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A MEAN VALUE RESULT FOR THE FOURTH MOMENT OF $|\zeta(\frac{1}{2}+it)|$ II.

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Abstract. We prove that, for a fixed $j \in \mathbb{N}$, there exists $\sigma_0 = \sigma_0(j)$ (< 1) such that

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 |\zeta(\sigma+it)|^{2j} \mathrm{d}t \ll_{j,\varepsilon} T^{1+\varepsilon}$$

holds for $\sigma > \sigma_0$. We also indicate how to obtain an asymptotic formula for the above integral, for the range of $\sigma > \sigma_1 = \sigma_1(j)$, where $\sigma_0 < \sigma_1 < 1$.

1. Introduction

Let as usual $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ $(\sigma > 1)$ denote the Riemann zeta-function, where $s = \sigma + it$ is a complex variable. Mean values of $\zeta(s)$ in the so-called "critical strip" $\frac{1}{2} \le \sigma \le 1$ represent a central topic in the theory of the zeta-function (see e.g., the monographs [9] and [10] for an extensive account).

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Of special interest are the moments on the so-called "critical" line $\sigma = \frac{1}{2}$. Unfortunately as of yet no bound of the form

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2m} \mathrm{d}t \ll_{\varepsilon,m} T^{1+\varepsilon} \qquad (m \in \mathbb{N}) \tag{1.1}$$

is known to hold when $m \geq 3$, while in the cases $m = 1, 2$ precise asymptotic formulas for the integrals in question are known (see op. cit.). Here and later ε denotes arbitrarily small, positive constants, not necessarily the same ones at each occurrence.

In the first part of this paper [11] the first author investigated the range of σ for which the bounds

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{4} |\zeta(\sigma+it)|^{2j} \mathrm{d}t \ll_{j,\varepsilon} T^{1+\varepsilon} \tag{1.2}$$

hold when $j = 1$ and $j = 2$. In particular, for $j = 1$ it was shown that (1.2) holds when $\sigma > 5/6 = 0.8\bar{3}$, while if (k, ℓ) is an exponent pair (see [4] or [9] for the definition and properties), then (1.2) holds when

$$\sigma > \max\left(\frac{\ell - k + 1}{2}, \frac{11k + \ell + 1}{8k + 2}\right), \tag{1.3}$$

and in particular (1.3) holds in this case when $\sigma \geq 1953/1984 = 0.984375$. At the end of [11] it was stated, as an open problem, to try to find $\sigma_0 = \sigma_0(j)$ (< 1) such that, for a fixed $j \in \mathbb{N}$, (1.2) holds for $\sigma > \sigma_0$.

In this paper we shall first improve the range for σ for which (1.2) holds, and prove that indeed there is a $\sigma_0 = \sigma_0(j)$ (< 1) such that (1.2) holds for $\sigma > \sigma_0$ for any given integer $j \geq 1$. Then, we shall sharpen the upper bound in (1.2) to an asymptotic formula in the range $\sigma > \sigma_1 = \sigma_1(j)\,(> \sigma_0)$, where $j \geq 1$ is any given integer.

2. Formulation of the theorems

Theorem 1. *For any given integer $j \geq 1$ there exists a number $\sigma_0 = \sigma_0(j)$ (< 1) such that, for $\sigma > \sigma_0$, we have*

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{4} |\zeta(\sigma+it)|^{2j} \mathrm{d}t \ll_{j,\varepsilon} T^{1+\varepsilon}. \tag{2.1}$$

In fact, if (k,ℓ) *is an exponent pair, then* (2.1) *holds for*

$$\sigma_0 = \frac{\ell+(6j-1)k}{1+4jk} \qquad \text{if } \ \ell+(2j-1)k<1. \tag{2.2}$$

Corollary 1. When $j=2$ we see that (2.1) holds when

$$\sigma > \sigma_0 = 37/38 = 0.97368\ldots .$$

This follows on choosing the exponent pair $(1/30, 26/30)$, and improves the range obtained in [11]. Further slight improvements may be obtained by a more judicious choice of the exponent pair. Firstly, there is an algorithm (see e.g., the monograph [4] of Graham–Kolesnik) for optimizing certain expressions containing exponent pairs. Secondly, there are (new) exponent pairs obtainable by a variant of the so-called Bombieri–Iwaniec method for the estimation of exponential sums (see e.g., M.N. Huxley [6], [7], [8]). An application of these procedures would lead to small improvements of the value 37/38.

