ON THE VALUES OF ARITHMETIC FUNCTIONS IN SHORT INTERVALS

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Dedicated to Professor Antal Iványi on his seventieth anniversary

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Abstract. In this short paper the following assertion is proved. For positive integer \( d \) and \( c > 0 \) let \( J_c(N) = \lfloor N, N + c\sqrt{N} \rfloor \) and \( K_d = \{ n \in \mathbb{N} \mid (n, d) = 1 \} \). Let \( 1 < N_1 < N_2 < \cdots \) be an infinite sequence of integers and \( \ell_1, \ell_2, \cdots \) be integers coprime to \( d \). Assume that \( f \) and \( g \) are completely additive functions defined on \( K_d \), for which \( f(n) = g(n) \) if \( n \equiv \ell_j \pmod{d} \), \( n \in J_{\ell_j}(N_j) \) \((j = 1, 2, \cdots)\). If \( c > 2d \), then \( f(n) = g(n) \) identically on \( K_d \).

1. Introduction

1.1. Notations

Let \( \mathbb{N}, \mathbb{R}, \mathbb{C} \) be the sets of positive integers, real and complex numbers, respectively. Let \( (G, +) \) and \( (\mathbb{H}, \cdot) \) be commutative semigroups. We shall denote by \( \mathcal{A}_G \) \((\mathcal{A}_\mathbb{C})\) the set of additive (completely additive) arithmetical functions.

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taking values on $G$. Similarly, let $\mathcal{M}_G (\mathcal{M}^*_G)$ be the set of multiplicative (completely multiplicative) arithmetical functions taking values on $H$. For $G = \mathbb{R}$ we write $\mathcal{A} (\mathcal{A}^*)$ instead of $\mathcal{A}_G (\mathcal{A}^*_G)$ and for $H = \mathbb{C}$ we write $\mathcal{M} (\mathcal{M}^*)$ instead of $\mathcal{M}_C (\mathcal{M}^*_C)$.

### 1.2. Known results

We state some of the known results in this direction.

**Theorem A.** ([1]) Let $f \in \mathcal{A}^*$, for which

$$f(n) = 0 \text{ holds for } n \in [N_j, N_j + 4\sqrt{N_j}]$$

$(j = 1, 2, \cdots)$, $1 < N_1 < N_2 < \cdots$ is an arbitrary infinite sequence of integers. Then $f(n) = 0$ identically.

**Theorem B.** ([3]) Let $f \in \mathcal{A}^*$. Let $\lambda(N) = (2 + \epsilon)\sqrt{N}$ for an arbitrary constant $\epsilon > 0$. Assume that $1 < N_1 < N_2 < \cdots$ is an infinite sequence of integers such that

$$f(n) \leq f(n + 1) \text{ holds for } n \in [N_j, N_j + \lambda(N_j)]$$

$(j = 1, 2, \cdots)$. Then $f(n) = c \log n$, where $c$ is a constant.

**Theorem C.** ([3]) There exists an $f \in \mathcal{A}^*$ which is not identically zero, and for $1 < N_1 < N_2 < \cdots$ satisfies

$$f(n) = A_j \text{ if } n \in [N_j, N_j + \varrho(N_j)]$$

for $j = 1, 2, \cdots$ and

$$\varrho(N) = \exp \left( c \sqrt{\log N} \left( \log \log \log N \right) \right),$$

where $c$ is a suitable positive constant and $A_j$ are arbitrary complex or reals.

