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SOME FURTHER RESULTS ON THE DEGREE OF APPROXIMATION OF CONTINUOUS FUNCTIONS

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Abstract. In this paper we establish some theorems on the degree of approximation of continuous functions by matrix means related to partial sums of a Fourier series, employing some known and new wider classes of null-sequences than those of Rest Bounded Variation Sequences or of Head Rest Bounded Variation Sequences. These new results give significantly better degrees than all results obtained previously by others.

1. Introduction

Let $f(x)$ be a 2π- periodic continuous function. Let $S_n(f;x)$ denote the n-th partial sum of its Fourier series at x and let $\omega(\delta) = \omega(\delta, f)$ denote the modulus of continuity of f.

Let $A := (a_{n,k})$ $(k, n = 0, 1, \ldots)$ be a lower triangular infinite matrix of real numbers and let the A-transform of $\{S_n(f;x)\}$ be given by

$$T_{n,A}(f) := T_{n,A}(f;x) := \sum_{k=0}^{n} a_{n,k} S_k(f;x) \quad (n = 0, 1, \ldots).$$

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The deviation $\|T_{n,A}(f) - f\|$ was estimated by P. Chandra [2] and [3] for monotonic sequences $\{a_{n,k}\}$, where $\|\cdot\|$ denotes the supnorm. Later on, these results are generalized by L. Leindler [4] who in his paper considered the sequences of Rest Bounded Variation and of Head Bounded Variation.

A sequence $\beta := \{c_n\}$ of nonnegative numbers tending to zero is called of Rest Bounded Variation, or briefly $\beta \in RBVS$, if it has the property

$$\sum_{n=m}^{\infty} |c_n - c_{n+1}| \leq K(\beta)c_m$$

for all natural numbers m, where $K(\beta)$ is a constant depending only on β.

A sequence $\beta := \{c_n\}$ of nonnegative numbers will be called of Head Bounded Variation, or briefly $\beta \in HBVS$, if it has the property

$$\sum_{n=0}^{m-1} |c_n - c_{n+1}| \leq K(\beta)c_m$$

for all natural numbers m, or only for all $m \leq N$ if the sequence β has only finite nonzero terms, and the last nonzero term is c_N.

Since Chandra's and Leindler's results are not connected directly to our results, here we shall not recall those. However we shall emphasize that some results on this topic are given recently by Leindler [6], and for interested reader we would like to mention that some generalizations of Leindler's results are made by present author in [7].

Very recently B. Wei and D. Yu [8] have generalized Leindler's results, and thus Chandra's results, without assumptions that $A \in RBVS$ or $A \in HBVS$. They verified there that Leindler's results are consequences of their results. Before we recall their results we shall suppose that

$$a_{n,k} \geq 0, \quad \sum_{k=0}^{n} a_{n,k} = 1, \tag{1.1}$$

and $\omega(t)$ is such that

$$\int_{u}^{\pi} t^{-2}\omega(t)dt = O(H(u)), \quad (u \to 0^+), \tag{1.2}$$

where $H(u) \geq 0$, and

$$\int_{0}^{t} H(u)du = O(tH(t)), \quad (t \to 0^+). \tag{1.3}$$

Using notation $\triangle a_{n,k} = a_{n,k} - a_{n,k+1}$ B. Wei and D. Yu's results read as follows:

Theorem 1.1. *Let (1.1) hold. Suppose that $\omega(t)$ satisfies (1.2). Then*

$$\|T_{n,A}(f) - f\| = O\Big(\omega(\pi/n) + \sum_{k=0}^{n} |\triangle a_{nk}| H(\pi/n)\Big).$$

If, in addition, $\omega(t)$ satisfies (1.3), then

$$\|T_{n,A}(f) - f\| = O\Big(\sum_{k=0}^{n} |\triangle a_{nk}| H\Big(\sum_{k=0}^{n} |\triangle a_{nk}|\Big)\Big),$$

$$\|T_{n,A}(f) - f\| = O\Big(\sum_{k=0}^{n} |\triangle a_{nk}| H(\pi/n)\Big).$$

Theorem 1.2 *Let $(a_{n,k})$ satisfies (1.1). Then*

$$\|T_{n,A}(f) - f\| = O\Big(\omega(\pi/n) + \sum_{k=1}^{n} k^{-1}\omega(\pi/k) \sum_{\mu=0}^{k+1} a_{n\mu} + \sum_{k=1}^{n} \omega(\pi/k) \sum_{\mu=k}^{n} |\triangle a_{n\mu}|\Big).$$

In 2009, B. Szal [1] introduced a new class of sequences as follows.

