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# ON THE VALUES OF ARITHMETIC FUNCTIONS IN SHORT INTERVALS

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*Dedicated to Professor Antal Iványi on his seventieth anniversary*

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**Abstract.** In this short paper the following assertion is proved. For positive integer $d$ and $c > 0$ let $J_c(N) = [N,\ N + c\sqrt{N}]$ and $\mathcal{K}_d = \{n \in \in \mathbb{N} \mid (n, d) = 1\}$. Let $1 < N_1 < N_2 < \cdots$ be an infinite sequence of integers and $\ell_1, \ell_2, \cdots$ be integers coprime to $d$. Assume that $f$ and $g$ are completely additive functions defined on $\mathcal{K}_d$, for which $f(n) = g(n)$ if $n \equiv \equiv \ell_j \pmod{d}$, $n \in J_c(N_j)$ $(j = 1, 2, \cdots)$. If $c > 2d$, then $f(n) = g(n)$ identically on $\mathcal{K}_d$.

## 1. Introduction

### 1.1. Notations

Let $\mathbb{N}, \mathbb{R}, \mathbb{C}$ be the sets of positive integers, real and complex numbers, respectively. Let $(\mathbb{G}, +)$ and $(\mathbb{H}, \cdot)$ be commutative semigroups. We shall denote by $\mathcal{A}_{\mathbb{G}}$ $(\mathcal{A}^*_{\mathbb{G}})$ the set of additive (completely additive) arithmetical functions

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taking values on $\mathbb{G}$. Similarly, let $\mathcal{M}_{\mathbb{H}}$ ($\mathcal{M}_{\mathbb{H}}^*$) be the set of multiplicative (completely multiplicative) arithmetical functions taking values on $\mathbb{H}$. For $\mathbb{G}=\mathbb{R}$ we write $\mathcal{A}$ ($\mathcal{A}^*$) instead of $\mathcal{A}_{\mathbb{R}}$ ($\mathcal{A}_{\mathbb{R}}^*$) and for $\mathbb{H}=\mathbb{C}$ we write $\mathcal{M}$ ($\mathcal{M}^*$) instead of $\mathcal{M}_{\mathbb{C}}$ ($\mathcal{M}_{\mathbb{C}}^*$).

### 1.2. Known results

We state some of the known results in this direction.

**Theorem A.** ([1]) *Let $f\in\mathcal{A}^*$, for which*

$$f(n)=0 \quad holds\ for \quad n\in[N_j, N_j+4\sqrt{N_j}]$$

*$(j=1,2,\cdots)$, $1<N_1<N_2<\cdots$ is an arbitrary infinite sequence of integers. Then $f(n)=0$ identically.*

**Theorem B.** ([3]) *Let $f\in\mathcal{A}^*$. Let $\lambda(N)=(2+\epsilon)\sqrt{N}$ for an arbitrary constant $\epsilon>0$. Assume that $1<N_1<N_2<\cdots$ is an infinite sequence of integers such that*

$$f(n)\le f(n+1) \quad holds\ for \quad n\in[N_j, N_j+\lambda(N_j)]$$

*$(j=1,2,\cdots)$. Then $f(n)=c\log n$, where $c$ is a constant.*

**Theorem C.** ([3]) *There exists an $f\in\mathcal{A}^*$ which is not identically zero, and for $1<N_1<N_2<\cdots$ satisfies*

$$f(n)=A_j \quad if \quad n\in[N_j, N_j+\varrho(N_j)]$$

*for $j=1,2,\cdots$ and*

$$\varrho(N)=\exp\left(c\sqrt{(\log N)\ (\log\log\log N)}\right),$$

*where $c$ is a suitable positive constant and $A_j$ are arbitrary complex or reals.*