**Theorem D.** ([6]) Let $\Phi_j(z) \in \mathbb{C}[z]$ be a sequence of polynomials with $\deg \Phi_j \leq h$ and $\alpha_h = \frac{h+1}{h+2}$. Assume that $b_j \to \infty$ is an infinite sequence of reals, $1 < N_1 < N_2 < \cdots$ is an infinite sequence of integers and $a_1, a_2, \cdots$ is a sequence of arbitrary complex numbers. For $f \in \mathcal{A}^*$ assume that

$$\Phi_j(E)f(n) = a_j \text{ holds for } n \in [N_j, N_j + b_j N_j^{\alpha_j}]$$

$(j = 1, 2, \cdots)$. Then $f(n) = 0$, identically.
Here $E$ is the shift operator. If $P(z) = a_0 + a_1 z + \cdots + a_k z^k$, then

$$P(E)f(n) = a_0 f(n) + a_1 f(n + 1) + \cdots + a_k f(n + k).$$

In [5] P. Erdős and I. Kátai showed the existence of a completely additive function which vanishes in particular short intervals but takes the value 1 in one interval. Here is their result.

**Theorem E.** ([5]) Let $x > x_0(\epsilon)$ for $\epsilon > 0$. Then there exists a function $f \in A^*$ for which

$$f(n) = 0 \text{ for } n \in [N + 1, N + \lambda(x)],$$

where $\frac{x}{2} \leq N \leq x$ and

$$\lambda(x) = \exp \left( \left( \frac{1}{2} - \epsilon \right) \frac{(\log x)(\log \log \log x)}{\log \log x} \right)$$

and which takes on a non-zero value in $[1, \sqrt{x}]$.

**Remark 1.** Existence of an $f(n)$ with infinitely many such intervals is yet to be established.

In [2] I. Kátai, examined arbitrary complex valued multiplicative functions which remain constant on square-free numbers in short intervals and proved the following theorem.

**Theorem F.** ([2]) Let $\theta = 0.6108$ and $J(N) = [N, N + N^\theta]$. Let $f$ be a multiplicative function defined on the set of square-free numbers and $f(n) \neq 0, (n \in \mathbb{N})$. Assume that there exists a sequence of complex numbers $a_1, a_2, \cdots$ and a sequence of positive integers $1 < N_1 < N_2 < \cdots$ such that

$$f(n) = a_j \text{ if } n \in J(N_j) \text{ (n is square-free}).$$

Then $f(n) = 1$ for every square-free $n$.

**Remark 2.** Theorem F remains valid with $\theta = 0.6$.

This comes from the following result of M. Filaseta:

Let $g(x)$ be a function, $1 \leq g(x) \leq \log x$ for $x$ sufficiently large, and set $h(x) = x^\frac{1}{6}g(x)^3$. Then the number of square-free integers belonging to the interval $[x, x + h(x)]$ is

$$\frac{h(x)}{\xi(2)} + O\left( \frac{h(x)\log x}{g(x)^3} \right) + O\left( \frac{h(x)}{g(x)} \right).$$

The interested reader can look at [7] for a complete proof.
2. Results

Let \( J_c(N) = [N, N + c\sqrt{N}] \), where \( c \) is a fixed constant. For each positive integer \( d \), let \( \mathcal{K}_d = \{ n \in \mathbb{N} \mid (n, d) = 1 \} \). Here we prove the following results which are variants of the results quoted in the previous section.

**Theorem 1.** Let \( f \in \mathcal{M}^* \) be defined on \( \mathcal{K}_d \), where \( d \in \mathbb{N} \) is given. Assume that there exists an infinite sequence of integers \( 1 < N_1 < N_2 < \cdots \), an infinite sequence of reduced residues \( \ell_1 \equiv 1 \pmod{d} \), \( \ell_2 \equiv 2 \pmod{d} \), \( \cdots \) and a sequence of nonzero complex numbers \( a_1, a_2, \cdots \) such that

\[
(2.1) \quad f(n) = a_{\ell_\nu} \quad \text{if} \quad n \in J_c(N_{\nu}) \quad \text{and} \quad n \equiv \ell_\nu \pmod{d}
\]

\( (\nu = 1, 2, \cdots) \). If \( c > 2d \), then \( f(n) = \chi(n) \) for a Dirichlet character \( \chi \) \( (\mod{d}) \).