Definition 1.1. *A sequence $\alpha := \{c_k\}$ of nonnegative numbers tending to zero is called of Rest Bounded Second Variation, or briefly $\{c_k\} \in RBSVS$, if it has the property*

$$\sum_{k=m}^{\infty} |c_k - c_{k+2}| \le K(\alpha) c_m$$

for all natural numbers m, where $K(\alpha)$ is positive, depending only on sequence $\{c_k\}$, and we assume it to be bounded.

In his paper Szal showed the following inclusions

$$RBVS \subset RBSVS \subset AMS \quad \text{and} \quad RBVS \neq RBSVS.$$

For further investigations we introduce the following definition.

Definition 1.2. *A null sequence $\alpha := \{c_k\}$ of nonnegative numbers is called of Head Bounded Second Variation Sequence, briefly $\{c_k\} \in HBSVS$, if it has the property*

$$\sum_{k=0}^{m-1} |c_k - c_{k+2}| \le K(\alpha) c_m$$

for all natural numbers n, where $K(\alpha)$ depends only on sequence $\{c_k\}$.

Note that

$$\begin{aligned} K(\alpha)c_m \geq & \sum_{k=0}^{m-1} |c_k - c_{k+2}| \geq \\ \geq & |c_0 + c_1 - c_m - c_{m+1}| \geq \\ \geq & |c_m + c_{m+1}| - |c_0 + c_1| \geq \\ \geq & |c_m + c_{m+1}| \geq c_m + c_{m+1} \Longrightarrow \\ \Longrightarrow & c_{m+1} \leq K(\alpha)c_m \Longrightarrow \\ \Longrightarrow & \{c_k\} \in MS, \end{aligned}$$

so $HBSVS \subset MS$, where MS denotes the monotone decreasing null-sequences.

Moreover, we easy can show that

$$\sum_{k=0}^{m-1} |c_k - c_{k+2}| \leq (K(\alpha) + 3) \sum_{k=0}^{m-1} |c_k - c_{k+1}|,$$

which means that $HBVS \subset HBSVS$, but not conversely.

The present paper offers some new and significant estimations of the deviation $T_{n,A}(f) - f$ in the supnorm. Also, we shall show that results obtained previously by others are consequences of ours as a special case (see the Section 4).

We emphasize here that throughout of this paper we write $u = O(v)$, if there exists a positive constant C such that $u \leq Cv$, and all the constants $K(\cdot)$ are assumed to be positive and bounded.

2. Helpful lemmas

To prove the main results we need some auxiliary statements.

Lemma 2.1. ([2]) *If (1.2) and (1.3) hold then*

$$\int_0^{\pi/n} \omega(t)dt = O\left(n^{-2}H(\pi/n)\right).$$

Lemma 2.2. ([3]) *If (1.2) and (1.3) hold then*

$$\int_0^r t^{-1}\omega(t)dt = O\left(rH(r)\right) \quad (r \to +0).$$

Lemma 2.3. *For any lower triangular infinite matrix $(a_{n,k})$, $k, n = 0, 1, 2, \ldots$ of nonnegative numbers, it holds uniformly in $0 < t < \pi$, that*

$$\sum_{k=0}^{n} a_{n,k} \sin\left(k + \frac{1}{2}\right) t = O\left(\sum_{k=0}^{\tau} a_{nk} + \frac{1}{t(\pi - t)} \sum_{k=\tau}^{n} |a_{n,k} - a_{n,k+2}|\right), \tag{2.4}$$

where τ denotes the integer part of $\frac{\pi}{t}$.

It also holds that

$$\sum_{k=0}^{n} a_{n,k} \sin\left(k + \frac{1}{2}\right) t = O\left(\frac{1}{t(\pi - t)} \sum_{k=0}^{n} |a_{n,k} - a_{n,k+2}|\right). \tag{2.5}$$

