

## ON ADDITIVE FUNCTIONS SATISFYING SOME RELATIONS

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*Dedicated to Dr. János Fehér on his 75th anniversary*

Communicated by L. Germán

(Received July 10, 2012)

**Abstract.** We prove that if an additive commutative semigroup  $\mathbb{G}$  (with identity element 0) and  $\mathbb{G}$ -valued completely additive functions  $f_0, f_1, f_2$  satisfy the relation  $f_0(n) + f_1(2n + 1) + f_2(n + 2) = 0$  for all  $n \in \mathbb{N}$ , then  $f_0(n) = f_1(2n + 1) = f_2(n) = 0$  for all  $n \in \mathbb{N}$ . The same result is proved when the relation  $f_0(n) + f_1(2n - 1) + f_2(n + 2) = 0$  holds for all  $n \in \mathbb{N}$ .

### 1. Introduction

Let  $\mathbb{G}$  be an additive commutative semigroup with identity element 0. Let  $\mathcal{A}_{\mathbb{G}}^*$  denote the set of those functions  $f : \mathbb{N} \rightarrow \mathbb{G}$ , for which  $f(nm) = f(n) + f(m)$  holds for all  $n, m \in \mathbb{N}$ . The domain of  $f \in \mathcal{A}_{\mathbb{G}}^*$  can be extended to  $\mathbb{Q}_+$  (the multiplicative group of positive rationals) by

$$f\left(\frac{n}{m}\right) = f(n) - f(m).$$

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<sup>1</sup>The last two authors have been supported for this project by the European Union and the European Social Fund under the grant agreement TÁMOP-4.2.1/B-09/1/KMR-2010-0003.

If we define  $f(-\alpha) := f(\alpha)$  for  $\alpha \in \mathbb{Q}_+$ , then the equation  $f(\alpha\beta) = f(\alpha) + f(\beta)$  remains valid for arbitrary nonzero rational numbers  $\alpha, \beta$ . Let  $\mathcal{P}$  be the set of primes.

In case  $\mathbb{G} = \mathbb{R}$ , then we simply write  $\mathcal{A}^*$  instead of  $\mathcal{A}_{\mathbb{R}}^*$ .

In an old paper written by Kátai I. [6] the following conjecture has been formulated:

**Conjecture 1.** *If  $f_0, f_1, \dots, f_k \in \mathcal{A}^*$  and,*

$$(1.1) \quad L_n = f_0(n) + f_1(n+1) + \dots + f_k(n+k) \equiv 0 \pmod{1}$$

*for all  $n \in \mathbb{N}$ , then*

$$(1.2) \quad f_0(n) \equiv f_1(n) \equiv \dots \equiv f_k(n) \equiv 0 \pmod{1}$$

*are satisfied for all  $n \in \mathbb{N}$*

This conjecture has been proved for  $k = 2, 3$  (see [4] and [5]) and in [3] the case  $k = 4$  assuming the fulfilment of relation (1.1) for every  $n \in \mathbb{Z}$ . Here we define  $f_j(0) = 0$  ( $j = 0, \dots, k$ ). P.D.T.A. Elliott investigated the case when  $f_j = f_0$  and  $f_j = -f_0$  for  $j = 1, \dots, k$  is arbitrary (see [1] and [2]), and even the case when  $f_j \in \{f_0, -f_0, f_1, -f_1\}$ , ( $j = 2, \dots, k$ ).

For other results we refer to works [7], [8], [9] and [10]

The following, more general problem seems to be interesting, also. Let  $A_0(n), A_1(n), \dots, A_k(n) \in \mathbb{Q}$  for all  $n \in \mathbb{N}$  and  $f_0, f_1, \dots, f_k \in \mathcal{A}^*$  for which

$$f_0(A_0(n)) + f_1(A_1(n)) + \dots + f_k(A_k(n)) \equiv 0 \pmod{1}$$

holds. Under what conditions can we assert that

$$f_0(n) \equiv f_1(n) \equiv \dots \equiv f_k(n) \equiv 0 \pmod{1}$$

are satisfied for all  $n \in \mathbb{N}$ .