**Theorem 2.** Let \( d \), the sequences \( N_\nu, \ell_\nu \) be as in Theorem 1. Let \( g \in \mathcal{A}^*_G \) be defined on \( \mathcal{K}_d \). Assume that

\[
(2.2) \quad g(n) \leq g(n + d) \quad \text{if} \quad n \in J_c(N_{\nu}) \quad \text{and} \quad n \equiv \ell_\nu \pmod{d}
\]

\( (\nu = 1, 2, \cdots) \). If \( c > 2d \), then there exists a constant \( A \) such that

\[
g(n) = A \log n \quad \text{for} \quad n \in \mathcal{K}_d.
\]

Now for the Abelian group \( G \) let

\[
\mathcal{X}_G = \{ \, g \in \mathcal{A}^*_{gg} \mid g(n) = g(m) \quad \text{all} \quad n, m \in \mathcal{K}_d, \quad n \equiv m \pmod{d} \}.
\]

**Theorem 3.** Let \( g \in \mathcal{A}^*_G \) be defined on \( \mathcal{K}_d \). Assume that there exists an infinite sequence of integers \( 1 < N_1 < N_2 < \cdots \), an infinite sequence of reduced residues \( \ell_1 \equiv 1 \pmod{d} \), \( \ell_2 \equiv 2 \pmod{d} \), \( \cdots \) and a sequence \( a_1, a_2, \cdots \) of elements of \( \mathbb{G} \) such that

\[
(2.3) \quad g(n) = a_{\ell_\nu} \quad \text{if} \quad n \in J_c(N_{\nu}) \quad \text{and} \quad n \equiv \ell_\nu \pmod{d}
\]

\( (\nu = 1, 2, \cdots) \). If \( c > 2d \), then \( g \in \mathcal{X}_G \).

We provide the proofs of Theorems 1, 2 and omit the proof of Theorem 3 as it is similar.
3. Proof of Theorem 1

Assume that \( c, d \), the sequences \( N_\nu, \ell_\nu \) are as in the statement of Theorem 1 with \( c > 2d \). As the sequence of reduced residues \( \ell_1 \pmod{d}, \ell_2 \pmod{d}, \cdots \) is infinite, there exists a reduced residue \( \ell \pmod{d} \) such that \( \ell_\nu \equiv \ell \pmod{d} \) holds for infinitely many \( \nu \). Consequently, the condition (2.1) can be replaced by the following:

\[
(3.1) \quad f(n) = a_\ell \text{ if } n \in J_c(N_\nu) \text{ and } n \equiv \ell \pmod{d}, \quad (\nu = 1, 2, \cdots),
\]

where \( \ell \in \mathcal{K}_d \) is fixed integer and \( a_\ell \) is a nonzero complex number.

First we deduce the following lemma.

**Lemma 1.** Assuming (3.1) and if

\[
du^2 - (c\sqrt{N_\nu} - d^2)u + N_\nu d \leq 0,
\]

then

\[
(3.2) \quad f(u) = f(u + d).
\]

**Proof.** Indeed, if \( du^2 - (c\sqrt{N_\nu} - d^2)u + N_\nu d \leq 0 \), then

\[
c\sqrt{N_\nu}u \geq du(u + d) + N_\nu d
\]

and so

\[
\frac{N_\nu + c\sqrt{N_\nu}}{u + d} \geq \frac{N_\nu}{u} \geq d.
\]

Thus there exists an \( r \) for which \( ur \equiv \ell \pmod{d} \) and

\[
r \in \left[ \frac{N_\nu}{u}, \frac{N_\nu + c\sqrt{N_\nu}}{u} \right], \quad r \in \left[ \frac{N_\nu}{u + d}, \frac{N_\nu + c\sqrt{N_\nu}}{u + d} \right],
\]

i.e

\[
r(\nu + d) \equiv \ell \pmod{d}, \quad \text{and} \quad ru, \; r(u + d) \in J_c(N_\nu).
\]

