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ON RANDOM ARITHMETICAL FUNCTIONS II.

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Abstract. Mean values of random multiplicative functions over polynomial values, and the mean values of random multiplicative functions defined on the set of Gaussian integers will be investigated.

1. Introduction

1.1.

This paper is continuation of [1]. The method we use is similar but somewhat more complicated.

1.2.

Let $\mathcal{P}$ be the set of prime numbers, the letters p with and without indices always denote prime numbers. Let $\mathcal{M}^*$ be the set of completely multiplicative functions. A function $f : \mathbb{N} \to \mathbb{C}$ belongs to $\mathcal{M}^*$ if $f(1) = 1$ and $f(nm) = f(n) \cdot f(m)$. Let $\tau(n)$ be the number of divisors of n, and $\tau_k(n)$ be the number of those positive integers $d_1, \ldots, d_k$ for which $n = d_1 \ldots d_k$.

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Let $\varrho(n)$ be the number of solutions of the congruence $x^2+1 \equiv 0 \pmod{n}$. It is clear that $\varrho(n)$ is a multiplicative function.

$$\varrho(p^\alpha) = 0 \quad \text{if} \quad p \equiv -1 \pmod 4 \quad (\alpha = 1, 2, \dots),$$

$$\varrho(p^\alpha) = 2 \ \text{ if } \ p \equiv 1 \pmod 4 \quad \text{and} \quad \varrho(2^\alpha) = \alpha \geq 2.$$

Let $\tau(n)$ be the number of solutions f the equation $n = u^2 + v^2, \ u, v \in \mathbb{Z}$.

1.3.

Let G be the set of Gaussian integers, i.e. $G = \{u + iv | u, v \in \mathbb{Z}\}$. Let G^* be the multiplicative semigroup defined over G, that is $G^* = G \setminus \{0\}$. Let I be the set of units in G^*, i.e. $I = \{1, -1, i, -i\}$. We say that α_1 and α_2 are associates if $\alpha_1 = \varepsilon\alpha_2$ with some $\varepsilon \in I$. Let furthermore G_+^* be the set of those $\alpha \in G^*$ for which $Re\ \alpha \geq 0$ and $Im\ \alpha > 0$. It is clear that

(1) if $\alpha, \beta \in G_+^*$, then $\alpha\beta \in G_+^*$,

(2) if $\gamma \in G^*$, then there is a unique $\varepsilon \in I$, such that $\varepsilon\gamma \in G_+^*$.

Let $\tilde{\mathcal{P}}$ be the set of primes in G^*. A general prime element is denoted by π. It is known that:

(1) if $p \in \mathcal{P}, \ p \equiv 3 \pmod 4$, then $p \in \tilde{\mathcal{P}}$,

(2) $1 + i \in \tilde{\mathcal{P}}$,

(3) if $p \equiv 1 \pmod 4$, $\quad p = u^2 + v^2$, then $u + iv \in \tilde{\mathcal{P}}$,

(4) the associates of the numbers listed in (1), (2), (3) belong to $\tilde{\mathcal{P}}$,

(5) all elements of $\tilde{\mathcal{P}}$ are listed in (1), (2), (3), (4).

Let $\tilde{\mathcal{P}_+}$ be the set of those primes which belong to G_+^*. One can see that every $\alpha \in G_+^*$ can be uniquely written as the product of primes $\pi_1, \dots, \pi_k$ where $\pi_l \in G_+^*$.

Let $\tilde{\mathcal{M}}^*$ be the set of completely multiplicative functions over G^*.

We shall say that $f : G \to \mathbb{C}$ belongs to $\mathcal{M}^*$, if $f(\varepsilon) = 1 \quad (\varepsilon \in I)$, and $f(\alpha\beta) = f(\alpha) \cdot f(\beta)$ holds for every $\alpha, \beta \in G^*$.

Let $T_k(\alpha) \quad (\alpha \in G^*)$ be defined as follows.