In this short paper we investigate the simple non-trivial case

$$(A_0(n), A_1(n), A_2(n)) = (n, 2n+1, n+2)$$

and

$$(A_0(n), A_1(n), A_2(n)) = (n, 2n-1, n+2).$$

## 2. Formulation of the theorems

We shall prove the following two theorems.

**Theorem 2.1.** *Let  $\mathbb{G}$  be an additive commutative semigroup with identity element 0. If  $f_0, f_1, f_2 \in \mathcal{A}_{\mathbb{G}}^*$  and*

$$\mathcal{A}(n) := f_0(n) + f_1(2n + 1) + f_2(n + 2) = 0$$

*holds for all  $n \in \mathbb{N}$ , then*

$$f_0(n) = f_1(2n + 1) = f_2(n) = 0$$

*hold for all  $n \in \mathbb{N}$ .*

**Theorem 2.2.** *Let  $\mathbb{G}$  be an additive commutative semigroup with identity element 0. If  $f_0, f_1, f_2 \in \mathcal{A}_{\mathbb{G}}^*$  and*

$$\mathcal{B}(n) := f_0(n) + f_1(2n - 1) + f_2(n + 2) = 0$$

*holds for all  $n \in \mathbb{N}$ , then*

$$f_0(n) = f_1(2n - 1) = f_2(n) = 0$$

*hold for all  $n \in \mathbb{N}$ .*

## 3. Lemmas

Firstly we prove a few lemmas.

**Lemma 1.** *Assume that  $f_0, f_1, f_2 \in \mathcal{A}_{\mathbb{G}}^*$  satisfy the condition  $\mathcal{A}(n) = 0$  in Theorem 2.1 for all  $n \in \mathbb{N}$ . Let  $f_1(2) = 0$ . Then*

$$f_0(n) = f_1(n) = f_2(n) = 0$$

*holds for all  $n \leq 5$ .*

**Proof.** Let  $\mathcal{B}$  be the subgroup of  $\mathbb{Q}_+^3$  generated by the element  $(1, 2, 1)$  and the sequences

$$L_n = \left( n, 2n + 1, n + 2 \right) \quad (n \in \mathbb{N}).$$

Since  $\mathcal{A}(n) = 0$ , therefore

$$f_0(a) + f_1(b) + f_2(c) = 0 \quad \text{for all } (a, b, c) \in \mathcal{B}.$$

We use the following notations for a prime  $p$ :

$$a_p = (p, 1, 1), \quad b_p = (1, p, 1) \quad \text{and} \quad c_p = (1, 1, p).$$

We show that  $a_p, b_p$ , and  $c_p$  are elements of  $\mathcal{B}$  for all primes  $p \leq 19$ . This assertion proves Lemma 1.

Using a simple Maple program and the relation  $\mathcal{A}(n) = 0$  for  $n = 4, 25, 38, 40$  and  $n = 42$ , we will get the following 5 equations.

$$(3.1) \quad E_1 := \frac{L_4}{L_2} = a_2^2 b_3 c_2 \in \mathcal{B},$$

$$(3.2) \quad E_2 := \frac{L_3 L_{12} L_{16} L_{25}}{L_1^4 L_2^2 L_5 L_8 L_{10}} = \frac{a_3^2}{b_3^3 c_2^5} \in \mathcal{B},$$

$$(3.3) \quad E_3 := \frac{L_1^3 L_6 L_{12} L_{38}}{L_2^2 L_3 L_{16} L_{19}} = \frac{a_3 b_3 c_2^2}{a_2^2} \in \mathcal{B},$$

$$(3.4) \quad E_4 := \frac{L_{16} L_{40}}{L_1^3 L_5} = a_2^7 b_3^2 c_2^2 \in \mathcal{B},$$

and

$$(3.5) \quad E_5 := \frac{L_1^3 L_2 L_{28} L_{42}}{L_7^2 L_8 L_9} = \frac{a_2 b_3^2 c_2^4}{a_3} \in \mathcal{B}.$$

This system has solutions in  $a_2, a_3, b_3, c_2$ , which are given in terms of  $E_1, \dots, E_5$ . Thus  $a_2, a_3, b_3, c_2$  are elements of  $\mathcal{B}$ .