Hence, from (3.1) we have

\[
f(r(u)) = a_\ell \quad \text{and} \quad f(r(u + d)) = a_\ell,
\]

which with \( a_\ell \neq 0 \) implies that \( f(u) = f(u + d) \). Thus the assertion follows.
Now we shall verify that
\[(3.3) \quad du^2 - (c\sqrt{N_\nu} - d^2)u + N_\nu d \leq 0.\]

Clearly the condition \(c > 2d\) implies that
\[
(c\sqrt{N_\nu} - d^2)^2 - 4d^2N_\nu \geq \left[(c^2 - 4d^2)\sqrt{N_\nu} - 2cd^2\right] + d^4 > 0
\]
for all \(\nu > \nu_0\), where
\[
N_{\nu_0} \geq \left(\frac{2cd^2}{c^2 - 4d^2}\right)^2.
\]

Let
\[
\xi_{1,2} = \left(\frac{c\sqrt{N_\nu} - d^2}{2d}\right) \pm \sqrt{\left(\frac{c\sqrt{N_\nu} - d^2}{2d}\right)^2 - 4d^2N_\nu}
\]
and let
\[
\lambda_1 = \frac{c - \sqrt{c^2 - 4d^2}}{2d}, \quad \lambda_2 = \frac{c + \sqrt{c^2 - 4d^2}}{2d}.
\]

It is clear that
\[
\xi_1 = (1 + o_{\nu}(1))\lambda_1 \sqrt{N_\nu} \quad \text{and} \quad \xi_2 = (1 + o_{\nu}(1))\lambda_2 \sqrt{N_\nu}
\]
and that (3.3) holds for all \(u \in [\xi_1, \xi_2]\).

Let \(\epsilon > 0\) be an arbitrary constant such that \(\lambda_1 + \epsilon < \lambda_2 - \epsilon\). Let
\[
S = [\lambda_1 + \epsilon, \lambda_2 - \epsilon].
\]

Next we denote by \(\nu_1\) the least index for which \(\nu_1 > \nu_0\) and
\[
S \sqrt{N_\nu} = [(\lambda_1 + \epsilon)\sqrt{N_\nu}, (\lambda_2 - \epsilon)\sqrt{N_\nu}] \subseteq [\xi_1, \xi_2]
\]
is satisfied for all \(\nu > \nu_1\).

Let \(t_j = \sqrt{N_{j+n_1}}\). Then (3.3) holds for each \(u \in S t_j\), which implies from (3.2) that \(f(u) = f(u + d)\). Thus, we have proved that
\[(3.4) \quad f(n_1) = f(n_2) \neq 0 \quad \text{if} \quad n_1 \equiv n_2 \pmod{d}, n_1, n_2 \in S t_j, n_1 \in K_d.
\]

Now we complete the proof of Theorem 1.

**Proof.** Let \(u_j = (\lambda_1 + \epsilon)t_j, \quad v_j = (\lambda_2 - \epsilon)t_j\). It is easy to check that if
\[
n \in K_d, \quad n > n_0 = \frac{(\lambda_1 + \epsilon)d}{\lambda_2 - \lambda_1 - 2\epsilon},
\]

then
\[ \frac{v_j}{n+d} - \frac{u_j}{n} = \frac{\left(\lambda_2 - \lambda_1 - 2\epsilon\right)n - (\lambda_1 + \epsilon)d}{n(n+d)} t_j \geq d \]
for all sufficiently large integer \(j\). This shows that
\[ \left[ \frac{u_j}{n}, \frac{v_j}{n} \right] \cap \left[ \frac{u_j}{n+d}, \frac{v_j}{n+d} \right] \]
contains an interval of length \(\geq d\), consequently there exists \(m \in K_d\) for which
\(nm \in [u_j, v_j]\) and \((n + d)m \in [u_j, v_j]\). Thus, (3.4) implies \(f(nm) = f((n + +d)m) \neq 0\), and so \(f(n) = f(n + d)\).

