JUSTIFICATION OF THE FOURIER METHOD FOR EQUATIONS OF HOMOGENEOUS STRING VIBRATION WITH RANDOM INITIAL CONDITIONS

A. Slyvka-Tylyshchak (Kiev, Ukraine)

Communicated by Yu.V. Kozachenko

(Received January 15, 2012)

Abstract. We give conditions for the existence of a twice differentiable solution of hyperbolic type partial differential equation of homogeneous string vibration with random strongly Orlicz initial conditions.

1. Introduction

The influence of random factors should often be taken into account in solving problems of mathematical physics. These factors can be of a diverse nature: random boundary conditions and random initial conditions, random forces acting on the system, random coefficients of differential operators, etc. This brings up the necessity of analyzing specific features of the problem in question. The keypoints usually are: existence and uniqueness of the solution, the possibility of a constructive approximation of the solution and type of convergence of approximating functions to the solution, behavior of different functionals of the solution, etc. Different methods are applied depending on the type of the problem, specifics of random factors involved and the questions to be studied.

Mathematics Subject Classification: 60G60, 60G17

We consider boundary problems of homogeneous string vibration with random strongly Orlicz initial conditions. The main objective of the paper is to propose a new approach for studying partial differential equations with random initial conditions and to apply this approach for the justification of the Fourier method for solving hyperbolic type problems. Similar problems for the hyperbolic type equations are considered in [1, 4, 6, 7, 8, 10, 11], parabolic type equations are considered in [9]. Further references can be found in [2, 5].

2. Stochastic processes of the Orlicz space

Definition 2.1. ([2]) A continuous even convex function $u(x)$ $(x \in \mathbb{R})$ is called a C-function if $u(x)$ is monotonically increasing for $x > 0$ and $u(0) = 0$.

Definition 2.2. ([2]) We say that a C-function u satisfies the g-condition if there exist constants $z_0 > 0$, $k > 0$ and $A > 0$ such that the inequality

$$u(x)u(y) \leq Au(kxy)$$

holds for all $x > z_0$ and $y > z_0$.

Definition 2.3. ([2]) Suppose that (T, ρ) is a nonempty metric space and $\varepsilon > 0$. Denote by $N_\rho(t, \varepsilon)$ the smallest number of points in ε-net for the set T with respect to the udometric ρ. The function $(N_\rho(T, \varepsilon), \ \varepsilon > 0)$ is called the massiveness of the set with respect to the udometric ρ.

Let $\{\Omega, \Im, P\}$ be a probability space.

Definition 2.4. ([1]) The Orlicz space $L_u(\Omega)$ of random variables generated by a C-function $u(x)$ is defined to be the space of random variables $\xi(\omega) = \xi$, $\omega \in \Omega$ such that there exists a constant r_ξ with $Eu\left(\frac{\xi}{r_\xi}\right) \leq \infty$.

The Orlicz space $L_u(\Omega)$ is a Banach space with the norm

$$\|\xi\|_{L_u} = \inf\left\{r > 0 : Eu\left(\frac{\xi}{r}\right) \leq 1\right\}.$$

Definition 2.5. ([1]) A stochastic process $X = \{X(t), t \in T\}$ is said to be from the Orlicz space $L_u(\Omega)$ if for all $t \in T$ the random variable $X(t)$ belongs to $L_u(\Omega)$.

Definition 2.6. ([1]) Let $u(x)$ be a C-function such that $u(x)$ is stronger than $V(x) = x^2$ that is $V(x) > cx^2$ as $x > x_0, c > 0$. The set of random variables ξ $(E\xi = 0)$ from the space $L_u(\Omega)$ is called strongly Orlicz family of random variables if there exists a constant C_Δ such that for $\xi_i \in \Delta$, $i \in I$ and for all $\lambda_i \in \mathbb{R}^1$ the following inequality holds (I is any finite set)

$$\left\| \sum_{i \in I} \lambda_i \xi_i \right\|_{L_u} \le C_\Delta \left(E \left(\sum_{i \in I} \lambda_i \xi_i \right)^2 \right)^{1/2}.$$

Definition 2.7. ([1]) A stochastic process

$$X = \{X(t), t \in T\}, \quad (X \in L_u(\Omega))$$

is called a strongly Orlicz process if the family of random variables $X = \{X(t), t \in T\}$ is a strongly Orlicz family.