The solutions of the above equations (3.1)-(3.5) are:

$$a_2 = \frac{E_1^{98} E_2^{24} E_5^{32}}{E_3^{16} E_4^{37}}, \quad a_3 = \frac{E_1^{558} E_2^{136} E_5^{181}}{E_3^{90} E_4^{211}}$$

$$b_3 = \frac{E_3^5 E_4^{11}}{E_1^{28} E_2^8 E_5^{11}} \quad \text{and} \quad c_2 = \frac{E_3^{27} E_4^{63}}{E_1^{167} E_5^{53} E_2^{40}}.$$

Finally, we express  $a_5$ ,  $b_5$ ,  $c_3$  and  $c_5$  in the terms of  $a_2, a_3, b_3, c_2$  and  $L_n$ . We have

$$a_5 = \frac{L_1^2 L_2^2 L_5 a_2^4 a_3}{L_{12} L_{16} b_3 c_2^2}, \quad b_5 = \frac{L_2}{a_2 c_2^2}$$

and

$$c_3 = \frac{L_1}{b_3} \quad \text{and} \quad c_5 = \frac{L_1^3 L_2^2 L_3 L_5 a_2^5}{L_{10} L_{12} L_{16} b_3}$$

are elements of  $\mathcal{B}$ . This completes the proof of Lemma 1.

**Lemma 2.** *Assume that  $f_0, f_1, f_2 \in \mathcal{A}_{\mathbb{G}}^*$  satisfy the condition  $\mathcal{B}(n) = 0$  in Theorem 2.2 for all  $n \in \mathbb{N}$ . Let  $f_1(2) = 0$ . Then*

$$f_0(n) = f_1(n) = f_2(n) = 0$$

holds for all  $n \leq 7$ .

**Proof.** The proof is similar to the proof of Lemma 1. Let  $\mathcal{D}$  be the subgroup of  $\mathbb{Q}_+^3$  generated by the element  $(1, 2, 1)$  and the sequences

$$D_n := (n, 2n - 1, n + 2) \quad (n \in \mathbb{N}).$$

From our assumption  $\mathcal{B}(n) = 0$  for all  $n \in \mathbb{N}$ , we have

$$f_0(a) + f_1(b) + f_2(c) = 0 \quad \text{for all } (a, b, c) \in \mathcal{D}.$$

We shall use the following notations ( $p$  is prime):

$$A_p := (p, 1, 1) \in \mathcal{D}, \quad B_p := (1, p, 1) \in \mathcal{D} \quad \text{and} \quad C_p := (1, 1, p) \in \mathcal{D}.$$

By using a simple Maple program and the relation  $\mathcal{B}(n) = 0$  for  $n = 8, 18, 26, 28$  and  $n = 63$ , we obtain the following 5 equations in  $A_2, A_3, B_5, C_2$ :

$$(3.6) \quad F_1 := \frac{D_8}{D_2 D_3} = \frac{A_2^2}{A_3 C_2} \in \mathcal{D},$$

$$(3.7) \quad F_2 := \frac{D_1 D_{18}}{D_3 D_4} = \frac{A_3 C_2}{A_2} \in \mathcal{D},$$

$$(3.8) \quad F_3 := \frac{D_1^3 D_3 D_{14} D_{20} D_{26}}{D_2^3 D_5 D_7 D_9 D_{13}} = \frac{A_2 C_2}{A_3 B_5} \in \mathcal{D},$$

$$(3.9) \quad F_4 := \frac{D_2^3 D_{28}}{D_1 D_3 D_6 D_{14}} = \frac{A_2^3}{A_3^2} \in \mathcal{D},$$

$$(3.10) \quad F_5 := \frac{D_2^5 D_3^3 D_6 D_{10}^2 D_{63}}{D_1^2 D_{14} D_{48}^2 D_{50}} = \frac{A_3^4 B_5^4 C_2^9}{A_2^2} \in \mathcal{D}.$$

This system has solutions in  $A_2, A_3, B_5, C_2$ , which are given in terms of  $F_1, \dots, F_5$ . Thus  $A_2, A_3, B_5, C_2$  are elements of  $\mathcal{D}$ .

