ON THE PRIME DIVISORS OF THE EULER PHI AND THE SUM OF DIVISORS FUNCTIONS

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Abstract. The following assertion is proved. Given an arbitrary constant $\lambda > 2$, let $x_1 = \log x$, $x_{k+1} = \log x_k$ $(k = 1, 2, \ldots)$, $\varphi(n)$ be Euler's totient, and $\sigma(n)$ the sum of divisors function. Let $I_x = \left[\frac{\lambda x_2}{x_3}, x_2\right]$, $Q_1, Q_2 \in I_x$ be primes,

$$E_{Q_1,Q_2}(x) := \#\{n \leq x \mid Q_1 \not| \varphi(n),\ Q_2 \not| \varphi(n+1)\}.$$

Then, uniformly for $Q_1, Q_2 \in I_x$,

$$\frac{1}{x} E_{Q_1,Q_2}(x) = (1 + o_x(1)) \frac{B}{2} \kappa_1 \kappa_2,$$

where $\kappa_j = \exp\left(-\frac{x_2}{Q_j - 1}\right)$ $(j = 1, 2)$, B is a given constant. Some other assertions are formulated without proof.

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1. Introduction

Let $\varphi(n)$ be Euler's totient function, $\sigma(n)$ be the sum of divisors function. We shall define the iterates of φ and σ as follows:

$$\varphi_{k+1}(n)=\varphi(\varphi_k(n)),\quad \sigma_{k+1}(n)=\sigma(\sigma_k(n)),$$
$$\varphi_1(n)=\varphi(n),\quad \sigma_1(n)=\sigma(n).$$

Let $\mathcal{P}$ be the set of primes. It is known that φ and σ are multiplicative functions, and if p^α is a prime power, then $\varphi(p^\alpha)=p^{\alpha-1}(p-1)$, $\sigma(p^\alpha)=$ $=1+p+\ldots+p^\alpha$. In particular, $\varphi(p)=p-1$, $\sigma(p)=p+1$. The letters p,Q with and without suffixes always denote prime numbers. As usual let $p(n)$ be the smallest and $P(n)$ be the largest prime divisor of n.

Let $x_1=\log x$, $x_2=\log x_1,\ldots$.

The letters $c,c_1,c_2,\ldots$ denote suitable, $d,d_1,d_2,\ldots$ be arbitrary positive constants, not necessarily the same at every occurrence.

Let $(\xi_p=)\ \xi_p(x)=e^{-\frac{x_2}{p-1}}\quad (p\in\mathcal{P})$,

$$\tau(Q\mid x)=\xi_Q(x)\prod_{\substack{p<Q\\ p\in\mathcal{P}}}(1-\xi_p(x)),\quad Q\in\mathcal{P}.$$

Let
$$\mathcal{P}_\pm(Q):=\{p\mid p\in\mathcal{P},\quad p\equiv\pm1\pmod Q\}.$$

Since $Q\nmid\varphi(n)$, $n\le x$ holds if and only if $(n,\mathcal{P}_+(Q))=1$ and $Q^2\nmid n$, thus, by the sieve of Eratosthenes-Brun we obtain that

$$\begin{aligned}\frac{E_Q(x)}{x}:=&\frac{1}{x}\#\{n\le x\mid Q\nmid\varphi(n)\}=\\ =&(1+o_x(1))\left(1-\frac{1}{Q^2}\right)\prod_{\substack{p\le x\\ p\in\mathcal{P}_+(Q)}}(1-1/p)\end{aligned}\tag{1.1}$$

if $Q\le x_2^d$.

Since
$$\begin{aligned}\pi(x,k,l)=&\#\{p\le x,\quad p\equiv l\pmod k\}=\\ =&\frac{1}{\varphi(k)}(1+\mathcal{O}(e^{-c\sqrt{x_1}}))\end{aligned}$$

holds uniformly as $(k,l)=1, \quad k \le x_1^{d_1}$, we obtain that the right hand side of (1.1) is $\xi_q(x)\left(1+\mathcal{O}\left(\frac{\log Q}{Q}\right)\right)(1+o_x(1))$. We shall list several assertions which can be deduced by immediate application of the Brun sieve (see Theorem 2.5 in [7]).