**Theorem D.** [6]) *Let $\Phi_j(z)\in\mathbb{C}[z]$ be a sequence of polynomials with $\deg\Phi_j\le h$ and $\alpha_h=\frac{h+1}{h+2}$. Assume that $b_j\to\infty$ is an infinite sequence of reals, $1<N_1<N_2<\cdots$ is an infinite sequence of integers and $a_1,a_2,\cdots$ is a sequence of arbitrary complex numbers. For $f\in\mathcal{A}^*$ assume that*

$$\Phi_j(E)f(n)=a_j \quad holds\ for \quad n\in[N_j, N_j+b_jN_j^{\alpha_h}]$$

*$(j=1,2,\cdots)$. Then $f(n)=0$, identically.*

Here $E$ is the shift operator. If $P(Z) = a_0 + a_1 z + \cdots + a_k z^k$, then

$$P(E)f(n) = a_0 f(n) + a_1 f(n+1) + \cdots + a_k f(n+k).$$

In [5] P. Erdős and I. Kátai showed the existence of a completely additive function which vanishes in particular short intervals but takes the value 1 in one interval. Here is their result.

**Theorem E.** ([5]) *Let $x > x_0(\epsilon)$ for $\epsilon > 0$. Then there exists a function $f \in \mathcal{A}^*$ for which*

$$f(n) = 0 \quad for \quad n \in [N+1, N+\lambda(x)],$$

*where $\frac{x}{2} \le N \le x$ and*

$$\lambda(x) = \exp\left(\left(\frac{1}{2} - \epsilon\right) \frac{(\log x)(\log\log\log x)}{\log\log x}\right)$$

*and which takes on a non-zero value in $[1, \sqrt{x}]$.*

**Remark 1.** *Existence of an $f(n)$ with infinitely many such intervals is yet to be established.*

In [2] I. Kátai, examined arbitrary complex valued multiplicative functions which remain constant on square-free numbers in short intervals and proved the following theorem.

**Theorem F.** ([2]) *Let $\theta = 0{,}6108$ and $J(N) = [N, N + N^\theta]$. Let $f$ be a multiplicative function defined on the set of square-free numbers and $f(n) \neq$ $\neq 0$, $(n \in \mathbb{N})$. Assume that there exists a sequence of complex numbers $a_1, a_2, \cdots$ and a sequence of positive integers $1 < N_1 < N_2 < \cdots$ such that*

$$f(n) = a_j \quad if \quad n \in J(N_j) \quad (n \quad is\ square-free).$$

*Then $f(n) = 1$ for every square-free $n$.*

**Remark 2.** *Theorem F remains valid with $\theta = 0{,}6$.*

This comes from the following result of M. Filaseta:

Let $g(x)$ be a function, $1 \le g(x) \le \log x$ for $x$ sufficiently large, and set $h(x) = x^{\frac{1}{5}} g(x)^3$. Then the number of square-free integers belonging to the interval $[x, x + h(x)]$ is

$$\frac{h(x)}{\xi(2)} + O\Big(\frac{h(x)\log x}{g(x)^3}\Big) + O\Big(\frac{h(x)}{g(x)}\Big).$$

The interested reader can look at [7] for a complete proof.

## 2. Results

Let $J_c(N) = [N, N + c\sqrt{N}]$, where $c$ is a fixed constant. For each positive integer $d$, let $\mathcal{K}_d = \{n \in \mathbb{N} \mid (n,d) = 1\}$. Here we prove the following results which are variants of the results quoted in the previous section.

**Theorem 1.** *Let $f \in \mathcal{M}^*$ be defined on $\mathcal{K}_d$, where $d \in \mathbb{N}$ is given. Assume that there exists an infinite sequence of integers $1 < N_1 < N_2 < \cdots$, an infinite sequence of reduced residues $\ell_1 \pmod d$, $\ell_2 \pmod d, \cdots$ and a sequence of nonzero complex numbers $a_1, a_2, \cdots$ such that*

$$f(n) = a_{\ell_\nu} \quad if \quad n \in J_c(N_\nu) \quad and \quad n \equiv \ell_\nu \pmod d \tag{2.1}$$

