2. **Preliminaries**

We will need an estimate for the mean value of shifted positive multiplicative functions defined on $G$

\[ M_n(g, d) := \frac{1}{Aq^n} \sum_{\partial(a) = n}^* g(ad). \]

The following lemma yields the result of this type. In some cases it intersects with the corresponding results in the papers [9], [10].

**Lemma 2.1.** Let $g : G \rightarrow [0, \infty)$ be a multiplicative function such that $g \in M(\kappa, C, c)$ with some positive constants $\kappa$, $C$ and $c$. Then, uniformly for all $d \in G$ and $n \geq 0$,

\[ M_n(g, d) = (A(n+1))^{\kappa-1} \left( \frac{L(\kappa)\tilde{g}(d)}{\Gamma(\kappa)} + O \left( \frac{\tilde{g}(d)}{n+1} \right) \right), \]
where $L(\kappa)$ and the multiplicative functions $\tilde{g}$ and $\hat{g}$ are defined by

\[
L(\kappa) := \prod_p \left( 1 - \frac{1}{q^{\gamma_p}} \right)^{\kappa} \sum_{k \geq 0} \frac{g(p^k)}{q^{k\gamma_p}},
\]

(2.1)

\[
\tilde{g}(p^m) := \left( \sum_{k \geq 0} \frac{g(p^k)}{q^{k\gamma_p}} \right)^{-1} \sum_{k \geq 0} \frac{g(p^{k+m})}{q^{k\gamma_p}},
\]

\[
\hat{g}(p^m) := \left( 1 + \frac{c_1}{q^{\gamma_p}/3} \right) \sum_{k \geq 0} \frac{g(p^{k+m})}{q^{2k\gamma_p}/3}.
\]

Here $c_1 \geq 0$ is a constant, depending on $\kappa$ and $C$.

**Proof.** Our proof is similar to that in [2] and based on the properties of the generating function

\[
F(z, d) := A \sum_{n \geq 0} M_n(g, d) z^n = \sum_a^* g(ad) \left( \frac{z}{q} \right)^{\partial(a)}.
\]

By the Euler identity, for $|z| < 1$, we have

\[
F(z, d) = \prod_p^* \psi(z, p, d).
\]

Here

\[
\psi(z, p, d) := \sum_{k \geq 0} g(p^{k+\gamma_p(d)} \left( \frac{z}{q} \right)^{k\gamma(p)}
\]

and $\gamma_p(d)$ is defined by $p^{\gamma_p(d)}||d$. Since

\[
\sum_{k \geq 1} \frac{1 - (-1)^k I(G)}{k} \cdot z^k = \ln \frac{1 + z I(G)}{1 - z},
\]

for $|z| < 1$, we have the representation

(2.2)

\[
F(z, d) = G(z, d, \kappa) W^{\kappa}(z, G) \left( \frac{1 + z I(G)}{1 - z} \right)^{\kappa},
\]

where

\[
G(z, d, \kappa) := \prod_p^* \psi(z, p, d) \exp \left\{ -\kappa \left( \frac{z}{q} \right)^{\partial(p)} \right\},
\]

\[
W(z, G) := \exp \left\{ \sum_{k \geq 1} \left( \frac{\pi(k)}{q^k} - 1 - (-1)^k I(G) \right) \frac{z^k}{k} \right\}.
\]
Let us consider the function $G(z, d, \kappa)$ when $|z| \leq r := \min(1 + c/2, \sqrt{q})$. Set $k_0 := 1 + [1.5 \log_q (1 + 2C)]$ and

$$
\delta(z, p) := \begin{cases} 
\exp \left\{ -\kappa \left( \frac{z}{d} \right)^{\partial(p)} \right\}, & \text{if } \partial(p) < k_0, \\
(\psi(z, p, a_0))^{-1}, & \text{if } \partial(p) \geq k_0.
\end{cases}
$$

In the disc $|z| \leq r$ we have that

$$
|\psi(z, p, a_0) - 1| < 1/2, \quad \text{when } \partial(p) \geq k_0.
$$

Moreover, there exists a constant $c_1 = c_1(C, \kappa)$ such that

$$
|\delta(z, p)| \leq 1 + \frac{c_1}{q^{2\partial(p)/3}},
$$

for all $p \in \mathfrak{P}$. Further, let $\mathfrak{P}'$ be the subset of prime elements

$$
\mathfrak{P}' := \mathfrak{P} \setminus \{ p \in \mathfrak{P} : p|d, \partial(p) < k_0 \}.
$$

Then the function $G(z, d, \kappa)$ can be written in such form

$$
G(z, d, \kappa) = \prod_{p \in \mathfrak{P}'}^* \psi(z, p, a_0) \exp \left\{ -\kappa \left( \frac{z}{d} \right)^{\partial(p)} \right\} \cdot \prod_{\partial(p) \geq k_0}^* \delta(z, p) \psi(z, p, d).
$$