Theorem 2. *For any given integer* $j\ge 1$ *there exists a number* $\sigma_1=\sigma_1(j)$ *for which* $3/4<\sigma_1<1$ *such that, when* $\sigma>\sigma_1$*, there exists an asymptotic formula for the integral in* (2.1)*, namely*

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 |\zeta(\sigma+it)|^{2j}\mathrm{d}t \sim T(a_0(\sigma,j)\log^4 T + a_1(\sigma,j)\log^3 T + \\ + a_2(\sigma,j)\log^2 T + a_3(\sigma,j)\log T + a_4(\sigma,j)), \tag{2.3}$$

as $T\to\infty$*, where all the coefficients* $a_\ell(\sigma,j)$*, which depend on* σ *and* j*, may be evaluated explicitly.*

The merit of these results is that (2.1) and (2.3) hold for values of σ less than one; of course one expects the bound in (2.1) to hold for any $\sigma\ge\frac{1}{2}$, in which case, for $j=1$ and $j=2$, we would obtain the (yet unproved) sixth and eighth moment of $|\zeta\left(\frac{1}{2}+it\right)|$ (namely (1.1) with $m=3$ and $m=4$, respectively). The truth of (2.1) for any j and $\sigma>\frac{1}{2}$ is equivalent to the famous, but yet unproved, Lindelöf hypothesis that $\zeta\left(\frac{1}{2}+it\right)\ll_\varepsilon |t|^\varepsilon$. Theorem 2 sharpens the bound of Theorem 1 to an asymptotic formula, but whereas the value σ_0 in (2.2) is explicit, the value of σ_1 in Theorem 2 is not. From the proofs it will be clear that $\sigma_0<\sigma_1<1$, and that the value of σ_1 would be rather poor.

3. Proof of Theorem 1

In the proof of both (2.1) and (2.3) it is sufficient to consider the integral over $[T, 2T]$, then to replace T by $T2^{-j}$ $(j = 1, 2, \ldots)$ and sum all the resulting expressions. Also, it is sufficient to suppose that $\sigma \le 1$, since one has (see e.g., [9])

$$\zeta(\sigma + it) \ll \log|t| \qquad (\sigma \ge 1).$$

To prove the bound on σ in (2.1) involving k, ℓ, we shall use the simple approximate functional equation for $\zeta(s)$ (see [9, Theorem 1.8]), which gives

$$\zeta(s) = \sum_{n\le T} n^{-s} + O(T^{-\sigma}) \qquad (s = \sigma + it,\, T \le t \le 2T). \tag{3.1}$$

As in [11], the essential tool in our considerations is the following theorem for the fourth moment of $|\zeta\left(\frac{1}{2} + it\right)|$, weighted by a Dirichlet polynomial. This is stated as

Lemma 3.1. *Let $a_1, a_2, \ldots$ be complex numbers. Then, for $\varepsilon > 0$, $M \ge 1$ and $T \ge 1$,*

$$\int_0^T \Big|\sum_{m\le M} a_m m^{it}\Big|^2 \left|\zeta\left(\frac{1}{2} + it\right)\right|^4 \mathrm{d}t \ll_\varepsilon T^{1+\varepsilon} M(1 + M^2 T^{-1/2}) \max_{m\le M} |a_m|^2. \tag{3.2}$$

This strong result is due to N. Watt [13]. It is founded on the earlier works of J.-M. Deshouillers and H. Iwaniec [1], [2], involving the use of Kloosterman sums, but Watt's bound is sharper.