Theorem 2.1. ([1]) *Let Δ be a strongly Orlicz family of random variables. Then the linear closure $\overline{\Delta}$ of the family Δ in the space $L_2(\Omega)$ is a strongly Orlicz family.*

Theorem 2.2. ([1]) *Let $X_i = \{X_i(t),\ t \in T,\ i \in I\}$ be a family of strongly Orlicz stochastic processes. Let (T, Θ, μ) is a measurable space. If*

$$\varphi_{k_i}(t),\ i \in I,\ k = 1, \ldots, \infty$$

is a family of measurable functions in (T, Θ, μ) and the integral

$$\xi_{ki} = \int_T \varphi_{k_i}(t) X_i(t) \, d\mu(t)$$

is well defined in the mean square sense, then the family of random variables

$$\Delta_\xi = \left\{ \xi_{ki}, i \in I, k = \overline{1, \infty} \right\}$$

is a strongly Orlicz family.

Theorem 2.3. ([11]) *Let $\mathbf{R}^k$ be the k-dimensional space,*

$$d(t, s) = \max_{1 \le i \le k} |t_i - s_i|,$$

$T = \{0 \le t_i \le T_i,\ i = 1, 2, \ldots, k\}$, $X_n = \{X_n(t), t \in T\}$, $n = 1, 2, \ldots$ be a sequence of stochastic processes belonging to the Orlicz space $L_u(\Omega)$, and let the function u satisfy the g-condition. Assume that the process $X_n(t)$ is separable and

$$\sup_{d(t,s) \le h} \sup_{n = \overline{1, \infty}} \|X_n(t) - X_n(s)\| \le \sigma(h),$$

where $\sigma = \{\sigma(h), h > 0\}$ is a monotonically increasing continuous function such that $\sigma(h) \to 0$ as $h \to 0$. We also assume that

$$\int_0^{\epsilon} u^{(-1)} \left(\prod_{i=1}^{k} \left(\frac{T_i}{2\sigma^{(-1)}(u)} + 1 \right) \right) du < \infty,$$

where $\sigma^{(-1)}(u)$ is the inverse function of $\sigma(u)$. If the processes $X_n(t)$ converge in probability to the process $X(t)$ for all $t \in T$, then $X_n(t)$ converge in probability in the space $C(T)$.

Theorem 2.4. ([3]) *Let $\xi(X)$ be an almost sure continuous random field such that $E\xi(X) = 0$ for $X \in T$, where*

$$T = \{(x, y) \mid a_i \le x_i \le b_i,\ i = 1, \dots, n\}.$$

Let

$$B(X, Y) = E\xi(X)\xi(Y)$$

be the correlation of the field $\xi(X)$, and suppose that the partial derivatives exist.

$$B_{ii}(X, Y) = \frac{\partial^2 B(X, Y)}{\partial X_i \partial Y_i}, \quad i = 1, \dots n,$$

$B_{ii}(X, Y)$ are the correlation functions of square mean derivatives $\frac{\partial \xi(X)}{\partial x_i}$. If there is a version of the field $\frac{\partial \xi(X)}{\partial x_i}$, $i = 1, \dots, n$, that is a continuous random field, then this version is an ordinary partial derivative of the random field $\xi(X)$.

The following result contains conditions for the existence of partial derivatives for stochastic processes of Orlicz space.