For $\alpha \in G_+^*$ let $T_k(\alpha)$ be the number of solutions of the equation $\alpha = \chi_1 \dots \chi_k$ where $\chi_1, \dots, \chi_k \in G_+^*$. Furthermore let $T_k(\varepsilon) = 1$ (if $\varepsilon \in I$) and for an arbitrary $\beta \in G^*$ let $T_k(\beta) = T_k(\varepsilon\beta)$, where ε is that element in I

for which $\varepsilon\beta \in G_+^*$. It is clear that T_k is a multiplicative function. If π is a non-rational prime, i.e.

$$\pi\bar{\pi} = p, \quad p = 2 \quad \text{or} \quad p \equiv 1 \pmod 4, \text{then}$$

$$T_k(\pi^l) = \tau_k(p^l), \quad \text{and if} \quad \pi = p(\equiv 3 \pmod 4),$$

then $T_k(\pi^l) = \pi_k(p^l)$.

1.4.

Let $Q \geq 2$ be an integer, $A_Q = \{\kappa | \kappa^Q = 1\}$ = group of complex unit roots of order Q. Let $(\Omega, \mathcal{A}, P)$ be a probability space. Let $\xi_p \quad (p \in \tilde{\mathcal{P}_+})$ be a sequence of independent random variables distributed as follows: $P(\xi_p \equiv \kappa) = = \frac{1}{Q} \quad (\kappa \in A_Q)$.

We define the random multiplicative function $f \in \tilde{\mathcal{M}}^*$ by $f(\pi|\omega) = f(\pi) = = \xi_\pi \quad (\pi \in \tilde{\mathcal{P}}_+)$ and investigate the sum

$$\sum_{\substack{\alpha \in G_+^* \\ |\alpha| \leq r}} f(\alpha)h(\alpha),$$

where $h(\alpha) \quad (\alpha \in G^*)$ is an arbitrary complex function satisfying $|h(\alpha)| \leq 1$ (see Theorem 2 in §4). In §5 (see Theorem 3) we count the number of those $\alpha \in G_+^*$, for which $|\alpha| \leq r$, and $f(\alpha + \beta_j) = \kappa_j \quad (j = 1, \ldots, k), \quad \kappa_j \in \in A_Q$, $\beta_1, \ldots, \beta_k$ are distinct elements of G^*.

2. Lemmas

2.1.

Lemma 1. [Borel-Cantelli] *Let $A_1, A_2, \ldots$ be an infinite sequence of sets in $(\Omega, \mathcal{A}, P)$ and let $\sum_j P(A_j) < \infty$. Then almost all $\omega \in \Omega$ are belonging to finitely many A_i only.*

This is a wellknown assertion, see e.g. in [2].

2.2.

Lemma 2. *Let $a \geq 1$ be a square-free integer, $x \geq 2$. Let $N(x|a)$ be the number of solutions of $n^2 - am^2 = -1$ in integers $n, m \in \mathbb{N}$ such that $n \leq x$. Then $N(x|a) \leq c \log x$, c is an absolute constant.*

This lemma is wellknown. Despite of it, we shall give a short proof for it. Let us consider the Pell-equation $U^2 - aV^2 = 1$, and let U_0, V_0 be the smallest positive solution pair of it. It is known that all the other positive solutions U_l, V_l can be computed from $U_l + \sqrt{a}V_l = (U_0 + \sqrt{a}V_0)^l \quad (l = 1, 2, \ldots)$. Since $U_0 \geq 2, V_0 \geq 2$, therefore $U_l \geq 2^l$ $(l = 1, 2, \ldots)$.

Let (n_1, m_1) be the smallest positive solution of $n^2 - am^2 = -1$, and (n_2, m_2) be another positive solution such that $n_2 \leq x$. We have

$$(n_1 - \sqrt{a}m_1)(n_1 + \sqrt{a}m_1) = -1, \quad (n_2 - \sqrt{a}m_2)(n_2 + \sqrt{a}m_2) = -1.$$

Multiplying these equations we obtain that $U^2 - aV^2 = 1$, where

$$U = n_1 n_2 + a m_1 m_2, \quad V = n_1 m_2 - u_2 m_1 \ (> 0).$$

Thus $(U, V) = (U_l, V_l)$ for some l, $U \leq 3x^2$, thus $2^l \leq 3x^2$, $\quad l \leq \frac{1}{\log 2} \log 3x^2 \leq c \log x$. The lemma is proved.