We write (3.1) as

$$\zeta(s) = \sum_{m\le Y} m^{-s} + \sum_{Y<n\le T} n^{-s} + O(T^{-\sigma}) = \sum\nolimits_1 + \sum\nolimits_2 + O(T^{-\sigma}), \tag{3.3}$$

say, where $1 \ll Y = Y(T) \le T$ will be suitably chosen. The sum $\sum_2$ is split into $O(\log T)$ subsums with $N < n \le N' \le 2N$, $Y < N \le T$. To estimate each of these subsums we use the theory of (one-dimensional) exponent pairs. Removing the (monotonically decreasing) factor $n^{-\sigma}$ by partial summation from each subsum, it remains to estimate

$$S(N, t) := \sum_{N<n\le N'\le 2N} n^{it} \qquad (Y \le N \le T,\ T \le t \le 2T).$$

If (k,ℓ) is an exponent pair, then since $n^{it} = \mathrm{e}^{iF(n,t)}$ with $\frac{\partial^r F(n,t)}{\partial n^r} \asymp_r TN^{-r}$ for $r \geq 1$, it follows that

$$S(N,t) \ll \left(\frac{T}{N}\right)^k N^\ell = T^k N^{\ell-k},$$

and consequently

$$\sum\nolimits_2 \ll T^k N^{\ell-k-\sigma} \log T \ll T^k Y^{\ell-k-\sigma} \log T \tag{3.4}$$

if $\sigma \geq \ell - k$. Now we choose

$$Y = T^{\frac{1}{j(6-4\sigma)}}. \tag{3.5}$$

This gives

$$\begin{aligned} \zeta(\sigma+it) &= \sum_{m\leq Y} m^{-s} + \sum_{Y<n\leq T} n^{-s} + O(1) \ll \\ &\ll \Big|\sum_{m\leq Y} m^{-s}\Big| + T^{k+\frac{\ell-k-\sigma}{j(6-4\sigma)}} \log T + 1 \ll \\ &\ll \Big|\sum_{m\leq Y} m^{-s}\Big| + \log T \end{aligned} \tag{3.6}$$

provided that our condition

$$\sigma \geq \frac{\ell + (6j-1)k}{1+4jk}$$

holds. From (3.6) we obtain

$$\begin{aligned} |\zeta(\sigma+it)|^{2j} &\ll \Big|\sum_{m\leq Y} m^{-s}\Big|^{2j} + \log^{2j} T = \\ &= \Big|\sum_{m\leq Y^j} b_m m^{-s}\Big|^2 + \log^{2j} T \end{aligned}$$

say, with

$$b_m := \sum_{d_1\ldots d_j = m;\ d_r\leq Y\ (r=1,\ldots,j)} 1 \ll_\varepsilon m^\varepsilon. \tag{3.7}$$

Setting

$$Y_1 := Y^j = T^{\frac{1}{6-4\sigma}} \tag{3.8}$$

we estimate now

$$\int_T^{2T} \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 \Big|\sum_{m\le Y_1} b_m m^{-s}\Big|^2 \mathrm{d}t$$

by Lemma 3.1, where clearly the interval of integration may be taken to be $[T,\,2T]$. The sum over m is split into $O(\log T)$ subsums over m satisfying $M/2 \le m \le M' \le M,\ M \le M_1$, and we take $a_m = b_m m^{-\sigma}$ in Lemma 3.1 if m lies in this range, and $a_m = 0$ otherwise. Then we obtain that

(3.9)

$$\int_T^{2T} \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 \Big|\sum_{m\le Y_1} b_m m^{-s}\Big|^2 \mathrm{d}t \ll_\varepsilon T^{1+\varepsilon} \max_{M\ll Y_1} M^{1-2\sigma}(1+M^2T^{-1/2}) \ll_\varepsilon$$

$$\ll_\varepsilon T^{1+\varepsilon}(1+Y_1^{3-2\sigma}T^{-1/2}) \ll_\varepsilon T^{1+\varepsilon}$$

in view of (3.8).