Theorem 2.5. *Let $T = \{a_i \le x_i \le b_i,\ i = 1, \dots, m\}$. $\xi(X)$, $X \in T$, be a separable random field such that $\xi(X)$ is a strongly Orlicz stochastic processes. Let $B_{0000}(X, Y) = E\xi(X)\xi(Y)$ and assume that the partial derivatives $B_{i0i0}(X, Y) = \frac{\partial^2 B(X,Y)}{\partial x_i \partial y_i}$, $i = 1, \dots, m$, and*

$$B_{ikik}(X, Y) = \frac{\partial^4 B(X, Y)}{\partial x_i \partial y_i \partial x_k \partial y_k}, \quad i = 1, \dots, m, \quad k = 1, \dots, m$$

exist. Suppose that there exist monotone increasing continuous functions $\sigma_z(h) > 0$, $h > 0$, such that $\sigma_z(h) \to 0$ as $h \to 0$ for $z = (0, 0, 0, 0)$,

$z = (i,0,i,0)$, $i = 1,\dots,m$ and $z = (i,k,i,k)$, $i = 1,\dots,m$, $k = 1,\dots,m$. Assume that

$$\sup_{\substack{|x_i-y_i|\le h\\ i=1,\dots,m}} \left(B_z(X,X)+B_z(Y,Y)-2B_z(X,Y)\right)^{\frac{1}{2}} \le \sigma_z(h). \tag{2.1}$$

If

$$\int_0^{\varepsilon} u^{-1}\left(\left(\frac{\pi}{2\sigma_z^{(-1)}}+1\right)\left(\frac{T}{2\sigma_z^{(-1)}}+1\right)\right)du < \infty \tag{2.2}$$

for all z and for sufficiently small ε then with probability one the partial derivatives

$$\frac{\partial\xi(X)}{\partial x_i},\quad \frac{\partial^2\xi(X)}{\partial x_i\partial x_j},\quad i,j = 1,\dots,m.$$

Proof. The proof of this theorem is analogous to that of Theorem 3.9 of [7].

3. Conditions on existence with probability one of twice continuously differentiated solution of the boundary-value problem of homogeneous string vibration

Consider the boundary-value problem of first kind for a homogeneous hyperbolic equation [12]. The problem is whether one can find a function $u = (u\,(x,y)\,,\ x\in[0,\pi]\,,\ \ t\in[0,t])$ satisfying the following conditions:

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right) - q(x)u - \rho(x)\frac{\partial^2 u}{\partial t^2} = 0; \tag{3.1}$$

$$x\in[0,\pi]\,,\ t\in[0,T]\,,\ T>0;$$

$$u(0,t) = u(\pi,t) = 0,\ t\in[0,T]\,; \tag{3.2}$$

$$u(x,0) = \xi(x),\quad \frac{\partial u(x,0)}{\partial t} = \eta(x),\quad x\in[0,\pi]\,. \tag{3.3}$$

The functions

$$p = (p(x),\ x \in [0, \pi])\,, \quad q = (q(x), \quad x \in [0, \pi])\,, \quad \rho = (\rho(x), \quad x \in [0, \pi])$$

satisfy the following conditions:

(i) $p(x) > 0,\ \rho(x) > 0,\ q(x) \geq 0,\ x \in [0, \pi]\,;$

(ii) $p(x)$ and $\rho(x)$ are twice continuously differentiable on $x \in [0, \pi]$;

(iii) $q(x)$ is continuously differentiable on $[0, \pi]$.

Derivatives at the endpoints of the segment are interpreted as one-sided derivatives.

Assume also that $(\xi(x),\ x \in [0, \pi])$ $(\eta(x),\ x \in [0, \pi])$ are strongly Orlicz stochastic processes defined on the same complete probability space $(\Omega,\ \Im,\ P)$ such that

$$\xi(0) = \xi(\pi) = \eta(0) = \eta(\pi) = 0 \tag{3.4}$$

almost surely. Additional restrictions on the processes $\xi(\bullet)$ and $\eta(\bullet)$ will be imposed later. Denote by

$$B_\xi(x, y) = E\xi(x)\xi(y), \quad x, y \in [0, \pi]\,,$$

$$B_\eta(x, y) = E\eta(x)\eta(y), \quad x, y \in [0, \pi]$$

the correlation functions of ξ and η. We assume that the functions B_ξ and B_η are continuous, that is $\xi(\bullet)$ and $\eta(\bullet)$ are mean square continuous. (3.4) implies that