The solutions of the above equations (3.6)-(3.10) are:

$$A_2 = F_1 F_2, \quad A_3 = \frac{F_2^{10} F_4^6}{F_1^3 F_3^4 F_5},$$

$$B_5 = \frac{F_3^7 F_5^2 F_1^8}{F_2^{17} F_4^{12}} \quad \text{and} \quad C_2 = \frac{F_3^4 F_5 F_1^4}{F_2^8 F_4^6}.$$

Now, we express  $A_5, A_7, B_3, B_7, C_3, C_5$  and  $C_7$  in the terms of  $A_2, A_3, B_5, C_2$  and  $L_n$ . We have

$$A_5 = \frac{L_3^2 L_{10} A_2^3}{L_1 L_{48} A_3 B_5 C_2}, \quad A_7 = \frac{L_{14} A_2^2 C_2^2}{L_2^3},$$

$$B_3 = \frac{L_2}{A_2 C_2^2}, \quad B_7 = \frac{L_4}{L_1 A_2^2 C_2}$$

and

$$C_3 = L_1, \quad C_5 = \frac{L_3}{A_3 B_5}, \quad C_7 = \frac{L_1 L_5 L_{48} A_3 B_5 C_2^5}{L_2^2 L_3^2 L_{10} A_2}$$

are elements of  $\mathcal{D}$ . This completes the proof of Lemma 2.

#### 4. Proof of Theorem 2.1

Let  $\mathbb{G}$  be an additive commutative semigroup with identity element 0. If  $f_0, f_1, f_2 \in \mathcal{A}_{\mathbb{G}}^*$  and

$$\mathcal{A}(n) := f_0(n) + f_1(2n+1) + f_2(n+2) = 0$$

holds for all  $n \in \mathbb{N}$ . By using Lemma 1, we have  $f_0(p) = f_1(q) = f_2(p) = 0$  for primes  $p \leq 5$  and  $q = 3, 5$ .

Assume indirectly that the theorem is not true. Let  $n_0$  be the smallest positive integer for which  $f_j(n_0) \neq 0$ . Then  $n_0 = P \in \mathcal{P}$ ,  $P > 5$  and either  $f_0(P) \neq 0$  or  $f_2(P) \neq 0$ .

**Case I.**  $f_0(P) = \xi (\neq 0)$ .

If  $P \equiv 1 \pmod{3}$ , then  $3|P+2, 3|2P+1$ , thus  $f_1(P+2) = 0, f_2(2P+1) = 0$ , consequently  $\mathcal{A}(P) = 0$  implies that  $f_0(P) = 0$ .

It remains to consider the case  $P \equiv -1 \pmod{3}$ . Let  $4P+1 = 3Q$ . Then it follows from the fact  $P > 5$  that  $\frac{Q-1}{2} < \frac{Q+3}{2} < P$ , consequently  $\mathcal{A}(\frac{Q-1}{2}) = 0$  implies that

$$0 = f_0\left(\frac{Q-1}{2}\right) + f_1(Q) + f_2\left(\frac{Q+3}{2}\right) = f_1(Q),$$

thus we infer from  $\mathcal{A}(2P) = 0$  that

$$0 = f_0(2P) + f_1(3Q) + f_2\left(4\frac{P+1}{2}\right)$$

and so  $f_0(P) = 0$ .