This proves Theorem 1, since it is easy to see that $\big(\ell+(6j-1)k\big)/(1+4jk) > > \ell - k$. It remains to show that the condition in (2.2) is fulfilled, namely that for $j \in \mathbb{N}$ there exists an exponent pair (k,ℓ) such that

$$\ell + (2j-1)k < 1. \tag{3.10}$$

To see this we consider the exponent pair (see [4, p. 39])

$$(k,\ell) = \left(\frac{16}{120Q-32}, \frac{120Q-16q-63}{120Q-32}\right) \quad (Q = 2^q,\ q \ge 2). \tag{3.11}$$

The condition in (3.10) reduces then to

$$16(2j-1) < 16q + 31.$$

Since j is fixed and q is arbitrary, we can take e.g.

$$q \ge 2j-2,$$

establishing (3.10).

4. Proof of Theorem 2

For the proof of Theorem 2 we start with (3.3), taking this time

$$Y \;:=\; T^{\frac{1}{11j}-\varepsilon_1} \tag{4.1}$$

with a fixed, small $\varepsilon_1 > 0$. Then, if $\sigma \ge \ell - k$, (3.4) gives

$$\sum\nolimits_2 \;\ll\; T^{k+(\frac{1}{11j}-\varepsilon_1)(\ell-k-\sigma)}\log T.$$

Here the exponent of T is negative if

$$11kj + (1-11j\varepsilon_1)(\ell-k-\sigma) < 0,$$

or

$$\sigma \;>\; \ell + (11j-1)k, \tag{4.2}$$

if ε_1 is taken to be sufficiently small. The condition (4.2) is necessary for the assertion of Theorem 2 to hold, provided that the exponent pair (k,ℓ) satisfies

$$(11j-1)k + \ell < 1. \tag{4.3}$$

The existence of exponent pairs satisfying (4.3) follows if one uses the exponent pair (3.11), since the condition (4.3) reduces then to $16(11j-1) < 16q+31$, which is certainly true if $q \ge 11j-2$.

Therefore we have, if (4.1) holds,

$$\zeta(\sigma+it) = \sum_{n\le Y} n^{-s} + O(T^{-\varepsilon_3}) \qquad (T \le t \le 2T)$$

with some positive ε_3 depending on ε_1, if

$$\sigma > (11j-1)k + \ell + \varepsilon_2, \qquad (11j-1)k+\ell < 1, \tag{4.4}$$

since $(11j-1)k + \ell + \varepsilon_2 > \ell - k$. It remains to evaluate

$$\begin{aligned} &\int_T^{2T} \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 \Big|\sum_{n\le Y} n^{-\sigma-it}\Big|^{2j} \mathrm{d}t = \\ &= \int_T^{2T} \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 \Big|\Big(\sum_{n\le Y} n^{-\sigma-it}\Big)^j\Big|^2 \mathrm{d}t. \end{aligned} \tag{4.5}$$

We write

$$\Big(\sum_{n\le Y} n^{-\sigma-it}\Big)^j = \sum{}' + \sum{}'',$$

say, where

$$\sum{}' := \sum_{n\le Y} d_j(n)n^{-\sigma-it}, \quad \sum{}'' := \sum_{Y<n\le T} b_n n^{-\sigma-it},$$

where $d_j(n)$ is the divisor function generated by $\zeta^j(s)$, and b_n is given by (3.7). Thus the right-hand side of (4.5) becomes

$$I_1 + 2\cdot \mathrm{Re} I_2 + I_3,$$

with

$$\begin{aligned} I_1 &:= \int_T^{2T} \Big|\zeta\Big(\frac12+it\Big)\Big|^4 \Big|\sum{}'\Big|^2 \mathrm{d}t,\\ I_2 &:= \int_T^{2T} \Big|\zeta\Big(\frac12+it\Big)\Big|^4 \overline{\sum{}'}\sum{}'' \mathrm{d}t,\\ I_3 &:= \int_T^{2T} \Big|\zeta\Big(\frac12+it\Big)\Big|^4 \Big|\sum{}''\Big|^2 \mathrm{d}t. \end{aligned} \tag{4.6}$$