Inequality (2.4) implies that, for $|z| \leq r$, the multiplicative functions $G_2(z, d, \kappa)$ and $\hat{g}(d)$ are related by the inequality

$$
|G_2(z, d, \kappa)| \leq \hat{g}(d).
$$

Taking exponent and logarithm, which is allowed by (2.3), in the routine way we obtain

$$
G_1(z, d, \kappa) = e^{L_g(z, \kappa)} G_3(z, d, \kappa).
$$

Here $G_3(z, d, \kappa)$ is analytic and bounded for $|z| \leq r$. Moreover, the assumptions of lemma allow us to assert that the function $L_g(z, \kappa)$ has an analytic continuation and is bounded in this domain. Thus we have that $G(z, d, \kappa)$ is analytic in the disc $|z| \leq r$, satisfies there the inequality

$$
|G(z, d, \kappa)| \ll \hat{g}(d),
$$

and

$$
G(1, d, \kappa) = \hat{g}(d) \prod_p \psi(1, p, a_0) \exp \left\{ -\kappa \left( \frac{1}{q^{\partial(p)}} \right) \right\}.
$$
From (1.1) it follows that \( W(z, G) \) is analytic in the disc \(|z| < q^{1-\mu}\). Moreover, it can be shown (see eg. [9]) that
\[
W(1, G) = \frac{A}{1 + H(\tilde{G})} \prod_p \left(1 - q^{-\partial(p)}\right) \exp \left\{ q^{-\partial(p)} \right\}.
\]

Therefore in virtue of (2.2), (2.5) and (2.6) we obtain
\[
F(z, d) = H(z, d, \kappa)(1 - z)^{-\kappa},
\]
where \( H(z, d, \kappa) \) is analytic and satisfies the inequality
\[
|H(z, d, \kappa)| \ll \hat{g}(d),
\]
when \(|z| \leq r_1 := \min(r, q^{(1-\mu)/2})\). Moreover,
\[
(2.7) \quad H(1, d, \kappa) = A\kappa L(\kappa)\hat{g}(d).
\]
Thus with the sole exception at point \( z = 1 \) for \(|z| \leq r_1 \) we have
\[
(2.8) \quad F(z, d) = H(1, d, \kappa)(1 - z)^{-\kappa} + O \left( \hat{g}(d)|1 - z|^{-\kappa} \right).
\]
According to Theorem 1 and Corollary 3 in [5] this estimate implies
\[
M_0(g, d)A = H(1, d, \kappa) \left( \frac{n + \kappa - 1}{n} \right) + O \left( \hat{g}(d)n^{\kappa-2} \right).
\]
Since \( M_0(g, d) = A^{-1}F(0, d) \ll \hat{g}(d) \) and
\[
\left( \frac{n + \kappa - 1}{n} \right) = \frac{(n + 1)^{-\kappa-1}}{\Gamma(\kappa)} \left( 1 + O \left( \frac{1}{n+1} \right) \right),
\]
the desired estimate follows from (2.8) and (2.7).

In addition, we provide some asymptotic formulas which will be useful in the sequel.

**Lemma 2.2** ([8] p. 86). Suppose that \( \sigma \in \mathbb{R} \). Then
\[
\sum_{m=1}^{n} m^\sigma q^m = \frac{q}{q-1} n^\sigma q^n + O \left( n^{\sigma-1} q^n \right).
\]
Lemma 2.3. For $0 \leq u \leq t \leq 1$, $n \geq 1$, $\gamma > 0$ and $\delta > 0$, we have
\[ \sum_{nu<k\leq nt} \frac{1}{(1+k)\gamma (1+n-k)\delta} = n^{1-\gamma-\delta} I(u,t;\gamma,\delta,n^{-1}) + O(\rho_n(u,t;\gamma,\delta)), \]
where
\[ I(u,t;\gamma,\delta,\eta) := \int_u^t \frac{dv}{(\eta + v)^\gamma (\eta + 1 - v)\delta}. \]
Moreover,
\[ I(u,t;\gamma,\delta,\eta) = I(u,t;\gamma,\delta,0) + O (\rho_n(u,t;\gamma,\delta)). \]

Proof. The first formula in this lemma follows from Euler-Maclaurin summation formula. The relations (2.9) and (2.10) follow from the definition of the integral $I(u,t;\gamma,\delta,\eta)$ by straightforward estimations (see, eg. in [2]).

3. Proof of Theorem 1.1

Assumptions of the theorem imply, that the multiplicative functions $1/T(a) \in \mathfrak{M}(\alpha,1,c_1)$ and $f(a)/T(a) \in \mathfrak{M}(\beta,1,c_2)$ with some positive constants $c_1$ and $c_2$. We have
\[ F_n(t) = S_n(t) + R_n(t), \]
where
\[ S_n(t) := q - 1 \sum_{0 \leq m \leq n} \sum_0^\infty T(a,nt) T(a), \]
\[ R_n(0) = 0 \quad \text{and} \quad R_n(t) \ll q^{-n} \sum_{\partial(d) \leq nt} f(d) \sum_{k \leq \partial(d)(1-t)/t} \sum_0^\infty \frac{1}{T(ad)}, \quad t \in (0,1]. \]
To evaluate the most inner sum we apply Lemma 2.1 with $g(a) = g_0(a) := 1/T(a)$. We have
\[ \sum_{\partial(a)=k} \frac{1}{T(ad)} = \frac{A^n q^k}{(1+k)^\beta} \left( \frac{L_0(\alpha)\tilde{g}_0(d)}{\Gamma(\alpha)} + O \left( \frac{\tilde{g}_0(d)}{1+k} \right) \right). \]
Here \( L_0(\alpha) \) and multiplicative functions \( \tilde{g}_0, \hat{g}_0 \) are defined in (2.1) by setting \( g = g_0 \). An easy calculation shows that