**Case II.**  $f_2(P) = \nu (\neq 0)$ .

|     | From $\mathcal{A}(n) = 0$ | we obtain that                   | and that                               |
|-----|---------------------------|----------------------------------|--|
| (1) | $\mathcal{A}(P-2) = 0$    | $2P-3 \in \mathcal{P}$           | $f_1(2P-3) = -\nu$                     |
| (2) | $\mathcal{A}(2P-2) = 0$   | $4P-3 \in \mathcal{P}$           | $f_1(4P-3) = -\nu$                     |
| (3) | $\mathcal{A}(6P-2) = 0$   | $4P-1 \in \mathcal{P}$           | $f_1(4P-1) = -\nu$                     |
| (4) | $\mathcal{A}(3P-5) = 0$   | $\frac{3P-5}{2} \in \mathcal{P}$ | $f_0\left(\frac{3P-5}{2}\right) = \nu$ |
| (5) |                           | $P \equiv 2 \pmod{3}$            |  |
| (6) | $\mathcal{A}(4P-2) = 0$   | $8P-3 \in \mathcal{P}$           | $f_1(8P-3) = -\nu$                     |
| (7) |                           | $P \equiv 3 \pmod{5}$            |  |

The assertions (1) and (2) are clear.

In order to show (3), let  $Q := \frac{3P-1}{2}$ . Then we have

$$Q \equiv 1 \pmod{3}, 3|Q+2, \frac{Q+2}{3} = \frac{P+1}{2} < P \quad \text{and} \quad 2Q+1 = 3P,$$

which with  $\mathcal{A}(Q) = 0$  shows that

$$f_0(Q) + f_1(2Q+1) + f_2(Q+2) = f_0(Q) + f_1(P) = 0.$$

It is clear from  $\mathcal{A}(\frac{P-1}{2}) = 0$  that  $f_1(P) = 0$ , consequently  $f_0(Q) = f_0(3P-1) = f_0(6P-2) = 0$ , thus  $\mathcal{A}(6P-2) = 0$  implies

$$0 = f_0(6P-2) + f_1(12P-3) + f_2(6P) = f_1(4P-1) + f_2(P),$$

which proves (3).

From  $\mathcal{A}(3P - 5) = 0$  we have

$$0 = f_0(3P - 5) + f_1(6P - 9) + f_2(3P - 3) = f_0\left(\frac{3P - 5}{2}\right) + f_1(2P - 3),$$

from (1) we obtain (4).

Since  $P \in \mathcal{P}$ , the assertion (5) follows from (3).

From (5), we have  $3|2P - 1$  and  $\frac{2P-1}{3} < P$ , consequently  $f_0(4P - 2) = f_0(2P - 1) = 0$ . Thus we obtain from  $\mathcal{A}(4P - 2) = 0$  that

$$0 = f_0(4P - 2) + f_1(8P - 3) + f_2(4P) = f_1(8P - 3) + f_2(P),$$

which proves (6). Since  $P \in \mathcal{P}$ , the assertion (7) follows from (1), (2) and (6).

Let  $T := \frac{3P-5}{2}$ . Then we infer from (4) and (7) that

$$(4.1) \quad T \in \mathcal{P}, \quad f_0(T) = \nu$$

and

$$(4.2) \quad T \equiv -1 \pmod{3}, \quad T \equiv 2 \pmod{5}.$$

From (4.2) we have  $5|2T + 1$ ,  $\frac{2T+1}{5} = \frac{3P-4}{5} < P$ , consequently we obtain from  $\mathcal{A}(T) = 0$  that

$$\mathcal{A}(T) = f_0(T) + f_1(2T + 1) + f_2(T + 2) = f_0(T) + f_2(T + 2) = 0.$$