I. Let $Q \in [x_3, x_2], \quad Q \in \mathcal{P}, \quad l \in \mathbb{Z}, \quad l \ne 0$. Then

$$\frac{1}{x}\#\{n \le x \mid \sigma(n) \not\equiv 0 \pmod{Q}\} = (1+o_x(1))\xi_Q(x), \tag{1.2}$$

$$\begin{aligned} &\frac{1}{x}\#\{n \le x \mid \varphi(n) \not\equiv 0 \pmod{Q},\ \sigma(n) \not\equiv 0 \pmod{Q}\} = \\ &\qquad = (1+o_x(1))\xi_Q^2(x), \end{aligned} \tag{1.3}$$

$$\frac{1}{lix}\#\{p \le x \mid \varphi(p+l) \not\equiv 0 \pmod{Q}\} = (1+o_x(1))\xi_Q(x), \tag{1.4}$$

$$\frac{1}{lix}\#\{p \le x \mid \sigma(p+l) \not\equiv 0 \pmod{Q}\} = (1+o_x(1))\xi_Q(x), \tag{1.5}$$

$$\begin{aligned} &\frac{1}{lix}\#\{p \le x \mid \varphi(p+l) \not\equiv 0 \pmod{Q}, \sigma(p+l) \not\equiv 0 \pmod{Q}\} = \\ &\qquad = (1+o_x(1))\xi_Q^2(x). \end{aligned} \tag{1.6}$$

II. Similar assertions can be proved for the set of integers $F(n)$ or $F(p)$, where $F \in \mathbb{Z}[x]$ is a polynomial the leading coefficient of which is positive.

III. Let $Q_1, \dots, Q_r; \quad Q_1^*, \dots, Q_s^* \in [x_3, x_2]$ be primes, $Q_i \ne Q_j$ if $i \ne j, \quad Q_u^* \ne Q_v^*$ if $u \ne v$. Let $l \ne 0$. Then

$$\begin{aligned} &\frac{1}{x}\#\{n \le x \mid (\varphi(n), Q_1 \dots Q_r) = 1, (\sigma(n), Q_1^* \dots Q_s^*) = 1\} = \\ &\qquad = (1+o_x(1))\left\{\prod_{j=1}^{r} \xi_{Q_j}(x)\right\}\left\{\prod_{l=1}^{s} \xi_{Q_l^*}(x)\right\}, \end{aligned} \tag{1.7}$$

furthermore

$$\begin{aligned} &\frac{1}{lix}\#\{p \le x \mid (\varphi(p+l), Q_1 \dots Q_r) = 1,\ (\sigma(p+l), Q_1^* \dots Q_s^*) = 1\} = \\ &\qquad = (1+o_x(1))\left\{\prod_{j=1}^{r} \xi_{Q_j}(x)\right\}\left\{\prod_{l=1}^{s} \xi_{Q_l^*}(x)\right\}, \end{aligned} \tag{1.8}$$

For fixed Q, (1.1), (1.2) and (1.3) can be improved (see [2], [1]).

2. Counting those integers n for which $\varphi(n)$ and $\sigma(n)$ each avoid a given prime as their smaller prime factor

Let $u(n)$ be the smallest prime Q for which $Q\nmid\varphi(n)$, and $v(n)$ be the smallest $Q\in\mathcal{P}$, for which $Q\nmid\sigma(n)$.