Proof. For arbitrary $\lambda_n \geq 0$ and for $n \geq m \geq 0$ we have

$$\mathbb{B}_{m,n}(t) :=$$

$$:= \sum_{k=m}^{n} \lambda_k \cos\left(k + \frac{1}{2}\right) t =$$

$$= \frac{1}{2} \sum_{k=m}^{n} (\lambda_k + \lambda_{k+1}) \cos\left(k + \frac{1}{2}\right) t + \frac{1}{2} \sum_{k=m}^{n} (\lambda_k - \lambda_{k+1}) \cos\left(k + \frac{1}{2}\right) t,$$

and whence

$$\frac{1}{2}\mathbb{B}_{m,n}(t) =$$

$$= \frac{1}{2} \sum_{k=m}^{n} (\lambda_k + \lambda_{k+1}) \cos\left(k + \frac{1}{2}\right) t - \frac{1}{2} \sum_{k=m+1}^{n+1} \lambda_k \cos\left(k + \frac{1}{2} - 1\right) t =$$

$$= \frac{1}{2} \sum_{k=m}^{n} (\lambda_k + \lambda_{k+1}) \cos\left(k + \frac{1}{2}\right) t -$$

$$- \frac{1}{2} \sum_{k=m+1}^{n} \lambda_k \cos\left(k + \frac{1}{2} - 1\right) t - \frac{1}{2}\lambda_{n+1} \cos\left(n + \frac{1}{2}\right) t =$$

$$=\frac{1}{2}\sum_{k=m}^{n}(\lambda_k+\lambda_{k+1})\cos\left(k+\frac{1}{2}\right)t-$$

$$-\frac{1}{2}\cos t\sum_{k=m+1}^{n}\lambda_k\cos\left(k+\frac{1}{2}\right)t-\frac{1}{2}\sin t\sum_{k=m+1}^{n}\lambda_k\sin\left(k+\frac{1}{2}\right)t-$$

$$-\frac{1}{2}\lambda_{n+1}\cos\left(n+\frac{1}{2}\right)t$$

or

$$\frac{1+\cos t}{2}\mathbb{B}_{m+1,n}(t)=$$

$$=\frac{1}{2}\sum_{k=m}^{n}(\lambda_k+\lambda_{k+1})\cos\left(k+\frac{1}{2}\right)t-\frac{1}{2}\sin t\sum_{k=m+1}^{n}\lambda_k\sin\left(k+\frac{1}{2}\right)t-$$

$$-\frac{1}{2}\lambda_{n+1}\cos\left(n+\frac{1}{2}\right)t-\frac{1}{2}\lambda_m\cos\left(m+\frac{1}{2}\right)t,$$

and therefore

$$\mathbb{B}_{m+1,n}(t)=$$

$$=\frac{1}{2\cos^2\frac{t}{2}}\left\{\sum_{k=m}^{n}(\lambda_k+\lambda_{k+1})\cos\left(k+\frac{1}{2}\right)t-\sin t\sum_{k=m+1}^{n}\lambda_k\sin\left(k+\frac{1}{2}\right)t-\right.$$

$$\left.-\lambda_{n+1}\cos\left(n+\frac{1}{2}\right)t-\lambda_m\cos\left(m+\frac{1}{2}\right)t\right\}.$$

Further

$$\mathbb{L}_{m,n}(t):=$$

$$:=\sum_{k=m}^{n}\lambda_k\sin\left(k+\frac{1}{2}\right)t=$$

$$=\frac{1}{2}\sum_{k=m}^{n}(\lambda_k+\lambda_{k+1})\sin\left(k+\frac{1}{2}\right)t+\frac{1}{2}\sum_{k=m}^{n}(\lambda_k-\lambda_{k+1})\sin\left(k+\frac{1}{2}\right)t,$$

and whence

$$\mathbb{L}_{m,n}(t)= \tag{2.6}$$

$$=\frac{1}{2}\sum_{k=m}^{n}(\lambda_k+\lambda_{k+1})\sin\left(k+\frac{1}{2}\right)t-\frac{1}{2}\sum_{k=m+1}^{n+1}\lambda_k\sin\left(k+\frac{1}{2}-1\right)t=$$

$$=\frac{1}{2}\sum_{k=m}^{n}(\lambda_k+\lambda_{k+1})\sin\left(k+\frac{1}{2}\right)t-$$
$$-\frac{1}{2}\sum_{k=m+1}^{n}\lambda_k\sin\left(k+\frac{1}{2}-1\right)t-\frac{1}{2}\lambda_{n+1}\sin\left(n+\frac{1}{2}\right)t=$$
$$=\frac{1}{2}\sum_{k=m}^{n}(\lambda_k+\lambda_{k+1})\sin\left(k+\frac{1}{2}\right)t-$$
$$-\frac{1}{2}\cos t\sum_{k=m+1}^{n}\lambda_k\sin\left(k+\frac{1}{2}\right)t+\frac{1}{2}\sin t\mathbb{B}_{m+1,n}(t)-$$
$$-\frac{1}{2}\lambda_{n+1}\sin\left(n+\frac{1}{2}\right)t.$$