This with (4.1) implies

$$(4.3) \quad f_2(T + 2) = -\nu.$$

From (4.2), we have  $5|3T + 4$ ,  $\frac{3T+4}{5} = \frac{9P-7}{10} < P$  and  $f_0(3T + 4) = f_0\left(\frac{3T+4}{5}\right) = 0$ . Thus we obtain from  $\mathcal{A}(3T + 4) = 0$  that

$$0 = f_0(3T + 4) + f_1(6T + 9) + f_2(3T + 6) = f_1(2T + 3) + f_2(T + 2),$$

which with (4.3) implies

$$(4.4) \quad f_1(2T + 3) = \nu.$$

Finally,  $\mathcal{A}(T + 1) = 0$  implies that

$$f_0(T + 1) + f_1(2T + 3) + f_2(T + 3) = 0.$$

Since  $\frac{T+1}{2} = \frac{3P-1}{4} < P$ ,  $\frac{T+3}{2} = \frac{3P+1}{4} < P$ , we deduce that  $f_1(2T + 3) = 0$ . This contradicts to (4.4).

The proof of Theorem 1 is complete.



**5. Proof of Theorem 2.2**

Let  $\mathbb{G}$  be an additive commutative semigroup with identity element 0. If  $f_0, f_1, f_2 \in \mathcal{A}_{\mathbb{G}}^*$  and

$$\mathcal{B}(n) := f_0(n) + f_1(2n + 1) + f_2(n + 2) = 0$$

holds for all  $n \in \mathbb{N}$ . By using Lemma 2, we have  $f_0(p) = f_1(q) = f_2(p) = 0$  for primes  $p \leq 5$  and  $q = 3, 5$ .

Assume indirectly that the theorem is not true. Let  $n_0$  be the smallest positive integer for which  $f_j(n_0) \neq 0$ . Then  $n_0 = P \in \mathcal{P}, P > 7$  and either  $f_0(P) \neq 0$  or  $f_2(P) \neq 0$ .

**Case I.**  $f_2(P) = \nu (\neq 0)$ .

We infer from  $\mathcal{B}(P - 2) = 0$  that  $f_1(2P - 5) = -\nu, 2P - 5 \in \mathcal{P}$  and so  $P \equiv -1 \pmod{3}$ . We have

$$\mathcal{B}(2P - 2) = f_0(2P - 2) + f_1(4P - 5) + f_2(2P) = 0.$$

Since  $f_0(2P - 2) = 0$ , therefore  $f_1(4P - 5) = -\nu, Q = \frac{4P-5}{3} \in \mathcal{P}$  and

$$\mathcal{B}\left(\frac{Q+1}{2}\right) = f_0\left(\frac{Q+1}{2}\right) + f_1(Q) + f_2\left(\frac{Q+5}{2}\right) = 0.$$

Since  $P > 5$ , we have  $\frac{Q+1}{2} = \frac{2P-1}{3} < P$  and  $\frac{Q+5}{2} = \frac{2P+5}{3} < P$ , consequently

$$f_1(Q) = f_1\left(\frac{4P-5}{3}\right) = f_1(4P - 5) = 0.$$

This cannot occur.

**Case II.**  $f_0(P) = \xi (\neq 0)$ .

Since  $4 \mid 2P + 2$  and  $\frac{2P+2}{4} = \frac{P+1}{2} < P$ , we infer from  $\mathcal{B}(2P) = 0$  that

$$0 = \mathcal{B}(2P) = f_0(2P) + f_1(4P - 1) + f_2(2P + 2) = \xi + f_1(4P - 1).$$

Thus  $f_1(4P - 1) = -\xi$  and either  $4P - 1 \in \mathcal{P}$  or  $\frac{4P-1}{3} \in \mathcal{P}$ .

If  $Q := \frac{4P-1}{3} \in \mathcal{P}$ , then  $P > 7$  shows that  $\frac{Q+1}{2} = \frac{2P+1}{3} < P$  and  $\frac{Q+5}{2} = \frac{2P+7}{3} < P$ , consequently

$$0 = \mathcal{B}\left(\frac{Q+1}{2}\right) = f_0\left(\frac{Q+1}{2}\right) + f_1(Q) + f_2\left(\frac{Q+5}{2}\right) = f_1(Q).$$

This cannot occur. Thus we have proved that  $4P - 1 \in \mathcal{P}$ , and so

$$(5.1) \quad P \equiv -1 \pmod{3}.$$

From (5.1) we have  $3|2P - 1$ , consequently  $\mathcal{B}(P) = 0$  implies that  $f_2(P + 2) = -\xi$ ,  $P + 2 \in \mathcal{P}$ .