Let $K_Q(x):=\#\{n\le x\mid u(n)=Q\}$, $\quad T_Q(x):=\#\{n\le x\mid v(n)=Q\}$,

$$S_{Q_1,Q_2}(x)=\#\{n\le x\mid u(n)=Q_1, v(n)=Q_2\}.$$

Theorem 1. *Assume that* $Q,Q_1,Q_2\in\left[x_3,\frac{x_2}{x_3}\right]$. *Then*

$$\frac{K_Q(x)}{x}=(1+o_x(1))\xi_Q(x), \tag{2.1}$$

$$\frac{T_Q(x)}{x}=(1+o_x(1))\xi_Q(x), \tag{2.2}$$

$$\frac{S_{Q_1,Q_2}(x)}{x}=(1+o_x(1))\xi_{Q_1}(x)\cdot\xi_{Q_2}(x). \tag{2.3}$$

Furthermore, if $l\ne 0$, *then*

$$\frac{1}{lix}\#\{p\le x\mid u(p+l)=Q\}=(1+o_x(1))\xi_Q(x), \tag{2.4}$$

$$\frac{1}{lix}\#\{p\le x\mid v(p+l)=Q\}=(1+o_x(1))\xi_Q(x), \tag{2.5}$$

$$\begin{aligned}\frac{1}{lix}\#\{p\le x\mid u(p+l)=Q_1, v(p+l)=Q_2\}=\\ =(1+o_x(1))\xi_{Q_1}(x)\cdot\xi_{Q_2}(x).\end{aligned} \tag{2.6}$$

Remark 1. Unfortunately we cannot extend Theorem 1 for the values $Q,Q_1,Q_2\ge x_2/x_3$.

Conjecture 1. *Let d be a positive constant. Then, uniformly as $x_3 < Q, Q_1, Q_2 \le dx_2$ we have*

$$\frac{1}{x}\#\{n \le x \mid u(n) = Q\} = (1 + o_x(1))\tau(Q|x), \tag{2.7}$$

$$\frac{1}{x}\#\{n \le x \mid v(n) = Q\} = (1 + o_x(1))\tau(Q|x), \tag{2.8}$$

$$\frac{1}{x}\#\{n \le x \mid u(n) = Q_1, v(n) = Q_2\} = = (1 + o_x(1))\tau(Q_1|x)\tau(Q_2|x). \tag{2.9}$$

Remark. Similar assertion seems to hold for the set of shifted primes as well.

In [2] we considered $N_k(Q|x)$, the number of those $n \le x$ for which $Q \nmid \varphi_{k+1}(n)$. We determined the asymptotic of $N_k(Q|x)$ in the range $Q \in (x_2^{k+\varepsilon}, x_2^{k+1-\varepsilon})$. By using the same method with some generalization we could prove

Theorem 2. *Let $\varepsilon > 0$, $k \ge 2$ be fixed, $l \ne 0$, $l \in \mathbb{Z}$, and let $x_2^{k+\varepsilon} \le Q \le x_2^{k+1-\varepsilon}$, $Q \in \mathcal{P}$. Then, setting $\eta_{k,Q}(x) := \exp\left(-\frac{x_2^{k+1}}{(k+1)!(Q-1)}\right)$, we obtain*

$$\frac{1}{x}\#\{n \le x \mid Q \nmid \sigma_{k+1}(n)\} = (1 + o_x(1))\eta_{k,Q}(x), \tag{2.10}$$

$$\frac{1}{lix}\#\{p \le x \mid Q \nmid \sigma_{k+1}(p + l)\} = (1 + o_x(1))\eta_{k,Q}(x), \tag{2.11}$$

$$\frac{1}{x}\#\{n \le x \mid Q \nmid \sigma_{k+1}(n), Q \nmid \varphi_{k+1}(n)\} = = (1 + o_x(1))\eta_{k,Q}^2(x), \tag{2.12}$$

and

$$\frac{1}{lix}\#\{p \le x \mid Q \nmid \sigma_{k+1}(p + l), Q \nmid \varphi_{k+1}(p + l)\} = = (1 + o_x(1))\eta_{k,Q}^2(x). \tag{2.13}$$

Theorem 3. *Let* $\varepsilon > 0$, $k \geq 2$, $r, s \geq 1$. *Let* $Q_1, \ldots, Q_r$ *and* $Q_1^*, \ldots, Q_s^*$ *be distinct primes from the interval* $\left[x_2^{k+\frac{1}{2}+\varepsilon}, x_2^{k+1-\varepsilon}\right]$. *Then*

$$\begin{aligned}&\frac{1}{x}\#\{n \leq x \mid (Q_1 \ldots Q_r, \varphi_{k+1}(n)) = 1,\ (Q_1^* \ldots Q_s^*, \sigma_{k+1}(n)) = 1\} = \\ &\qquad = (1 + o_x(1)) \left\{\prod_{j=1}^{r} \eta_{k,Q_j}(x)\right\} \left\{\prod_{l=1}^{s} \eta_{l,Q_l^*}(x)\right\},\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}&\frac{1}{lix}\#\{p \leq x \mid (Q_1 \ldots Q_r, \varphi_{k+1}(p+l)) = 1,\ (Q_1^* \ldots Q_s^*, \sigma_{k+1}(p+l)) = 1\} = \\ &\qquad = (1 + o_x(1)) \left\{\prod_{j=1}^{r} \eta_{k,Q_j}(x)\right\} \left\{\prod_{l=1}^{s} \eta_{l,Q_l^*}(x)\right\}.\end{aligned} \tag{2.15}$$

We shall not prove these theorems.