*($\nu = 1, 2, \cdots$). If $c > 2d$, then $f(n) = \chi(n)$ for a Dirichlet character $\chi$ $\pmod d$.*

**Theorem 2.** *Let $d$, the sequences $N_\nu$, $\ell_\nu$ be as in Theorem 1. Let $g \in \mathcal{A}^*$ be defined on $\mathcal{K}_d$. Assume that*

$$g\big(n\big) \;\le\; g\big(n+d\big) \quad if \quad n \in J_c(N_\nu) \quad and \quad n \equiv \ell_\nu \pmod d \tag{2.2}$$

*($\nu = 1, 2, \cdots$). If $c > 2d$, then there exists a constant $A$ such that*

$$g(n) = A \log n \quad for \quad n \in \mathcal{K}_d.$$

Now for the Abelian group $\mathbb{G}$ let

$$\mathcal{X}_{\mathbb{G}} = \{\, g \in \mathcal{A}^*_{gg} \;\mid\; g(n) = g(m) \quad \text{all} \quad n,\ m \in \mathcal{K}_d,\ n \equiv m \pmod d\}.$$

**Theorem 3.** *Let $g \in \mathcal{A}^*_{\mathbb{G}}$ be defined on $\mathcal{K}_d$. Assume that there exists an infinite sequence of integers $1 < N_1 < N_2 < \cdots$, an infinite sequence of reduced residues $\ell_1 \pmod d$, $\ell_2 \pmod d, \cdots$ and a sequence $a_1, a_2, \cdots$ of elements of $\mathbb{G}$ such that*

$$g(n) = a_{\ell_\nu} \quad if \quad n \in J_c(N_\nu) \quad and \quad n \equiv \ell_\nu \pmod d \tag{2.3}$$

*($\nu = 1, 2, \cdots$). If $c > 2d$, then $g \in \mathcal{X}_{\mathbb{G}}$.*

We provide the proofs of Theorems 1, 2 and omit the proof of Theorem 3 as it is similar.

## 3. Proof of Theorem 1

Assume that $c$, $d$, the sequences $N_\nu$, $\ell_\nu$ are as in the statement of Theorem 1 with $c > 2d$. As the sequence of reduced residues $\ell_1 \pmod d$, $\ell_2 \pmod d, \cdots$ is infinite, there exists a reduced residue $\ell \pmod d$ such that $\ell_\nu \equiv \ell \pmod d$ holds for infinitely many $\nu$. Consequently, the condition (2.1) can be replaced by the following:

$$f(n) = a_\ell \quad \text{if} \quad n \in J_c(N_\nu) \quad \text{and} \quad n \equiv \ell \pmod d, \quad (\nu = 1, 2, \cdots), \tag{3.1}$$

where $\ell \in \mathcal{K}_d$ is fixed integer and $a_\ell$ is a nonzero complex number.

First we deduce the following lemma.

**Lemma 1.** *Assuming (3.1) and if*

$$du^2 - (c\sqrt{N_\nu} - d^2)u + N_\nu d \le 0,$$

*then*

$$f(u) = f(u+d). \tag{3.2}$$

**Proof.** Indeed, if $du^2 - (c\sqrt{N_\nu} - d^2)u + N_\nu d \le 0$, then

$$c\sqrt{N_\nu} u \ge du(u+d) + N_\nu d$$

and so

$$\frac{N_\nu + c\sqrt{N_\nu}}{u+d} - \frac{N_\nu}{u} \ge d.$$

Thus there exists an $r$ for which $ur \equiv \ell \pmod d$ and

$$r \in \Big[\frac{N_\nu}{u}, \quad \frac{N_\nu + c\sqrt{N_\nu}}{u}\Big], \quad r \in \Big[\frac{N_\nu}{u+d}, \quad \frac{N_\nu + c\sqrt{N_\nu}}{u+d}\Big],$$

i.e

$$ru \equiv r(u+d) \equiv \ell \pmod d, \quad \textit{and} \;\; ru, \; r(u+d) \in J_c(N_\nu).$$

Hence, from (3.1) we have

$$f(ru) = a_\ell \quad \text{and} \quad f(r(u+d)) = a_\ell,$$

which with $a_\ell \ne 0$ implies that $f(u) = f(u+d)$. Thus the assertion follows.