3. The mean value of random multiplicative function over $n^2 + 1$

Let $f(n) = f(n|\omega)$ be defined as in §1.4. Let

$$S_N(\omega|h) := \sum_{n \leq N} f(n^2 + 1) h(n^2 + 1).$$

Theorem 1. *The following relations hold with probability 1:*

$$\lim_{N \to \infty} \frac{S_N(\omega|h)}{N^{\frac{3}{4}} (\log N)^2} = 0, \tag{3.1}$$

$$\lim_{N \to \infty} \frac{1}{N^{\frac{3}{4}} (\log N)^2} = \sum_{n \leq N} f(n^2 + 1) = 0, \tag{3.2}$$

$$\lim_{N \to \infty} \frac{1}{N^{\frac{3}{4}} (\log N)^2} = \sum_{p \leq N} f(p^2 + l) = 0. \tag{3.3}$$

Proof. (3.2), (3.3) are special cases of (3.1), by choosing $h(n^2+1) = 1$ ($n \in \mathbb{N}$), and by choosing

$$h(n^2+1) = \begin{cases} 1 & \text{if } n \in \mathcal{P}, \\ 0 & \text{otherwise.} \end{cases}$$

We shall prove (3.1). This is an easy consequence of

Lemma 3. *Let $N \geq 2$. Then $ES_N(\omega|h) = 0$, $E\left|S_N(\omega|h)\right|^2 \leq cN \log N$.*

First we deduce (3.1) from Lemma 3. Let

$$\lambda_N = \frac{1}{\log\log N}, \qquad T_N = \frac{S_N(\omega|h)}{N^{\frac{3}{4}}(\log N)^2}.$$

We have

$$\begin{aligned} P(|T_N| > \lambda_N) &\leq \int \frac{1}{\lambda_N} \frac{|S_N(\omega|h)|^2}{N^{\frac{3}{2}}(\log N)^4} dP \leq \\ &\leq \frac{cN(\log N)\log\log N}{N^{\frac{3}{2}}(\log N)^4} = \frac{c(\log\log N)}{N^{\frac{1}{2}}(\log N)^3}. \end{aligned} \tag{3.4}$$

Let now N run over $N_m = m^2$ ($m = 1, 2, \ldots$), $A_m := \{\omega \big| |T_N| > \lambda_{N_m}\}$. From (3.4), and Lemma 1 we obtain that

$$\lim T_{N_m} = 0. \tag{3.5}$$

Let $N_m \leq N < N_{m+1}$. Since $|T_N| \leq |T_{N_m}| + |T_N - T_{N_m}|$, and

$$|T_N - T_{N_m}| \leq \frac{|S_N(\omega|h) - S_{N_m}(\omega|h)|}{N_m^{\frac{3}{4}}(\log N_m)^2} \leq \frac{c}{m^{\frac{1}{2}}} \to 0 \qquad (m \to 0),$$

we obtain (3.1).

Finally we prove Lemma 3. It is clear that $Ef(n^2+1) = 0$ for every $n \geq 1$, since n^2+1 cannot be a square. Therefore $ES_N(\omega, h) = 0$.

We have

$$E|S_N(\omega|h)|^2 \leq \sum_{n_1, n_2 \leq N} E(f(n_1^2+1)f(n_2^2+1)).$$

A summand on the right hand side can be different from zero, and in that case it equals to 1, if there is a square-free a such that $n_1^2+1=am_1^2,\ n_2^2+1=am_2^2$.

Let n_1 be run over the integers $1,2,\dots,N$. For every n_1 the number of possible $n_2 \le N$ with the same a is at most $c\log N$ (see Lemma 2), therefore Lemma 3 is true.

Remark. We can prove similar theorems for quadratic irreducible polynomial $P(x) \in \mathbb{Z}[x]$ instead of x^2+1. Perhaps analogous result holds for polynomials $P(x)$ the degree of which is larger than 2. We hope to return to this question in another paper.

4. Mean values of random multiplicative functions over the Gaussian integers

We shall keep the notations defined in §1.4.