3. Counting those integers n for which $\varphi(n)$ and $\varphi(n+1)$ each avoid given primes in their respective prime factorizations

The problem of giving the asymptotic of those $n \leq x$ for which $Q_1 \nmid \varphi(n)$ and $Q_2 \nmid \varphi(n+1)$ simultaneously for given primes Q_1, Q_2 seems to be much harder. We are unable to determine it for example if $Q_1 = Q_2 = 3$.

Theorem 4. *Let* $\lambda > 2$ *be an arbitrary constant. Let* $\mathcal{I}_x = \left[\frac{\lambda x_2}{x_3}, x_2\right]$,

$$B = \prod_{p \geq 3} \left(1 - \frac{2}{p(p-1)}\right).$$

Let $Q_1, Q_2 \in \mathcal{I}_x$ *be arbitrary primes. Let*

$$E_{Q_1,Q_2}(x) := \#\{n \leq x \mid Q_1 \nmid \varphi(n),\ Q_2 \nmid \varphi(n+1)\}. \tag{3.1}$$

Then, uniformly as $Q_1, Q_2 \in \mathcal{I}_x$,

$$\frac{1}{x} E_{Q_1,Q_2}(x) = (1 + o_x(1)) \frac{B}{2} \kappa_1 \kappa_2, \tag{3.2}$$

where $\kappa_1 = \exp\left(-\frac{x_2}{Q_1-1}\right), \quad \kappa_2 = \exp\left(-\frac{x_2}{Q_2-1}\right)$.

Remark. A similar assertion can be proved for $\sigma(n)$ instead of $\varphi(n)$.

Proof. It is clear that

$$\frac{\kappa_1\kappa_2}{Q_1} \to 0, \quad \frac{\kappa_1\kappa_2}{Q_2} \to 0, \qquad \text{as} \quad x \to \infty. \tag{3.3}$$

Let $\mathcal{P}_j = \{p \mid p \equiv 1 \pmod{Q_j}\}$ $(j = 1,2)$, $\mathcal{N}(\mathcal{P}_j) = \{n \mid p|n \Rightarrow p \in \in \mathcal{P}_j\}$, $\mathcal{N}_{Q_j}(\mathcal{P}_j) = \{n \mid n \in \mathcal{N}(\mathcal{P}_j) \text{ and } Q_j^2 \nmid n\}$, $(j = 1,2)$.

Let $\mathcal{E}_{Q_1,Q_2} = \{n \mid n \le x,\ n \in \mathcal{N}_{Q_1}(\mathcal{P}_1),\ n+1 \in \mathcal{N}_{Q_2}(\mathcal{P}_2)\}$. Let $Y = = x^{1/\gamma_x}$, $\gamma_x = 40x_2$.

For some $n \in \mathcal{E}_{Q_1,Q_2}$ we write $n = \xi u$, $n+1 = \eta v$, where $\xi \in \mathcal{N}_{Q_1}(\mathcal{P}_1)$, $\eta \in \in \mathcal{N}_{Q_2}(\mathcal{P}_2)$, $P(\xi) \le Y$, $P(\eta) \le Y$, $p(u) > Y$, $p(v) > Y$. Let $\mathcal{T}(\xi,\eta)$ be the set of those $n \in \mathcal{E}_{Q_1,Q_2}$ for which ξ and η are fixed. Let $T(\xi,\eta) = \#\mathcal{T}(\xi,\eta)$. If $T(\xi,\eta) \ne 0$, then $(\xi,\eta) = 1$ and $2 \mid \xi\eta$.