Since

$$0 = \mathcal{B}(2P) = f_0(2P) + f_1(4P - 1) + f_2(2P + 2)$$

and  $f_2(2P + 2) = 0$ , we have

$$0 = \mathcal{B}(2P) = f_0(2P) + f_1(4P - 1) + f_2(2P + 2) = \xi + f_1(4P - 1),$$

consequently  $f_1(4P - 1) = -\xi$ ,  $4P - 1 \in \mathcal{P}$ .

On the other hand, we have

$$\mathcal{B}(2P + 2) = f_0(2P + 2) + f_1(4P + 3) + f_2(2P + 4) = 0,$$

which implies that  $f_1(4P + 3) = \xi$ ,  $4P + 3 \in \mathcal{P}$ . Since  $P, P + 2, 4P - 1, 4P + 3 \in \mathcal{P}$ , therefore  $P \equiv 1 \pmod{5}$  or  $P \equiv 2 \pmod{5}$ .

**Case II.a.**  $P \equiv 2 \pmod{5}$ .

Since  $15|8P - 1, 5|2P + 1$ , therefore

$$\mathcal{B}(4P) = f_0(4P) + f_1\left(\frac{8P - 1}{15}\right) + f_2\left(\frac{2P + 1}{5}\right) = 0,$$

therefore  $f_0(P) = 0$ .

**Case II.b.**  $P \equiv 1 \pmod{5}$ .

In this case  $6P - 1 = 5Q$ ,  $\frac{Q+1}{2} = \frac{3P+2}{5} < P$  and  $\frac{Q+5}{2} = \frac{3P+12}{5} < P$ . Thus

$$\mathcal{B}\left(\frac{Q+1}{2}\right) = f_0\left(\frac{Q+1}{2}\right) + f_1(Q) + f_2\left(\frac{Q+5}{2}\right) = f_1(Q).$$

Hence

$$(5.2) \quad f_1(Q) = f_1(6P - 1) = 0.$$

Since  $5|3P + 2$ ,  $f_2(3P + 2) = 0$ , therefore  $\mathcal{B}(3P) = 0$  with (5.2) implies that  $f_0(P) = 0$ .

The proof of Theorem 2 is complete.

## 6. Final remarks

**Theorem 6.1.** Let  $\alpha_0 = \beta_0 = (1, 2, 1)$  and  $\alpha_n = (n, 2n + 1, n + 2)$ ,  $\beta_n = (n, 2n - 1, n + 2)$ . Let  $\mathcal{B}$  be the subgroup of  $\mathbb{Q}_+^3$  generated by  $\alpha_n$  ( $n = 0, 1, 2, \dots$ ) and  $\mathcal{D}$  be the subgroup of  $\mathbb{Q}_+^3$  generated by  $\beta_n$  ( $n = 0, 1, 2, \dots$ ). Then

$$\mathcal{B} = \mathbb{Q}_+^3 \quad \text{and} \quad \mathcal{D} = \mathbb{Q}_+^3.$$

It means that for every  $(r_1, r_2, r_3) \in \mathbb{Q}_+^3$  there exist  $n_1, n_2, \dots, n_k \in \mathbb{N}_0$ ,  $\epsilon_1, \epsilon_2, \dots, \epsilon_k \in \{-1, 1\}$  and  $m_1, m_2, \dots, m_l \in \mathbb{N}_0$ ,  $\delta_1, \delta_2, \dots, \delta_l \in \{-1, 1\}$  such that

$$(r_1, r_2, r_3) = \prod_{i=1}^k \alpha_{n_i}^{\epsilon_i},$$

and that

$$(r_1, r_2, r_3) = \prod_{i=1}^l \alpha_{m_i}^{\delta_i}.$$

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