$$B_\xi(0, y) = B_\xi(x, 0) = B_\xi(\pi, y) = B_\xi(x, \pi) = 0,$$

$$B_\eta(0, y) = B_\eta(x, 0) = B_\eta(\pi, y) = B_\eta(x, \pi) = 0.$$

The particular equation (3.1) describes the oscillation of a nonhomogeneous string with fixed ends (3.2) and random initial conditions (3.3). In this case the stochastic process $\xi(\bullet)$ describes the initial position of the string, and the process $\eta(\bullet)$ represents the initial velocity. If the initial position and initial velocity are nonrandom, then the boundary-value problem (3.1)-(3.3) is classical and studied in detail. We are interested in the probabilistic aspects of the problem, and therefore we assume that the processes $\xi(\bullet)$ and $\eta(\bullet)$ have zero means.

Independently of whether the initial conditions are deterministic or random the Fourier method is about looking for a solution

$$u\,(x, t) = \sum_{k=1}^{\infty} X_k(x) \left[A_k \cos \sqrt{\lambda_k} t + \frac{B_k}{\sqrt{\lambda_k}} \sin \sqrt{\lambda_k} t \right], \tag{3.5}$$

$$x \in [0, \pi]\,,\ t \in [0, T]\,,\ T > 0,$$

where

$$A_k = \int_0^\pi \xi(x) X_k(x) \rho(x) dx, \quad k \geq 1,$$

$$B_k = \int_0^\pi \eta(x) X_k(x) \rho(x) dx, \quad k \geq 1,$$

and where $\lambda_k,\ k \geq 1$ are eigenvalues, and $X_k = (X_k)(x),\ x \in [0, \pi]),\ k \geq 1$, the corresponding orthonormal, with weight $\rho(\bullet)$, eigenfunctions of the following Sturm-Liouville problem:

$$\frac{d}{dx}\left(p(x)\frac{dX_k(x)}{dx}\right) - q(x)x(x) + \lambda\rho(x)X(x) = 0, \tag{3.6}$$

$$x \in [0, \pi],$$

$$X(0) = X(\pi) = 0. \tag{3.7}$$

The assumptions imposed on the function $p(x),\ \ \rho(x)$ and $q(x)$ make all eigenvalues $\lambda_k,\ k \geq 1$ positive, and we can assume that $\lambda_1 < \lambda_2 < \lambda_3 < \, < \ldots < \lambda_n < \ldots$. Observe also that the eigenfunctions $X_k,\ k \geq 1$ are twice continuously differentiable on $[0, \pi]$.

Suppose that $D = [0, \pi] \times [0, T]$, and let $C(D)$ be the space of functions continuous on D. This space is a separable Banach space.

Lemma 3.1. [12] *Assume that $\lambda_k,\ k \geq 1$ are eigenvalues and $X_k,\ k \geq 1$ are the corresponding eigenfunctions of the Sturm-Liouville problem (3.6)-(3.7) with the functions p, q, and ρ satisfying (i)-(iii). Then*

$$\sqrt{\lambda_k} = k + O\left(\frac{1}{k}\right)$$

as $k \to \infty$, and

$$X_k(x) = \sqrt{\frac{2}{\pi}} \sin k \left(\int_0^x \left(\frac{\rho(u)}{p(u)}\right)^{\frac{1}{2}} du \right) + \frac{\beta_k}{k},$$

for all $x \in [0, \pi]$, where

$$\sup_{k \geq 1} \sup_{x \in [0, \pi]} |\beta_k(x)| < \infty.$$

Lemma 3.2. [12] *Suppose that the functions* p, q, *and* ρ *satisfy conditions (i)-(iii). Then the eigenvalues* $\lambda_k,\ k \geq 1$ *and the corresponding eigenfunctions* $X_k,\ k \geq 1$ *of the Sturm-Liouville problem (3.6)-(3.7) are the eigenvalues and the corresponding eigenfunctions of the integral equation*

$$X(x) = \lambda \int_0^\pi G(x,s)\rho(s)X(s)ds, \tag{3.8}$$

where $G(x,s),\ x,s \in [0,\pi]$ *is the influence function of the boundary-value problem (3.6)-(3.7) defined as follows:*

$$G(x,s) = \begin{cases} u(x)v(s), & x \leq s; \\ u(s)v(x), & x > s, \end{cases}$$

$u(x)$ *and* $v(x)$ *are twice continuously differentiable on* $[0,\pi]$.