Let $D_r = \{\alpha | \alpha \in G_+^*, |\alpha| < r\}$

$$T(r) = T(r|\omega) = \sum_{\alpha \in D_r} f(\alpha)h(\alpha).$$

For some $\beta \in G$ let γ^Q be the "largest" Q'th power divisor of β, such that $\gamma \in G_+^*$. The largest means that if $\gamma_1^Q|\beta,\ \gamma_1 \in G_+^*$, then $\gamma_1|\gamma$.

Let $a(\beta)$ be defined by $\frac{\beta}{\gamma^Q}$. It is clear that $a(\beta) \in G_+^*$.

It is clear that for $\beta, \beta_1, \beta_2 \in G_+^*$:

$$Ef(\beta) = \begin{cases} 1 & \text{if} \quad a(\beta) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$Ef(\beta_1)f(\beta_2) = \begin{cases} 1 & \text{if} \quad a(\beta_1) = a(\beta_2), \\ 0 & \text{otherwise.} \end{cases}$$

Hence we obtain that

$$E|T(r)|^{2k} = \sum_{a(\alpha_1 \dots \alpha_k) = a(\beta_1 \dots \beta_k)} h(\alpha_1)\dots h(\alpha_k)\bar{h}(\beta_1)\dots\bar{h}(\beta_k).$$

Here $\alpha_1, \dots, \alpha_k,\ \beta_1, \dots, \beta_k$ run over D_r. Let us write $A_j = a(\alpha_j),\ \alpha_j = A_j \gamma_j^Q,\ A_j \in G_+^*,\ \gamma_j \in G_+^*$.

We have $|\gamma_j|^Q \le \frac{|\alpha_j|}{|A_j|} \le \frac{r}{|A_j|}$, consequently for fixed A_j the number of γ, for which $A_j \gamma_j^Q \in D_r$ holds in less than $c\left(\frac{r}{|A_j|}\right)^{\frac{2}{Q}}$, where c is absolute positive constant. Consequently,

$$E|T(r)|^{2k} \le cT^{\frac{4k}{Q}} \Sigma^* \frac{1}{|A_1 \dots A_k| \cdot |B_1 \dots B_k|}, \tag{4.1}$$

where $*$ on the right hand side of (4.1) means that we have to sum over those Q-free Gaussian integers $A_1, \dots, A_k,\ B_1, \dots, B_k$ for which $|A_j| \le r,\ |B_j| \le r$, and

$$a(A_1 \dots A_k) = a(B_1 \dots B_k).$$

Let us write $A_1 \dots A_k = D \cdot e^Q$, where D is Q-free, $|e^Q D| \le r^k$. For fixed D and e the number $A_1, \dots, A_k$ satisfying $A_1 \dots A_k = De^a$ is no more than the number of possible solutions of $D = c_1 \dots c_k \quad (c_j \in G_+^*),\ e^Q = \nu_1 \dots \nu_k \quad (\nu_j \in G_+^*)$. Thus for fixed D and e we have $T_k(D) \cdot T_k(e^Q)$ solutions. Since $|D| \le r^k,\ |e^Q| \le r^k;\ D, e^Q \in G_+^*$, we obtain that

$$E|T(r)|^{2k} \le cr^{\frac{4k}{Q}} \Sigma_1 \cdot \Sigma_2^2,$$

where

$$\Sigma_1 = \sum_{\substack{|D| \le r^k \\ D \in G_+^*}} \frac{T_k^2(D)}{|D|^{\frac{4}{Q}}}; \qquad \Sigma_2 = \sum_{\substack{|e^Q| \le r^k \\ e \in G_+^*}} \frac{T_k(e^Q)}{|e^Q|^{\frac{2}{Q}}}.$$

To estimate Σ_1, Σ_2, we observe that $|D|^2 = n$ holds for $\frac{r(n)}{4}$ integers $D \in G_+^*$, where $r(n)$ is defined in §1.1. Thus

$$\Sigma_1 \le \sum_{n \le r^{2k}} \frac{\tau_k^2(n) r(n)}{n^{\frac{2}{Q}}}, \qquad \Sigma_2 \le \sum_{n \le r^{2k}} \frac{\tau_k(n^Q) r(n)}{n}.$$