It is well-known that $\psi(x,y) \ll xe^{-u/2}$, $u = \frac{\log x}{\log y}$, where

$$\psi(x,y) = \#\{n \le x \mid P(n) \le y\}.$$

It is clear that

$$E_{Q_1,Q_2}(x) \le \sum_{\xi,\eta} T(\xi,\eta). \tag{3.4}$$

We shall overestimate the contribution of those terms standing on the right hand side of (3.4), for which $\xi > x^{1/10}$, or $\eta > x^{1/10}$ holds. This is less than

$$2x \sum_{\substack{m > x^{1/10} \\ P(m) < Y}} 1/m \le 2x \sum_{j=0}^{\infty} \frac{1}{2^j x^{1/10}} \psi(2^{j+1}x^{1/10}, Y) \le$$

$$\le 2x \sum_{j \ge 0} \exp\left(-\frac{\frac{1}{10}x_1 + j\log 2}{2\log Y}\right) = 2xe^{-\frac{\gamma_x}{20}} \cdot \frac{1}{1 - e^{-\frac{\log 2}{\log Y}}} \le$$

$$\le 4xe^{-\frac{\gamma_x}{20}} \cdot \frac{x_1}{\log 2} \cdot \frac{1}{\gamma_x} \ll \frac{x}{x_1^{3/2}}.$$

Thus

$$E_{Q_1,Q_2}(x) \le \sum_{\max(\xi,\eta) \le x^{1/10}} T(\xi,\eta) + \mathcal{O}\left(\frac{x}{x_1^{3/2}}\right). \tag{3.5}$$

We shall estimate $T(\xi,\eta)$ for $\max(\xi,\eta) < x^{1/10}$. We have to count those $n \le x$, for which $n = \xi u$, $n+1 = \eta v$, $n \le x$ and $p(u) > Y$, $p(v) > Y$. Let u_0, v_0 be the smallest pair of those positive integers u, v, for which $\eta v - \xi u = 1$. Let $F_1(t) = u_0 + \eta t$, $F_2(t) = v_0 + \xi t$. Then

$$T(\xi,\eta) = \#\left\{ t \le \frac{x}{\xi\eta} \;\middle|\; p(F_1(t)) > Y,\ \ p(F_2(t)) > Y \right\}.$$

By using Theorem 2.6 in Halberstam - Richert [7], we deduce that

$$T(\xi,\eta) = (1 + o_x(1))x \cdot \frac{1}{2} \prod_{2<p<Y} \left(1 - \frac{2}{p}\right) \cdot \theta_{\xi,\eta} \tag{3.6}$$

uniformly for all possible ξ, η, where

$$\theta_{\xi,\eta} = \frac{1}{\xi\eta} \prod_{\substack{p\mid\xi\eta \\ p\ne 2}} \frac{1-1/p}{1-2/p}. \tag{3.7}$$

The implied constants in the error term of (8.4) of Theorem 2.6 in [7] may depend on the coefficients of F_1, F_2, i.e. on ξ and on η, but reading the proof carefully one can see that estimate (3.6) will hold for all ξ, η provided that we add an error term, thus implying that (3.6) holds with that particular restriction. Hence, from (3.5), we obtain that

$$E_{Q_1,Q_2}(x) \le (1 + o_x(1))\frac{x}{2} \prod_{2<p<Y} \left(1 - \frac{2}{p}\right) \Sigma_1, \tag{3.8}$$

where

$$\Sigma_1 = \sum \frac{1}{\xi\eta} \prod_{\substack{p\mid\xi\eta \\ p>2}} \frac{1-1/p}{1-2/p}. \tag{3.9}$$

Let Σ_2 be the sum of those terms on the right hand side of (3.9), for which additionally $Q_1 \nmid \xi$, $Q_2 \nmid \eta$ holds. It is clear that

$$0 < \Sigma_1 - \Sigma_2 \ll \left(\frac{1}{Q_1} + \frac{1}{Q_2}\right) \Sigma_1 \ll \frac{x_3}{x_2} \Sigma_1. \tag{3.10}$$