It implies that \(f(n)\) is a periodic function \((\mod d)\) on the set \(n \in K_d\).
Since \(f(n) \neq 0\), it should be a character. This completes the proof.

4. Proof of Theorem 2

The basic idea is the same as that of the proof of Theorem 1. Let us start with the following lemma.

**Lemma 2.** Let \(J_{N,M} = [N, N + M]\). Assume that

\[ g(u + d) - g(u) \geq 0 \] holds for \(u \equiv \ell \pmod{d}\),

\(u \in J_{N,M}\), where \((\ell, d) = 1\).

Assume that \(ur \equiv \ell \pmod{d}\), and that \(ur \in J_{N,M}\), \((u + d)r \in J_{N,M}\).

Then

\[ g(u + d) - g(u) \geq 0. \]

**Proof.** Assume that \(ur \equiv \ell \pmod{d}\), and that \(ur \in J_{N,M}\), \((u + d)r \in J_{N,M}\). Then \(ur + kd \equiv \ell \pmod{d}\) and \(ur + kd \in J_{N,M}\) for \(k = 0, 1, \ldots r\). Thus, we infer from (4.1) that
\[ g(ur + dr) - g(ur) = \sum_{k=1}^{r} \left( g(ur + kd) - g(ur + (k - 1)d) \right) \geq 0. \]
Proof. (Proof of Theorem 2) Repeating the argument used in the proof of Theorem 1, in this case we obtain the following assertion:

\[(4.3) \quad g(n + d) - g(n) \geq 0 \quad \text{if} \quad n \in St_j, \quad n \in \mathcal{K}_d, \]

where \( St_j = [u_j, v_j] = [(\lambda_1 + \epsilon)t_j, (\lambda_2 - \epsilon)t_j] \).

As in the proof of Theorem 1, by using (4.3) we can prove that if \( j \) is a sufficiently large integer, then for each \( n \in \mathcal{K}_d \) there exists \( m \in \mathcal{K}_d \) for which \( nm \in [u_j, v_j] \) and \( (n + d)m \in [u_j, v_j] \). Then we have

\[ nm + kd \in [u_j, v_j], \quad nm + kd \in \mathcal{K}_d, \quad (k = 0, 1, \ldots, m) \]

and so (4.3) implies that

\[ g(nm + dm) - g(nm) = \sum_{k=1}^{r} \{ g(nm + kd) - g(nm + (k - 1)d) \} \geq 0, \]

Consequently, \( g(n + d) - g(n) \geq 0 \) for every \( n \in \mathcal{K}_d \). Now we can deduce easily that \( g(n) = c \log n \), if \( (n, d) = 1 \).

Let \( p \) and \( q \) be primes, \( p \neq q \), \( (pq, d) = 1 \). Let \( P = p^{k_0} \equiv 1 \pmod{d}, Q = q^{\ell_0} \equiv 1 \pmod{d} \). Let \( u_h, v_h \) be such a sequence of integers for which \( P^{u_h} < Q^{v_h} < P^{u_h+1} \). It is obvious that \( \frac{u_h}{v_h} \rightarrow \frac{\log Q}{\log P} \) as \( h \rightarrow \infty \).

Furthermore \( g(P^{u_h}) \leq g(Q^{v_h}) \leq g(P^{u_h+1}) \), whence

\[ \frac{u_h}{v_h} \leq \frac{g(Q)}{g(P)} \leq \frac{u_h + 1}{v_h}, \]

and so

\[ \frac{k_0 u_h}{\ell_0 v_h} \leq \frac{g(q)}{g(p)} \leq \frac{k_0 u_h + 1}{\ell_0 v_h}. \]

Consequently,

\[ \frac{g(q)}{g(p)} = \frac{\log q}{\log p}. \]

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