By using routine estimates in number theory we obtain that

$$\sum_{n \le x} \tau_k^2(n) r(n) \le cx (\log x)^{d(k)},$$

$$\sum_{n \le x} \tau_k(n^Q) r(n) \le cx (\log x)^{d(k)}$$

with some suitable positive constants $d(k)$, and c, therefore

$$\Sigma_2 \le c_1(k)(\log\log r)\log r,$$

furthermore

$$\Sigma_1 \le c_2(k)(\log\log r)\log r, \qquad \text{if } Q = 2,$$

and

$$\Sigma_1 \le c_3(k)(\log r)^{d(k)}(r^{2k})^{1-\frac{2}{Q}}.$$

We proved

Lemma 4. *Let $k \ge 1$ be an arbitrary integer. Then there are positive numbers $c(k), d(k)$ for which*

$$E\left|T(r|\omega)\right|^{2k} \le c(k) r^{2k}(\log r)^{d(k)}, \tag{4.2}$$

if $r \ge 2$.

Hence we obtain

Theorem 2. *Let $\epsilon > 0$ be an arbitrary small constant. Then*

$$\lim_{r\to\infty}\frac{T(r,\omega)}{r^{1+\varepsilon}} = 0$$

with probability 1.

Indeed, let k be so large that $k\varepsilon > 1$. From (4.2) with $\lambda_r = \frac{1}{\log\log r}$ $(r \ge 4)$ we have

$$P\left(\left|\frac{T(r,\omega)}{r^{1+\varepsilon}}\right| \ge \lambda_r\right) \le \frac{1}{\lambda_r^{2k}}\int\left|\frac{T(r,\omega)}{r^{1+\varepsilon}}\right|^{2k} dP \le$$
$$\le (\log\log r)^{2k}\frac{1}{r^{2k\varepsilon}}.$$

Let us apply this for $r = n$ $(n = 4, 5, \ldots)$ and use the Borel-Cantelli lemma. We obtain that

$$\frac{T(n,\omega)}{n^{1+\varepsilon}} \to 0 \qquad (n \to \infty, \quad n \in \mathbb{N}).$$

Finally we observe that if $n \le r < n+1$, then the number of Gaussian integers α in the ring $n \le |\alpha| < r$ is bounded by cn an $n \to \infty$, therefore (4.3) is true.

5. On random subset of the Gaussian integers defined by the values of random multiplicative functions

Let us keep the notation used earlier. Let ξ_π be independent random variables, $P(\xi_\pi = \kappa) = \frac{1}{Q}$ $(\kappa \in A_Q)$. Let $\beta_1, \dots, \beta_k$ be fixed distinct Gaussian integers. Let $f(\alpha|\omega) \in \mathcal{M}^*$ defined on the set of $\tilde{\mathcal{P}}_+$ by $f(\pi) = \xi_\pi$. Let

$$S := \{\alpha \mid \alpha + \beta_j \in \mathcal{G}_+^*,\ j = 1, \dots, k\},$$

$\kappa_1, \dots, \kappa_k$ be fixed elements of A_Q,

$$\Delta := \{\alpha | \alpha + \beta_j \in \mathcal{G}_+^*,\ f(\alpha + \beta_j) = \kappa_j,\ j = 1, \dots, k\}.$$

Let $h(\alpha)$ be a complex valued function defined on S, such that $|h(\alpha)| \le 1$. Let

$$R(r) := \sum_{\substack{\alpha \in S \\ |\alpha| \le r}} h(\alpha), \qquad R(r|\Delta) := \sum_{\substack{\alpha \in S \\ |\alpha| \le r \\ \alpha \in \Delta}} h(d).$$

Let

$$\Lambda(r) = \left| R(r|\Delta) - \frac{R(r)}{Q^k} \right|.$$

We shall prove

Theorem 3. *Let ε be an arbitrary constant. Then with probability 1,*

$$\lim_{r \to 0} \frac{\Lambda(r)}{r^{5/3+\varepsilon}} = 0.$$

Let $u_\kappa(x) = \frac{x^Q - 1}{x - \kappa}$ be defined for every $\kappa \in A_Q$. Easy to see that $u_\kappa(\kappa) = Q\bar{\kappa}$, and $u_\kappa(\lambda) = 0$ if $\lambda \neq \kappa, \lambda \in A_Q$.