After inserting of $\mathbb{B}_{m+1,n}(t)$ into (2.6) and performing some elementary transformations we obtain

$$\mathbb{L}_{m,n}(t)=\frac{1}{2\cos\frac{t}{2}}\left\{\sum_{k=m}^{n}(\lambda_k+\lambda_{k+1})\sin(k+1)t+\lambda_m\sin mt-\lambda_{n+1}\sin(n+1)t\right\}.$$

Moreover, using the summation by parts we get

$$\mathbb{L}_{m,n}(t)=$$
$$=\frac{1}{2\cos\frac{t}{2}}\left\{\sum_{k=m}^{n-1}(\lambda_k-\lambda_{k+2})\sum_{i=0}^{k}\sin(i+1)t+(\lambda_n+\lambda_{n+1})\sum_{i=0}^{n}\sin(i+1)t-\right.$$
$$\left.-(\lambda_m+\lambda_{m+1})\sum_{i=0}^{m-1}\sin(i+1)t+\lambda_m\sin mt-\lambda_{n+1}\sin(n+1)t\right\}.$$

Thus, since

$$\sum_{i=0}^{k}\sin(i+1)t=\frac{\cos\frac{t}{2}-\cos\left(k+\frac{3}{2}\right)t}{2\sin\frac{t}{2}}=-\frac{\sin(k+2)\frac{t}{2}\sin(k+1)\frac{t}{2}}{\sin\frac{t}{2}},$$

we have

$$\mathbb{L}_{m,n}(t)=$$

$$= -\frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\left\{\sum_{k=m}^{n-1}(\lambda_k-\lambda_{k+2})\sin(k+2)\frac{t}{2}\sin\left(k+1\right)\frac{t}{2}+\right.$$
$$+\left(\lambda_n+\lambda_{n+1}\right)\sin(n+2)\frac{t}{2}\sin\left(n+1\right)\frac{t}{2}-$$
$$-\left(\lambda_m+\lambda_{m+1}\right)\sin(m+1)\frac{t}{2}\sin\frac{mt}{2}+$$
$$\left.+\lambda_m\sin mt\sin\frac{t}{2}-\lambda_{n+1}\sin(n+1)t\sin\frac{t}{2}\right\}.$$

Therefore using the inequalities $\sin\frac{t}{2}\geq\frac{t}{\pi}$ and $\cos\frac{t}{2}\geq 1-\frac{t}{\pi}$ for $t\in(0,\pi)$ we obtain

$$(2.7)\quad |\mathbb{L}_{m,n}(t)|\leq\frac{\pi^2}{t(\pi-t)}\left\{\frac{1}{2}\sum_{k=m}^{n-1}|\lambda_k-\lambda_{k+2}|+\lambda_n+\lambda_{n+1}+\lambda_m+\lambda_{m+1}\right\}.$$

Now by (2.7), supposing that $n\geq\tau$, we have

$$|\mathbb{L}_{0,n}(t)|\leq$$
$$\leq\sum_{k=0}^{\tau}a_{nk}+\left|\sum_{k=\tau}^{n}a_{nk}\sin\left(k+\frac{1}{2}\right)t\right|\leq$$
$$\leq\sum_{k=0}^{\tau}a_{nk}+O\left(\frac{1}{t(\pi-t)}\left(a_{n,\tau}+a_{n,\tau+1}+\frac{1}{2}\sum_{k=\tau}^{n-1}|a_{n,k}-a_{n,k+2}|+a_{n,n}\right)\right).$$

Since $(a_{n,k})$ is a lower triangular infinite matrix, that is, $a_{n,k}=0$ for $k>n$, then

$$a_{n,\tau}+a_{n,\tau+1}\leq\sum_{k=\tau}^{n}|a_{n,k}-a_{n,k+2}|,$$

and

$$a_{n,n}=a_{n,n}+a_{n,n+1}=\sum_{k=n}^{n}|a_{n,k}-a_{n,k+2}|\leq\sum_{k=\tau}^{n}|a_{n,k}-a_{n,k+2}|.$$

Consequently,

$$|\mathbb{L}_{0,n}(t)|\leq\sum_{k=0}^{\tau}a_{nk}+O\left(\frac{1}{t(\pi-t)}\sum_{k=\tau}^{n}|a_{n,k}-a_{n,k+2}|\right),$$

which completely proves (2.4).