Now we shall verify that

$$du^2 - (c\sqrt{N_\nu} - d^2)u + N_\nu d \le 0. \tag{3.3}$$

Clearly the condition $c > 2d$ implies that

$$(c\sqrt{N_\nu} - d^2)^2 - 4d^2 N_\nu \ge \left[(c^2 - 4d^2)\sqrt{N_\nu} - 2cd^2\right] + d^4 > 0$$

for all $\nu > \nu_0$, where

$$N_{\nu_0} \ge \left(\frac{2cd^2}{c^2 - 4d^2}\right)^2.$$

Let

$$\xi_{1,2} = \frac{(c\sqrt{N_\nu} - d^2) \mp \sqrt{(c\sqrt{N_\nu} - d^2)^2 - 4d^2 N_\nu}}{2d}$$

and let

$$\lambda_1 = \frac{c - \sqrt{c^2 - 4d^2}}{2d}, \quad \lambda_2 = \frac{c + \sqrt{c^2 - 4d^2}}{2d}.$$

It is clear that

$$\xi_1 = (1 + o_\nu(1))\lambda_1 \sqrt{N_\nu} \quad \text{and} \quad \xi_2 = (1 + o_\nu(1))\lambda_2 \sqrt{N_\nu}$$

and that (3.3) holds for all $u \in [\xi_1,\ \xi_2]$.

Let $\epsilon > 0$ be an arbitrary constant such that $\lambda_1 + \epsilon < \lambda_2 - \epsilon$. Let

$$S = [\lambda_1 + \epsilon, \lambda_2 - \epsilon].$$

Next we denote by $\nu_1$ the least index for which $\nu_1 > \nu_0$ and

$$S\sqrt{N_\nu} = [(\lambda_1 + \epsilon)\sqrt{N_\nu}, (\lambda_2 - \epsilon)\sqrt{N_\nu}] \subseteq [\xi_1,\ \xi_2]$$

is satisfied for all $\nu > \nu_1$.

Let $t_j = \sqrt{N_{j+\nu_1}}$. Then (3.3) holds for each $u \in St_j$, which implies from (3.2) that $f(u) = f(u + d)$. Thus, we have proved that

$$f(n_1) = f(n_2) \ne 0 \quad \text{if} \quad n_1 \equiv n_2 \pmod d,\ n_1, n_2 \in St_j, n_1 \in \mathcal{K}_d. \tag{3.4}$$

Now we complete the proof of Theorem 1.

**Proof.** Let $u_j = (\lambda_1 + \epsilon)t_j,\ v_j = (\lambda_2 - \epsilon)t_j$. It is easy to check that if

$$n \in \mathcal{K}_d, \quad n > n_0 = \frac{(\lambda_1 + \epsilon)d}{\lambda_2 - \lambda_1 - 2\epsilon},$$

then

$$\frac{v_j}{n+d} - \frac{u_j}{n} = \frac{\Big[(\lambda_2 - \lambda_1 - 2\epsilon)n - (\lambda_1 + \epsilon)d\Big]t_j}{n(n+d)} \geq d$$

for all sufficiently large integer $j$. This shows that

$$\Big[\frac{u_j}{n}, \ \frac{v_j}{n}\Big] \bigcap \Big[\frac{u_j}{n+d}, \ \frac{v_j}{n+d}\Big]$$

contains an interval of length $\geq d$, consequently there exists $m \in \mathcal{K}_d$ for which $nm \in [u_j, v_j]$ and $(n+d)m \in [u_j, v_j]$. Thus, (3.4) implies $f(nm) = f\big[(n+ +d)m\big] \neq 0$, and so $f(n) = f(n+d)$.