Let $\alpha \in S$,

$$\Delta(\alpha) := u_{\kappa_1}(f(\alpha + \beta_1)) \cdots u_{\kappa_k}(f(\alpha + \beta_k)).$$

Then

$$\Delta(\alpha) = \begin{cases} Q^k \bar{\kappa}_1 \cdots \bar{\kappa}_k & \text{if } \alpha \in \Delta, \\ 0 & \text{otherwise.} \end{cases}$$

We can write $\Delta(\alpha)$ as a polynomial of $f(\alpha + \beta_1), \dots, f(\alpha + \beta_k)$; the degree in each variable is limited in $Q - 1$, and the coefficients of which do depend only on Q and k.

Let

$$s(\alpha) = \sum_{l_1=0}^{Q-1} \cdots \sum_{l_k=0}^{Q-1} d(l_1,\dots,l_k) f(\alpha+\beta_1)^{l_1} \cdots f(\alpha+\beta_k)^{l_k} =$$
$$= d(0,\dots,0) + \sum_{(l_1,\dots,l_k)\neq(0,\dots,0)} d(l_1,\dots,l_k) f(\alpha+\beta_1)^{l_1} \cdots f(\alpha+\beta_k)^{l_k}.$$

Then

$$\sum_{\substack{\alpha\in S\\ |\alpha|\le r}} h(\alpha)s(\alpha) = Q^k \bar{\kappa_1}\cdots\bar{\kappa_k} R(r|\Delta) =$$
$$= \bar{\kappa_1}\cdots\bar{\kappa_k} R(r) + \sum_{(l_1,\dots,l_k)\neq(0,\dots,0)} d(l_1,\dots,l_k) X_{l_1,\dots,l_k}(r),$$

where

$$X_{l_1,\dots,l_k}(r) = \sum_{\substack{\alpha\in S\\ |\alpha|\le r}} h(\alpha) f(\alpha+\beta_1)^{l_1} \cdots f(\alpha+\beta_k)^{l_k}.$$

We shall estimate $E|X_{l_1,\dots,l_k}(r)|^4$ for $(l_1,\dots,l_k)\neq(0,\dots,0)$.

Let us write $(\alpha+\beta_1)^{l_1}\cdots(\alpha+\beta_k)^{l_k}$ in the form $(\alpha+\beta_{j_1})^{m_1}\cdots(\alpha+\beta_{j_t})^{m_t}$, where $j_1,\dots,j_t$ is a non-empty subset of $\{1,\dots,k\}$, and $1\le m_j\le Q-1$. Let $L(\alpha) = (\alpha+\beta_{j_1})^{m_1}\cdots(\alpha+\beta_{j_t})^{m_t}$.

We shall write every γ as $a(\gamma)\cdot m(\gamma)^Q$, where $m(\gamma)^Q$ is the largest Q-th power divisor of Q and $a(\gamma)$ is Q-free. It is assumed that $\gamma\in\mathcal{G}_+^*$, $m(\gamma)\in\in\mathcal{G}_+^*$, $a\in\mathcal{G}_+^*$. It is clear that, if $\gamma_1,\gamma_2\in\mathcal{G}_+^*$, then

$$Ef(\gamma_1)\bar{f}(\gamma_2) = \begin{cases} 1 & \text{if } a(\gamma_1)=a(\gamma_2),\\ 0 & \text{otherwise.}\end{cases}$$

We shall write that $\gamma_1\sim\gamma_2$ if $a(\gamma_1)=a(\gamma_2)$.

Assume first that $t\ge 2$. We can write

$$X_{l_1,\dots,l_k} = \sum_{\substack{\alpha\in S\\ |\alpha|\le r}} h(d) f(L(\alpha)) =$$
$$= \sum_{j\ge 0}^{2^j\le r} \Theta\left(\frac{r}{2^i}\right),$$

where

$$\Theta(r) = \sum_{\substack{\alpha \in S \\ \frac{r}{2} < |\alpha| \le r}} h(d) f(L(\alpha)).$$

To estimate $\Theta(r)$ we denote by $\mathcal{N}_2$ the set of those integers $\alpha \in S$ for which there exists a squareful Gaussian integer $\mu \in G_+^*$, for which $|\mu| > \sqrt{r}/(\log r)^2$, and $\mu|(\alpha + \beta_u)$ for at least one $u \in \{j_1, \dots, j_t\}$, and let $\mathcal{N}_1$ be the set of those α which do not belong to $\mathcal{N}_2$.