By a similar technique we have

$$|\mathbb{L}_{0,n}(t)| = O\left(\frac{1}{t(\pi-t)}\left(a_{n,0}+a_{n,1}+\frac{1}{2}\sum_{k=0}^{n-1}|a_{n,k}-a_{n,k+2}|+a_{n,n}\right)\right) = \\ = O\left(\frac{1}{t(\pi-t)}\sum_{k=0}^{n}|a_{n,k}-a_{n,k+2}|\right)$$

which completes (2.5), and with this the proof of the lemma.

3. Main results

We establish the following

Theorem 3.3. *Let $(a_{n,k})$ satisfy conditions (1.1) and assume that $\omega(t)$ satisfies condition (1.2). Then*

$$\|T_{n,A}(f)-f\| = O\left(\omega(\pi/n)+H(\pi/n)\sum_{k=0}^{n}|a_{n,k}-a_{n,k+2}|\right). \tag{3.8}$$

If, in addition, $\omega(t)$ satisfies (1.3), then

$$\|T_{n,A}(f)-f\| = O\left(\sum_{k=0}^{n}|a_{n,k}-a_{n,k+2}|H\left(\sum_{k=0}^{n}|a_{n,k}-a_{n,k+2}|\right)\right), \tag{3.9}$$

$$\|T_{n,A}(f)-f\| = O\left(\sum_{k=0}^{n}|a_{n,k}-a_{n,k+2}|H(\pi/n)\right). \tag{3.10}$$

Proof. Denoting

$$\phi_x(t) := \frac{f(x+t)+f(x-t)-2f(x)}{2},$$

we easily obtain

$$T_{n,A}(f;x)-f(x) = \frac{2}{\pi}\int_0^{\pi}\phi_x(t)\left(2\sin\frac{t}{2}\right)^{-1}\sum_{k=0}^{n}a_{n,k}\sin\left(k+\frac{1}{2}\right)t\,dt. \tag{3.11}$$

By (3.11) we have

$$\|T_{n,A}(f;x)-f(x)\| \leq \frac{2}{\pi}\left(\int_{0}^{\pi/n}+\int_{\pi/n}^{\pi}\right) := \mathcal{B}_1(n)+\mathcal{B}_2(n). \tag{3.12}$$

According to (1.1) and the inequality $|\sin t| \leq t$ for $0 \leq t \leq \pi/n$, we have

$$\left|\sum_{k=0}^{n} a_{n,k}\sin\left(k+\frac{1}{2}\right)t\right| \leq 2nt.$$

Thus,

$$\mathcal{B}_1(n) = O(n)\int_{0}^{\pi/n}\omega(t)dt = O(\omega(\pi/n)). \tag{3.13}$$

Also, by (2.5), (1.2) and the obvious inequality $\frac{1}{t^2(\pi-t)} < \frac{1}{t^2}$, for $0 < t < \pi$, we obtain

$$\mathcal{B}_2(n) = \tag{3.14}$$

$$= O\left(\sum_{k=0}^{n}|a_{n,k}-a_{n,k+2}|\right)\int_{\pi/n}^{\pi} t^{-2}\omega(t)dt = O\left(H(\pi/n)\sum_{k=0}^{n}|a_{n,k}-a_{n,k+2}|\right).$$

Therefore (3.8) follows from (3.12)-(3.14).

Then according to (1.1), and

$$\sum_{k=0}^{n}|a_{n,k}-a_{n,k+2}| \leq \sum_{k=0}^{n}a_{n,k}+\sum_{k=0}^{n-2}a_{n,k+2} \leq 2\sum_{k=0}^{n}a_{nk} = 2 < \pi,$$

we get

$$\|T_{n,A}(f;x)-f(x)\| \leq \tag{3.15}$$

$$\leq \frac{2}{\pi}\left(\int_{0}^{\sum_{k=0}^{n}|a_{n,k}-a_{n,k+2}|} + \int_{\sum_{k=0}^{n}|a_{n,k}-a_{n,k+2}|}^{\pi}\right) := \mathcal{D}_1(n)+\mathcal{D}_2(n).$$