We shall write

$$\Sigma_2 = A(Y)B(Y)C(Y), \tag{3.11}$$

where

$$A(Y) = \prod_{\substack{3\le p<Y\\ (p-1,Q_1Q_2)=1}} \left(1+\frac{2}{p}+\frac{2}{p^2}+\dots\right) = \tag{3.12}$$
$$= \prod_{\substack{3\le p<Y\\ (p-1,Q_1Q_2)=1}} \left(1+\frac{2}{p-1}\right),$$

$$B(Y) = \prod_{\substack{p\equiv 1 \pmod{Q_1}\\ (p-1,Q_2)=1\\ p<Y}} (1+1/p); \quad C(Y) = \prod_{\substack{p\equiv 1 \pmod{Q_2}\\ (p-1,Q_1)=1\\ p<Y}} (1+1/p). \tag{3.13}$$

Thus we proved that

$$E_{Q_1,Q_2}(x) \le (1+o_x(1))\frac{x}{2}\prod_{2<p<Y}\left(1-\frac{2}{p}\right)A(Y)B(Y)C(Y)+\mathcal{O}\left(\frac{x}{x_1^3}\right). \tag{3.14}$$

Let $\mathcal{T}^*(\xi,\eta)$ be the set of those $n=\xi u \le x$, for which $n+1=\eta v$, $p(u)>$ $>Y$, $p(v)>Y$, and $u\in\mathcal{N}(\mathcal{P}_1)$, $v\in\mathcal{N}(\mathcal{P}_2)$. Let

$$\Delta(\xi,\eta) = \#(\mathcal{T}(\xi,\eta)\setminus\mathcal{T}^*(\xi,\eta)). \tag{3.15}$$

If $n\in\mathcal{T}(\xi,\eta)\setminus\mathcal{T}^*(\xi,\eta)$, then there exists $p_1|n$ such that $Y<p_1$, $p_1\equiv 1 \pmod{Q_1}$, or $p_2|n+1$, such that $Y<p_2$, $p_2\equiv 1 \pmod{Q_2}$.

We shall prove that

$$\sum_{\xi,\eta<x^{1/10}} \Delta(\xi,\eta) = o_x(1)x\kappa_1\kappa_2. \tag{3.16}$$

Let $\xi,\eta<x^{1/10}$ be fixed. By using Theorem 2.6 in [7] we can overestimate those solutions of $n=\xi u$, $n+1=\eta v$ counted in $\mathcal{T}(\xi,\eta)$ for which there exists either a $p_1\in\mathcal{P}_1$ such that $p_1|u$, and $p_1<x^{0,75}$, or a $p_2\in\mathcal{P}_2$, such that $p_2|v$ and $p_2<x^{0,75}$. We consider the first case. The second case is similar. If $p_1|n$, then let $u=p_1m$. For fixed ξ,p_1,η we should estimate those m,v for which $\eta v-(p_1\xi)m=1$, $p(v)>Y$, $p(m)>Y$. Arguing as above, by using Theorem 2.6 in [7] we obtain the number of the integers is less than $\frac{c}{p_1}T(\xi,\eta)$. Since

$$\sum_{\substack{Y<p_1<x\\ p_1\equiv 1(Q_1)}} 1/p_1 \le \frac{c_1}{Q_1}\log\frac{\log x}{\log Y} \ll \frac{x_3}{Q_1},$$

the contribution of these types of integers to (3.16) is less than

$$\frac{x_3}{Q_1}\sum_{\xi,\eta<x^{1/10}} T(\xi,\eta). \tag{3.17}$$

Let us observe that the number of those $n \le x$ for which there exists $p_1 \in \mathcal{P}_1$, $p_1|n$, $p_1 > \sqrt{x}$, or $p_2 \in \mathcal{P}_2$, such that $p_2|n+1$, $p_2 > \sqrt{x}$ is $o_x(1)x\kappa_1\kappa_2$. The number of these integers is less than

$$\sum_{\substack{p_1\equiv 1(Q_1)\\ \sqrt{x}<p_1<x}} \frac{x}{p_1} + \sum_{\substack{p_2\equiv 1(Q_2)\\ \sqrt{x}<p_2<x}} \frac{x}{p_2} \ll x\left(\frac{1}{Q_1}+\frac{1}{Q_2}\right),$$

and the right hand side is $o_x(1)x\kappa_1\kappa_2$.