$\gamma|\alpha + \beta_u$ implies that $\gamma\delta = \alpha + \beta_u$, $\frac{r}{2} - |\beta_u| \le |\gamma\delta| \le r + |\beta_u|$, thus the number of possible $\delta \in G^*$ is less than $c\left(\frac{r}{|\gamma|}\right)^2$, and so

$$\begin{aligned} \#\{N_2\} &\le c \sum_{|\gamma| \ge \sqrt{r}/(\log r)^2} \left(\frac{r}{|\gamma|}\right)^2 \le \\ &\le cr^2 \sum_{\substack{n > r/(\log r)^2 \\ n \text{square-full}}} \frac{r(n)}{n} \le \\ &\le cr^{3/2} (\log r)^3. \end{aligned}$$

Let

$$\Theta_1(r) = \sum_{\substack{\alpha \in S \\ \frac{r}{2} < |\alpha| \le r \\ \alpha \in \mathcal{N}_1}} h(\alpha) f(L(\alpha)).$$

We proved that

$$\Theta(r) = \Theta_1(r) + O(r^{3/2} (\log r)^3).$$

Applying the Cauchy-Schwarz inequality, we obtain that

$$(5.1) \qquad |X_{r_1,\dots,r_k}|^4 \le (\log r)^3 \sum \left|\Theta_1\left(\frac{r}{2^j}\right)\right|^4 + O(r^6 (\log r)^{12}).$$

Let $Ef(L(\alpha_1))f(L(\alpha_2))\bar{f}(L(\alpha_3))\bar{f}(L(\alpha_4)) \ne 0$ (and then $= 1$). It holds if and only if $a(L(\alpha_1)L(\alpha_2)) = a(L(\alpha_3)L(\alpha_4))$.

Let $H(E)$ be the number of those $\alpha_1, \alpha_2 \in \mathcal{N}_1$, $\frac{r}{2} \le |\alpha_2| \le r$ for which $e(L(\alpha_1)L(\alpha_2)) = E$.

It is clear that

$$E|\Theta_1(r)|^4 \le \sum H^2(E) \le \max H(E) \sum H(E).$$

Since $\sum H(E)$ is clearly $\le \#\{\alpha_1, \alpha_2 \in \mathcal{N}_1\} \le cr^4$, we have

$$E|(\Theta_1(r)|^4 \le cr^4 \max H(E).$$

Let us estimate $H(E)$. For a general Q-free integer A let $G(A)$ be the number of those $\alpha \in \mathcal{N}_1$ for which $L(\alpha) = AY^Q$ with some suitable integer A.

Let

$$\alpha + \beta_{i_{l_1}} = R_{l_1} C_{l_1} M_{l_1}, \quad \alpha + \beta_{i_{l_2}} = R_{l_2} C_{l_2} M_{l_2},$$

$$\vdots$$

$$\alpha + \beta_{i_{l_t}} = R_{l_t} C_{l_t} M_{l_t}$$

where $R_{l_j} C_{l_j}$ is the square-free part of $\alpha + \beta_{i_{l_j}}$, the prime divisors π in $\prod R_{l_j}$ satisfy $|\pi| \le K$, and the prime divisors ρ of $\prod C_{l_j}$ are such that $|\rho| > K$, where

$$K = \max_{u \ne v} |\beta_u - \beta_v|.$$

It is clear that $(C_{l_i}, C_{l_j}) = 1$ if $l_i \ne l_j$. Then

$$L(\alpha) = C_{l_1}^{m_1} \cdots C_{l_t}^{m_t} \nu, \quad (\nu, C_{l_1} \cdots C_{l_t}) = 1.$$