It is obvious from (1.1) that

$$\left|\sum_{k=0}^{n} a_{n,k} \sin\left(k+\frac{1}{2}\right)t\right| \le 1.$$

Thus, by Lemma 2.2 we have

$$\mathcal{D}_1(n) = O(1) \int_0^{\sum_{k=0}^{n}|a_{n,k}-a_{n,k+2}|} t^{-1}\omega(t)dt = \tag{3.16}$$

$$= O\left(\sum_{k=0}^{n}|a_{n,k}-a_{n,k+2}| H\left(\sum_{k=0}^{n}|a_{n,k}-a_{n,k+2}|\right)\right).$$

Using (2.5) and (1.2), and the inequality $\frac{1}{t^2(\pi-t)} < \frac{1}{t^2}$, for $0 < t < \pi$, we obtain

$$\begin{aligned} \mathcal{D}_2(n) =& O\left(\sum_{k=0}^{n}|a_{n,k}-a_{n,k+2}| \int_{\sum_{k=0}^{n}|a_{n,k}-a_{n,k+2}|}^{\pi} t^{-2}\omega(t)dt\right) = \\ =& O\left(\sum_{k=0}^{n}|a_{n,k}-a_{n,k+2}| H\left(\sum_{k=0}^{n}|a_{n,k}-a_{n,k+2}|\right)\right). \end{aligned} \tag{3.17}$$

From (3.15), (3.16) and (3.17) follows (3.9).

Now we turn back to prove (3.10). Since $a_{nk} = 0$ for $k > n$, we deduce that

$$a_{n,\ell} \le |a_{n,\ell} + a_{n,\ell+1}| - |a_{n,n}| \le \sum_{k=\ell}^{n}|a_{n,k}-a_{n,k+2}|$$

for $\ell = 0, 1, 2, \ldots, n$, which implies

$$1 = \sum_{\ell=0}^{n} a_{n\ell} \le (n+1)\sum_{k=0}^{n}|a_{n,k}-a_{n,k+2}|,$$

i.e.

$$\sum_{k=0}^{n}|a_{n,k}-a_{n,k+2}| \ge \frac{1}{2n}.$$

Whence, according to Lemma 2.2 we obtain

$$\mathcal{B}_1(n) = O\Big(\frac{1}{n}H(\pi/n)\Big) = O\Big(\sum_{k=0}^{n} |a_{n,k} - a_{n,k+2}| H(\pi/n)\Big). \tag{3.18}$$

Therefore, by (3.12), (3.14) and (3.18), (3.10) is proved.

Theorem 3.4. *Let* $(a_{n,k})$ *satisfies (1.1). Then*

$$\|T_{n,A}(f) - f\| = \tag{3.19}$$

$$= O\Big(\omega(\pi/n) + \sum_{k=1}^{n} k^{-1}\omega(\pi/k) \sum_{\mu=0}^{k+1} a_{n\mu} + \sum_{k=1}^{n} \omega(\pi/k) \sum_{\mu=k}^{n} |a_{n,\mu} - a_{n,\mu+2}|\Big).$$

Proof. According to (2.4), the inequality $1/t^2(\pi - t) < 1/t^2$, $0 < t < \pi$, and the property of the monotonicity of $\omega(t)$, we have

$$\mathcal{B}_2(n) = \tag{3.20}$$

$$= \frac{2}{\pi} \int_{\pi/n}^{\pi} \phi_x(t) \left(2 \sin \frac{t}{2}\right)^{-1} \sum_{k=0}^{n} a_{n,k} \sin\left(k + \frac{1}{2}\right) t \, dt =$$

$$= O\Bigg(\int_{\pi/n}^{\pi} t^{-1}\omega(t) \Big(\sum_{\mu=0}^{\tau} a_{n\mu} + \frac{1}{t(\pi - t)} \sum_{\mu=\tau}^{n} |a_{n,\mu} - a_{n,\mu+2}|\Big) dt\Bigg) =$$

$$= O\Bigg(\sum_{k=1}^{n-1} \int_{\pi/(k+1)}^{\pi/k} t^{-1}\omega(t) \Big(\sum_{\mu=0}^{\tau} a_{n\mu} + \frac{1}{t(\pi - t)} \sum_{\mu=\tau}^{n} |a_{n,\mu} - a_{n,\mu+2}|\Big)\Bigg) dt =$$

$$= O\Bigg(\sum_{k=1}^{n} k^{-1}\omega(\pi/k) \sum_{\mu=0}^{k+1} a_{n\mu} + \sum_{k=1}^{n} \omega(\pi/k) \sum_{\mu=k}^{n} |a_{n,\mu} - a_{n,\mu+2}|\Bigg).$$

Combining (3.12), (3.13) and (3.20), we immediately obtain (3.19). The proof of the theorem is completed.