It implies that $f(n)$ is a periodic function $\pmod d$ on the set $n \in \mathcal{K}_d$. Since $f(n) \neq 0$, it should be a character. This completes the proof.

## 4. Proof of Theorem 2

The basic idea is the same as that of the proof of Theorem 1. Let us start with the following lemma.

**Lemma 2.** *Let* $J_{N,M} = [N, N+M]$. *Assume that*

$$g(\nu + d) - g(\nu) \geq 0 \quad \textit{holds for} \quad \nu \equiv \ell \pmod d, \tag{4.1}$$

$\nu \in J_{N,M}$, *where* $(\ell, d) = 1$.

*Assume that* $ur \equiv \ell \pmod d$, *and that* $ur \in J_{N,M}$, $(u+d)r \in J_{N,M}$. *Then*

$$g(u+d) - g(u) \geq 0. \tag{4.2}$$

**Proof.** Assume that $ur \equiv \ell \pmod d$, and that $ur \in J_{N,M}$, $(u+d)r \in \in J_{N,M}$. Then $ur + kd \equiv \ell \pmod d$ and $ur + kd \in J_{N,M}$ for $k = 0, 1, \cdots r$. Thus, we infer from (4.1) that

$$g(ur + dr) - g(ur) = \sum_{k=1}^{r} \big\{ g\big(ur + kd\big) - g\big(ur + (k-1)d\big)\big\} \geq 0.$$

**Proof.** (Proof of Theorem 2) Repeating the argument used in the proof of Theorem 1, in this case we obtain the following assertion:

$$g(n+d)-g(n)\geq 0 \quad \text{if} \quad n\in St_j,\ n\in\mathcal{K}_d, \tag{4.3}$$

where $St_j=[u_j,v_j]=\Big[(\lambda_1+\epsilon)t_j,\ \ (\lambda_2-\epsilon)t_j\Big]$.

As in the proof of Theorem 1, by using (4.3) we can prove that if $j$ is a sufficiently large integer, then for each $n\in\mathcal{K}_d$ there exists $m\in\mathcal{K}_d$ for which $nm\in[u_j,v_j]$ and $(n+d)m\in[u_j,v_j]$. Then we have

$$nm+kd\in[u_j,v_j],\quad nm+kd\in\mathcal{K}_d,\quad (k=0,1,\cdots,m)$$

and so (4.3) implies that

$$g\big(nm+dm\big)-g(nm)=\sum_{k=1}^{r}\big\{g\big(nm+kd\big)-g\big(nm+(k-1)d\big)\big\}\geq 0,$$

Consequently, $g(n+d)-g(n)\geq 0$ for every $n\in\mathcal{K}_d$. Now we can deduce easily that $g(n)=c\log n$, if $(n,d)=1$.

Let $p$ and $q$ be primes, $p\neq q,\ (pq,d)=1$. Let $P=p^{k_0}\equiv 1 \pmod d$, $Q= =q^{\ell_0}\equiv 1 \pmod d$. Let $u_h,v_h$ be such a sequence of integers for which $P^{u_h}< <Q^{v_h}<P^{u_h+1}$. It is obvious that $\frac{u_h}{v_h}\to\frac{\log Q}{\log P}$ as $h\to\infty$.

Furthermore $g(P^{u_h})\leq g(Q^{v_h})\leq g(P^{u_h+1})$, whence

$$\frac{u_h}{v_h}\leq\frac{g(Q)}{g(P)}\leq\frac{u_h+1}{v_h},$$

and so

$$\frac{k_0}{\ell_0}\frac{u_h}{v_h}\leq\frac{g(q)}{g(p)}\leq\frac{k_0}{\ell_0}\frac{u_h+1}{v_h}.$$

Consequently,

$$\frac{g(q)}{g(p)}=\frac{\log q}{\log p}.$$

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