Therefore the problem reduces to the asymptotic evaluation of the integrals in (4.6). We begin with I_1, and as a technical convenience, we consider instead of I_1 the weighted integral

$$J_1 := \int_{-\infty}^{\infty} w(t)\Big|\zeta\Big(\frac12+it\Big)\Big|^4 \Big|\sum_{n\le Y} d_j(n)n^{-\sigma-it}\Big|^2 \mathrm{d}t, \tag{4.7}$$

where $w(t)$ (≥ 0) is a smooth function majorizing or minorizing the characteristic function of the interval $[T, 2T]$. The fact that the integrand in I_1 is nonnegative makes this effective. We take $w(t)$ to be supported in $[T/2, 4T]$ (see e.g., Chapter 4 of the first author's monograph [10] for an explicit construction of $w(t)$). We further have $w^{(r)}(t) \ll_r T_0^{-r}$ for all $r = 0, 1, 2, \ldots$, where T_0 is a parameter which satisfies $T^{1/2+\varepsilon} \ll T_0 \ll T$. We write the square of the sum in (4.7) as

(4.8)
$$\Big|\sum_{n\le Y} d_j(n)n^{-\sigma-it}\Big|^2 =$$
$$= \sum_{m,n\le Y} d_j(m)d_j(n)\Big(\frac{m}{n}\Big)^{-it}(mn)^{-\sigma} = \qquad (\text{here } m=\delta h, n=\delta k, (h,k)=1)$$
$$= \sum_{\delta\le Y}\delta^{-2\sigma}\sum_{h\le Y/\delta,\ k\le Y/\delta,\ (h,k)=1} d_j(\delta h)d_j(\delta k)(hk)^{-\sigma}\left(\frac{h}{k}\right)^{-it}.$$

With the aid of (4.8) it follows that I_1 reduces to the summation of

$$I_3(h,k) := \int_{-\infty}^{\infty} w(t)\left|\zeta\left(\frac{1}{2}+it\right)\right|^4\left(\frac{h}{k}\right)^{-it}\mathrm{d}t \qquad ((h,k)=1).$$

To evaluate integrals of the type $I_3(h,k)$, C.P. Hughes and M.P. Young [5] considered the more general "twisted fourth moment integral", namely

$$I(h,k) := \int_{-\infty}^{\infty}\left(\frac{h}{k}\right)^{-it}\zeta\left(\frac{1}{2}+\alpha+it\right)\zeta\left(\frac{1}{2}+\beta+it\right)\times \tag{4.9}$$
$$\times\zeta\left(\frac{1}{2}+\gamma-it\right)\zeta\left(\frac{1}{2}+\delta-it\right)w(t)\mathrm{d}t,$$

where $(h,k)=1$, and $\alpha,\beta,\gamma,\delta$ are complex numbers $\ll 1/\log T$, with the idea of letting eventually $\alpha,\beta,\gamma,\delta$ all tend to zero, in which case $I(h,k)$ becomes $I_3(h,k)$. We formulate their result as

Lemma 4.1. *With the notation introduced above we have, for* $(h,k) = 1$, $hk \le T^{2/11-\varepsilon}$ *and complex numbers* $\alpha,\beta,\gamma,\delta \ll 1/\log T$,

(4.10)
$$I(h,k) = \frac{1}{\sqrt{hk}}\int_{-\infty}^{\infty} w(t)\Bigg\{Z_{\alpha,\beta,\gamma,\delta,h,k}(0) + \left(\frac{t}{2\pi}\right)^{-\alpha-\beta-\gamma-\delta} Z_{-\gamma,-\delta,\alpha,-\beta,h,k}(0) +$$
$$+\left(\frac{t}{2\pi}\right)^{-\alpha-\gamma} Z_{-\gamma,\beta,-\alpha,\delta,h,k}(0) + \left(\frac{t}{2\pi}\right)^{-\alpha-\delta} Z_{-\delta,\beta,-\gamma,-\alpha,h,k}(0) +$$
$$+\left(\frac{t}{2\pi}\right)^{-\beta-\gamma} Z_{\alpha,-\gamma,-\beta,\delta,h,k}(0) + \left(\frac{t}{2\pi}\right)^{-\beta-\delta} Z_{\alpha,\delta,\gamma,-\beta,h,k}(0)\Bigg\}\mathrm{d}t +$$
$$+ O_\varepsilon\Big(T^{3/4+\varepsilon}(hk)^{7/8}(T/T_0)^{9/4}\Big).$$

The function $Z_{\dots}(0)$ is given in term of explicit, albeit complicated Euler products.