Since C_{l_j} are coprime square-free numbers, $m_\nu < Q$, therefore $C_{l_1}^{m_1} \cdots C_{l_t}^{m_t}$ is a divisor of A. Observe that R_{l_ν} are bounded, $M_{l_j} < r^{1/2}/(\log r)^{1/2}$, therefore

$$\frac{r}{2} - |\beta_{l_{i_j}}| \le |\alpha + \beta_{l_{i_j}}| \le |C_{l_j}||R_{l_j}| r^{1/2} (\log r)^{-2}$$

whence we obtain that $|R_{l_j}| > \sqrt{r}(\log r)$ for every large r. It implies that

$$\alpha + \beta_{l_1} \equiv 0 \pmod{R)_{l_1}}, \quad \alpha + \beta_{l_2} \equiv 0 \pmod{R)_{l_2}}$$

has at most one solution α. Hence we obtain that $G(A) \le T_3(A) \le \tau_3(|A|^2)$. Furthermore we have that

$$H(E) = \sum_{E_1 E_2 = E} \sum_U G(E_1 U) G(E_2 V(U)),$$

where U runs over the Q-free integers, and if $U = \prod_{j=1}^{h} \pi_j^{u_j}$, then $V(U) = \prod \pi_j^{Q-u_j}$. Since $G(E_1 U) \le \tau_3(|E_1 U|^2)$, and

$$\sum_U G(E_2 V(U)) \le cr^2,$$

we obtain that $H(E) \le cr^{2+\varepsilon}$, where $\varepsilon > 0$ is an arbitrary constant, $c = c(\varepsilon)$.

We proved that

$$E|\Theta_1(r)|^4 \ll r^{6+\varepsilon}.$$

From (5.1) we obtain that

$$E|X_{l_1,\ldots,l_k}|^4 \ll r^{6+\varepsilon}.$$

Let us consider the case $t = 1$. We have to estimate a sum of type

$$Z(r) = \sum_{\substack{\alpha\in S\\ |\alpha|\le r}} h(\alpha) f(\alpha+\beta_l)^m,$$

where $1 \le m \le Q_1$. Defining $g(\gamma) := f(\alpha)^m$, g is a random multiplicative function $g(\pi) = \xi_p^r$. ξ_p^r takes the values of unit roots of order $\frac{Q}{(Q,m)} = Q_m$, each with probability $\frac{1}{Q_m}$. Since $m < Q$, therefore we can apply Lemma 4 and prove that $E|Z(r)|^4 \ll r^{4+\varepsilon}$.

Let

$$\Lambda(r) := \left|\frac{R(r)}{Q^k} - R(r|\Delta)\right|.$$

We proved that

$$E(|\Lambda(r)|^4) \le cr^{6+\varepsilon}.$$

This implies that

$$P(|\Lambda(r)| > r^\sigma) \le \int \frac{|\Lambda(r)|^4}{r^{4\sigma}} dP \le$$
$$\le cr^{6-4\sigma+\varepsilon}.$$

Let $N_m = m^3$, $\sigma = \frac{5}{3} + \varepsilon$. Then

$$\sum P(|\Lambda(N_m)| > N_m^\sigma) \le c \sum \frac{1}{m^{1+\varepsilon}} \le \infty.$$

From the Borel-Cantelli lemma we obtain that

$$\lim_{m\to\infty} \frac{\Lambda(N_m)}{N_m^{5/3+\varepsilon}} = 0.$$

Let $N_m \le r \le N_{m+1}$. Then

$$|\Lambda(r) - \Lambda(N_m)| \le \#\{\alpha | N_m \le |\alpha| \le N_{m+1}\} \le cm^5.$$

Since

$$\frac{|\Lambda(r)|}{r^{5/3+\varepsilon}} \le \frac{|\Lambda(N_m)|}{N_m^{5/3+\varepsilon}} + \frac{cm^5}{N_m^{5/3+\varepsilon}},$$

and the last summand tends to zero as $m \to \infty$, we obtain that

$$\lim_{r\to\infty} \frac{\Lambda(r)}{r^{5/3+\varepsilon}} = 0$$

holds for almost all ω. Thus Theorem 3 is true.

Remark. The assertions in Theorem 2, 3 remain valid, if we extend the summation for all $\alpha \in G^*$, $|\alpha| \le r$. This is clear since $f(\varepsilon\alpha) = f(\alpha)$ $(\varepsilon \in I)$ holds for the function $f \in \tilde{\mathcal{M}}^*$.

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