\[
\tilde{g}_0(p^m) = g_0(p^m) \left(1 + O\left(q^{-\vartheta(p)}\right)\right),
\hat{g}_0(p^m) = g_0(p^m) \left(1 + O\left(q^{-2\vartheta(p)}\right)\right).
\]

Thus

\[
R_n(t) \ll q^{-n} \sum_{\vartheta(d) \leq nt}^* f(d) \tilde{g}_0(d) \sum_{k \leq \vartheta(d)(1-t)/t} \frac{q^k}{(1+k)^\beta}.
\]

By the Lemma 2.2

\[
R_n(t) \ll \frac{q^{-nt}}{(1+n(1-t))^\beta} \sum_{m \leq nt} \sum_{\vartheta(a) = m}^* f(a) \hat{g}_0(a).
\]

Since \( f(a) \hat{g}_0(a) \in \mathcal{M}(\beta, C_1, c_3) \) with some positive \( C_1 \) and \( c_3 \), for the inner sum we can apply Lemma 2.1 by setting \( g = f \cdot \hat{g}_0 \) and \( d = a_0 \). Then employing Lemma 2.2 again we obtain that

\[
R_n(t) \ll (1+n(1-t))^{-\beta}(1+nt)^{-\alpha},
\]

for all \( 0 \leq t \leq 1 \). Setting \( S_n(u,t) := S_n(t) - S_n(u) \), from this and (3.1) we deduce

\[
F_n(t) - F_n(u) = S_n(u,t) + O(\rho_n(u,t;\alpha,\beta)).
\]

It remains to evaluate the sum \( S_n(u,t) \). Changing order of summation we have

\[
S_n(u,t) = \frac{q^{-1}}{Aq^{n+1}} \sum_{nu < \vartheta(d) \leq nt} f(d) \sum_{m=0}^{n-\vartheta(d)} \sum_{\vartheta(a) = m}^* \frac{1}{T(ad)}.
\]

Since \( \tilde{g}_0(a) \leq \hat{g}_0(a) \), applying (3.2) and Lemma 2.2 we get

\[
S_n(u,t) = S_1(n;u,t) + O(R_1(n;u,t)),
\]

where

\[
S_1(n;u,t) := \frac{L_0(\alpha)A^{-\beta}}{\Gamma(\alpha)} \sum_{nu < m \leq nt} \frac{q^{-m}}{(1+n-m)^{\beta}} \sum_{\vartheta(d) = m}^* f(d) \tilde{g}_0(d)
\]

and

\[
R_1(n;u,t) := \sum_{nu < m \leq nt} \frac{q^{-m}}{(1+n-m)^{\beta+1}} \sum_{\vartheta(d) = m}^* f(d) \hat{g}_0(d).
\]
It is easy to see, that \( f(a) \tilde{g}_0(a) \in M(\beta, C_2, c_4) \) with some positive \( C_2 \) and \( c_4 \). Therefore the inner sums in the expressions of \( S_1(n; u, t) \) and \( R_1(n; u, t) \) we can evaluate by means of Lemma 2.1. So we have

\[
S_n(u, t) = \frac{L_0(\alpha)L_1(\beta)}{\Gamma(\alpha)\Gamma(\beta)} S(\alpha, \beta) + O(S(\alpha + 1, \beta) + S(\alpha, \beta + 1)),
\]

where

\[
S(\gamma, \delta) := \sum_{nu \leq j \leq nt} \frac{1}{(1 + n - j)^\delta(1 + n)^\gamma}
\]

for short. Now Lemma 2.3 implies

\[
S_n(u, t) = \frac{L_0(\alpha)L_1(\beta)}{\Gamma(\alpha)\Gamma(\beta)} I(u, t; \alpha, \beta, 0) + O(\rho_n(u, t; \alpha, \beta)).
\]

Here \( L_0(\alpha) \) and \( L_1(\beta) \) are defined in (2.1) by setting \( g = g_0 \) and \( g = f \cdot \tilde{g}_0 \) respectively. The routine calculation yields that \( L_0(\alpha) \cdot L_1(\beta) = 1 \) (see, e.g. [1, 2]). Finally the proof of the theorem follows from (3.4) and (3.3).

The estimate (1.2) is an easy consequence of Theorem 1.1 with \( u = 0 \), since \( F_n(0) \ll n^{-\beta} \) by (3.2) and Lemma 2.2.

References


G. Bareikis and A. Mačiulis
Vilnius University
Vilnius
Lithuania
gintautas.bareikis@mif.vu.lt
algirdas.maciulis@mif.vu.lt