(3.17) is proved, whence we obtain that

$$\begin{aligned} E_{Q_1,Q_2}(x) =&(1+o_x(1))\frac{x}{2}\prod_{2<p<Y}\left(1-\frac{2}{p}\right)A(Y)B(Y)C(Y)+\\ &+o_x(1)x\kappa_1\kappa_2. \end{aligned} \tag{3.18}$$

We have

$$\begin{aligned} \prod_{2<p<Y}\left(1-\frac{2}{p}\right)A(Y) = &\prod_{3\le p<Y}\left(1-\frac{2}{p(p-1)}\right)\cdot \prod_{\substack{3\le p<Y\\ p\equiv 1(Q_1)}}\frac{1}{1+\frac{2}{p-1}}\times\\ &\times \prod_{\substack{3\le p<Y\\ p\equiv 1(Q_2)}}\frac{1}{1+\frac{2}{p-1}}\cdot \prod_{\substack{3\le p<Y\\ p\equiv 1(Q_1,Q_2)}}\left(1+\frac{2}{p-1}\right). \end{aligned} \tag{3.19}$$

Furthermore,

$$\prod_{\substack{3\le p<Y\\ p\equiv 1 \pmod D}}\frac{1}{1+\frac{2}{p-1}} = e^{-\frac{2\log\log Y}{\varphi(D)}+\mathcal{O}\left(\frac{1}{D}\right)} \tag{3.20}$$

and

$$\prod_{\substack{3\le p<Y\\ p\equiv 1 \pmod D}}(1+1/p) = e^{\frac{\log\log Y}{\varphi(D)}+\mathcal{O}\left(\frac{1}{D}\right)} \tag{3.21}$$

uniformly as $D \le x_2^2$. Let

$$B = \prod_{p \ge 3} \left(1 - \frac{2}{p(p-1)}\right). \tag{3.22}$$

From (3.19), (3.20), (3.21) we have:

(i) the right hand side of (3.19) equals to

$$\begin{aligned}(1+o_x(1))B \cdot \kappa_1^2\kappa_2^2 e^{\frac{2x_2}{(Q_1-1)(Q_2-1)}} &= \\ = (1+o_x(1))B \cdot \kappa_1^2\kappa_2^2,\end{aligned} \tag{3.23}$$

(ii)

$$B(Y) = (1+o_x(1))\frac{1}{\kappa_1}, \tag{3.24}$$

(iii)

$$C(Y) = (1+o_x(1))\frac{1}{\kappa_2}. \tag{3.25}$$

Hence the theorem follows immediately.

4. Final remark

The distribution of the prime power divisors of the iterates of $\varphi(n)$, $\sigma(n)$ are investigated in [8].

References

[1] **Kátai, I.,** On sets characterized by the values of some multiplicative functions, *Annales Univ. Sci. Budapest. Sect. Comp.,* **32** (2010), 63-72.

[2] **Spearman, B.K. and Williams, K.S.,** Values of the Euler phi function not divisible by a given odd prime, *Ark. Mat.,* **44** (2006), 166-181.

[3] **B. V. Levin and A. S. Fainleib**, Applications of some integral equations to problems in number theory, Uspekhi Mat. Nauk., **22**(3), 135 (1967), 119-198 (in Russian), Russian Math. Surveys **22**(3) (1967), 119-204 (in English)

[4] **Wirsing, E.,** Das asymptotische Verhalten von Summen bei multiplikative Funktionen, *Math. Ann.,* **143** (1961), 75-102.

[5] **Bassily, N.L., Kátai, I. and Wijsmuller, M.V.,** Number of prime divisors of $\varphi_k(n)$, where φ_k is the k-fold iterate of φ, *J. Number Theory,* **65** (2) (1997), 226-239.

[6] **Indlekofer, K.-H. and Kátai, I.,** On the normal order of $\varphi_{k+1}(n)/\varphi_k(n)$, where φ_k is the k-fold iterate of Euler's function, *Liet. matem. rink.,* **44** (1) (2004), 68-84.

[7] **Halberstam, H. and Richert, H.,** *Sieve methods,* Academic Press, London, 1974.

[8] **Kátai, I.,** On the prime power divisors of the iterates of $\varphi(n)$ and $\sigma(n)$, *Siauliai Math. Semin.,* **4** (12) (2009), 125-143.

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