The proof of (4.10) is long and technically quite involved. The main aim in [5] is to obtain an asymptotic formula for

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 \left|M\left(\frac{1}{2}+it\right)\right|^2 \mathrm{d}t, \tag{4.11}$$

where

$$M(s) = \sum_{h\le T^\theta} a(h)h^{-s}$$

is a Dirichlet polynomial of length T^θ with coefficients $a(h)$. This is an important problem, since a good bound for (4.10) with $\theta = 1/2$ would give the hitherto unproved sixth moment of zeta-function in the form

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^6 \mathrm{d}t \ll_\varepsilon T^{1+\varepsilon},$$

which is (1.1) with $m = 3$. Two of the chief ingredients in the proof of (4.10) are an approximate functional equation for the product of four zeta values appearing in (4.9), and the so-called "delta method" of Duke, Friedlander and Iwaniec [3]. The authors analyze the consequences of their result. It may be compared to previous results on (4.10), due to J.-M. Deshouillers and H. Iwaniec [1], N. Watt [13] (cf. (3.2)), and most recently by Y. Motohashi [12], all of whom used powerful methods from the spectral theory of the non-Euclidean Laplacian, while the authors' approach is based on complex integration techniques and classical methods. In particular, when using the delta symbol method to deal with the shifted divisor problem they only use the classical Weil bound for Kloosterman sums, and not the more advanced and difficult bounds involving spectral theory. The accent is on obtaining an asymptotic formula, where the condition $hk \le T^{2/11-\varepsilon}$ needed for (4.10) to hold sets the limit to the range in which the asymptotic formula holds. On the other hand, Watt's result (3.2) gives the expected upper bound in the range $M \ll T^{1/4}$, but does not produce an asymptotic formula for the integral in (3.2). It does not seem possible to obtain unconditionally (1.1) for any $m > 2$ by the techniques used by the authors in [5] for the proof of (4.10).

We continue now the proof of Theorem 2, and we multiply (4.10) by $d_j(\delta h)d_j(\delta k)(hk)^{-\sigma}$ and insert the resulting expression in (4.7). The error term in (4.10) makes a contribution which will be, since $d_j(n) \ll_\varepsilon n^\varepsilon$,

$$\ll_\varepsilon \sum_{\delta\le Y} \delta^{\varepsilon-2\sigma} \sum_{h\le Y/\delta,\ k\le Y/\delta} T^{3/4+\varepsilon}(hk)^{7/8-\sigma}(T/T_0)^{9/4} \ll_\varepsilon$$
$$\ll_\varepsilon T^{3/4+\varepsilon}Y^{15/4-2\sigma}(T/T_0)^{9/4}.$$

Since $Y^{15/4-2\sigma} < Y^{11/4}$ because $\sigma > 1/2$ and (4.1) holds we see, as in the discussion made in [5], that we obtain first the desired asymptotic formula, with an error term $O(T^{1-\delta})$, for the twisted integral. Finally, if $\alpha,\beta,\gamma,\delta$ all tend to zero, we obtain the desired asymptotic formula for I_1, which clearly provides the main term in (2.3).

It remains to deal with the integrals I_2 and I_3 in (4.6). By using (3.9) we find that

$$I_3 \ll T^{1-\delta_1} \qquad (\delta_1 = \delta_1(j) > 0). \tag{4.12}$$

Finally by the Cauchy-Schwarz inequality for integrals, the result for I_1 and (4.12) we have

$$I_2 \le (I_1I_3)^{1/2} \ll T^{1-\delta_1/2}\log^2 T.$$

Collecting all the preceding estimates we obtain the assertion of Theorem 2. The value of σ_1 is determined by the condition (4.4) and the exponents of various error terms appearing in the course of the proof.

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