4. Conclusion

In this section we are going to show that the main results contain all results obtained previously by others. We begin first with the following remark.

Remark 4.1. *Because of the inequality*

$$\sum_{k=0}^{n} |a_{n,k} - a_{n,k+2}| \leq 2 \sum_{k=0}^{n} |\triangle a_{nk}|,$$

Theorem 1.1 and Theorem 1.2 (from [8]) follow immediately from ours.

Secondly, since

$$\sum_{k=0}^{n} |a_{n,k} - a_{n,k+2}| = \sum_{k=0}^{n-1} |a_{n,k} - a_{n,k+2}| + a_{n,n}$$

and

$$\begin{aligned}\sum_{k=0}^{n-1} |a_{n,k} - a_{n,k+2}| \geq & |a_{n,0} + a_{n,1} - a_{n,n}| \geq \\ \geq & |a_{n,n} - a_{n,0}| - |a_{n,1}| \geq \\ \geq & |a_{n,n} - a_{n,0}| \geq a_{n,n} - a_{n,0},\end{aligned}$$

then, if $\{a_{n,k}\} \in RBSVS$, we have

$$\sum_{k=0}^{n} |a_{n,k} - a_{n,k+2}| \leq 2 \sum_{k=0}^{n-1} |a_{n,k} - a_{n,k+2}| + a_{n,0} \leq (2K+1)a_{n,0}.$$

But, if $\{a_{n,k}\} \in HBSVS$, we have

$$\sum_{k=0}^{n} |a_{n,k} - a_{n,k+2}| = \sum_{k=0}^{n-1} |a_{n,k} - a_{n,k+2}| + a_{n,n} \leq (K+1)a_{n,n}.$$

Therefore the following immediate corollary of our results hold of true.

Corollary 4.1. *Let $(a_{n,k})$ satisfies conditions (1.1) and assume that $\omega(t)$ satisfies condition (1.2). Then:*

(i) If $\{a_{n,k}\} \in RBSVS$ we have

$$\|T_{n,A}(f) - f\| = O\left(\omega(\pi/n) + H(\pi/n)a_{n,0}\right).$$

If, in addition, $\omega(t)$ satisfies (1.3), then

$$\begin{aligned}\|T_{n,A}(f) - f\| &= O\left(a_{n,0}H(a_{n,0})\right), \\ \|T_{n,A}(f) - f\| &= O\left(a_{n,0}H(\pi/n)\right).\end{aligned}$$

(ii) If $\{a_{n,k}\} \in HBSVS$ *we have*

$$\|T_{n,A}(f) - f\| = O\left(\omega(\pi/n) + H(\pi/n)a_{n,n}\right).$$

If, in addition, $\omega(t)$ *satisfies (1.3), then*

$$\|T_{n,A}(f) - f\| = O\left(a_{n,n}H(a_{n,n})\right),$$
$$\|T_{n,A}(f) - f\| = O\left(a_{n,n}H(\pi/n)\right).$$

Remark 4.2. *Since* $RBVS \subset RBSVS$ *and* $HBVS \subset HBSVS$*, then Corollary 4.1 contains the results obtained in [4] and [8], and therefore we have obtained the same degrees on sup-norm approximation for two wider classes of numerical sequences.*

In [5] Leindler has extended the definition of $RBVS$ to the so-called $\gamma RBVS$. That definition can be stated as follows:

For a fixed n, let $\gamma_n := \{\gamma_{n,k}\}$, $(k = 0, 1, \ldots)$ be a nonnegative sequence. If a null-sequence $\alpha_n := \{a_{n,k}\}$, $(k = 0, 1, \ldots)$ of real numbers has the property

$$\sum_{k=m}^{\infty} |a_{n,k} - a_{n,k+1}| \leq K(\alpha_n)\gamma_{n,m}$$

for every positive integer m, then we call the sequence $\alpha_n := \{a_{n,k}\}$ a $\gamma RBVS$, briefly denoted by $\alpha_n \in \gamma RBVS$.

Similarly, in [8] was introduced a new kind of sequences $\gamma HBVS$ as follows:

For a fixed n, let $\gamma_n := \{\gamma_{n,k}\}$, $(k = 0, 1, \ldots)$ be a nonnegative sequence. If a null-sequence $\alpha_n := \{a_{n,k}\}$, $(k = 0, 1, \ldots)$ of real numbers has the property

$$\sum_{k=0}^{m-1} |a_{n,k} - a_{n,k+1}| \leq K(\alpha_n)\gamma_{n,m}$$

for every positive integer m, then we call the sequence $\alpha_n := \{a_{n,k}\}$ a $\gamma HBVS$, briefly denoted by $\alpha_n \in \gamma HBVS$.

We introduce here two new kind of sequences $\gamma RBSVS$ and $\gamma HBSVS$ as follows:

For a fixed n, let $\gamma_n := \{\gamma_{n,k}\}$, $(k = 0, 1, \ldots)$ be a nonnegative sequence. If a null-sequence $\alpha_n := \{a_{n,k}\}$, $(k = 0, 1, \ldots)$ of real numbers has the property

$$\sum_{k=m}^{\infty} |a_{n,k} - a_{n,k+2}| \leq K(\alpha_n)\gamma_{n,m}$$

$$\left(\sum_{k=0}^{m-1} |a_{n,k} - a_{n,k+2}| \leq K(\alpha_n)\gamma_{n,m}\right)$$

for every positive integer m, then we call the sequence $\alpha_n := \{a_{n,k}\}$ a $\gamma RBSVS$ ($\gamma HBSVS$), briefly denoted by $\alpha_n \in \gamma RBSVS$ ($\alpha_n \in \gamma HBSVS$).

It is obvious that if $\gamma_n = \alpha_n$, then $\gamma RBSVS \equiv RBSVS$ and $\gamma HBVS \equiv$ $\equiv HBSVS$.

Using a similar technique, as in the proof of Theorem 3.3 and Theorem 3.4, we have the following generalizations:

Theorem 4.5. *Let $(a_{n,k})$ satisfies conditions (1.1) and assume that $\omega(t)$ satisfies condition (1.2). Then*

(i) If $\{a_{n,k}\} \in \gamma RBSVS$ we have

$$\|T_{n,A}(f) - f\| = O\left(\omega(\pi/n) + H(\pi/n)\gamma_{n,0}\right).$$

If, in addition, $\omega(t)$ satisfies (1.3), then

$$\|T_{n,A}(f) - f\| = O\left(\gamma_{n,0}H(\gamma_{n,0})\right),$$
$$\|T_{n,A}(f) - f\| = O\left(\gamma_{n,0}H(\pi/n)\right).$$

(ii) If $\{a_{n,k}\} \in \gamma HBSVS$ we have

$$\|T_{n,A}(f) - f\| = O\left(\omega(\pi/n) + H(\pi/n)\gamma_{n,n}\right).$$

If, in addition, $\omega(t)$ satisfies (1.3), then

$$\|T_{n,A}(f) - f\| = O\left(\gamma_{n,n}H(\gamma_{n,n})\right),$$
$$\|T_{n,A}(f) - f\| = O\left(\gamma_{n,n}H(\pi/n)\right).$$

References

[1] **Szal, B.,** Generalization of a theorem on Besov-Nikol'skiĭ classes, *Acta Math. Hungar.,* **125** (1-2) (2009), 161-181.

[2] **Chandra, P.,** On the degree of approximation of a class of functions by means of Fourier series, *Acta Math. Hungar.,* **52** (1988), 199-205.

[3] **Chandra, P.,** A note on the degree of approximation of continuous functions, *Acta Math. Hungar.,* **62** (1993), 21-23.

[4] **Leindler, L.,** On the degree of approximation of continuous functions, *Acta Math. Hungar.,* **104** (2004), 105-113.

[5] **Leindler, L.,** Embedding results regarding strong approximation of sine series, *Acta Sci. Math. Szeged,* **71** (2005), 93-103.

[6] **Leindler, L.,** On the degree of approximation of continuous functions, Ann. Univ. Sci. Budapest. Sect. Comp., **29** (2008), 93-99.

[7] **Krasniqi, Xh.Z.,** On the degree of approximation of continuous functions that pertains to the sequence-to-sequence transformation, *Aust. J. Math. Anal. Appl.,* **7** (2) (2011), 1-10.

[8] **Wei, B. and Yu, D.,** On the degree of approximation of continuous functions by means of Fourier series, *Math. Commun.,* **17** (2012), 211-219.

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