Theorem 3.1. *Let* $(\xi(x),\ x \in [0,\pi])$, *and* $(\eta(x),\ x \in [0,\pi])$ *be strongly Orlicz stochastic processes. In order that a twice continuously differentiable solution of the problem (3.1)-(3.3) exist with probability one in the domain* D, *and be represented in the form of a uniformly convergent in probability series (3.5), it is sufficient that*

(i) the continuous derivatives

$$\frac{d^2\xi(x)}{dx^2}, \quad \frac{d\eta(x)}{dx}, \quad 0 \leq x \leq \pi$$

exist with probability one;

(ii) for all $0 \leq x \leq \pi,\ 0 \leq t \leq T$ *the series (3.5) and the series*

$$\sum_{k=1}^{\infty} \sqrt{\lambda_k} X_k(x) \left[-A_k \sin \sqrt{\lambda_k} t + \frac{B_k}{\sqrt{\lambda_k}} \cos \sqrt{\lambda_k} t \right], \tag{3.9}$$

$$\sum_{k=1}^{\infty} \lambda_k X_k(x) \left[A_k \cos \sqrt{\lambda_k} t + \frac{B_k}{\sqrt{\lambda_k}} \sin \sqrt{\lambda_k} t \right] \tag{3.10}$$

converge uniformly in probability.

Proof. From Theorem 4.1 of [7] follows, that for the existence of twice continuously differentiable solution of the problem (3.1)-(3.3) in the set D, with probability one it is sufficient the condition (i) to be satisfied and for all

$x \in [0, \pi]$, $t \in [0, T]$ series converge uniformly in probability come out from (3.5) differentiable by x and t and double differentiable by x and t, that is series (3.9), (3.10), and series

$$\sum_{k=1}^{\infty} \frac{dX_k(x)}{dx} \left[A_k \cos \sqrt{\lambda_k} t + \frac{B_k}{\sqrt{\lambda_k}} \sin \sqrt{\lambda_k} t \right], \tag{3.11}$$

$$\sum_{k=1}^{\infty} \frac{d^2 X_k(x)}{dx^2} \left[A_k \cos \sqrt{\lambda_k} t + \frac{B_k}{\sqrt{\lambda_k}} \sin \sqrt{\lambda_k} t \right]. \tag{3.12}$$

Substituting $X_n(x)$ and λ_n in (3.8) we have

$$X_n(x) = \lambda_n \int_0^{\pi} G(x,s)\rho(s) X_n(s) ds. \tag{3.13}$$

By differentiation of (3.13) we obtain

$$X_n'(x) = \lambda_n \int_0^{\pi} G^*(x,s)\rho(s) X_n(s) ds,$$

(3.14)

$$X_n''(x) = \lambda_n \left[\int_0^{\pi} G^{**}(x,s)\rho(s) X_n(s) ds + \left(v'(x)u(x) - v(x)u'(x) \right) \rho(x) X_n(x) \right],$$

where

$$G^*(x,s) = \begin{cases} u'(x)v(s), & x \le s; \\ u(s)v'(x), & x > s, \end{cases}$$

$$G^{**}(x,s) = \begin{cases} u''(x)v(s), & x \le s; \\ u(s)v''(x), & x > s, \end{cases}$$

By substituting (3.13) in (3.11), we have for all $m \ge 1$, $n \ge m$

$$\sum_{k=m}^{n} \frac{dX_k(x)}{dx} \left[A_k \cos \sqrt{\lambda_k} t + \frac{B_k}{\sqrt{\lambda_k}} \sin \sqrt{\lambda_k} t \right] =$$

$$= \int\limits_0^\pi G^*(x,s)\rho(s)\left[\sum_{k=m}^n \lambda_k X_k(s)\left[A_k \cos\sqrt{\lambda_k}t + \frac{B_k}{\sqrt{\lambda_k}}\sin\sqrt{\lambda_k}t\right]\right] ds.$$

Then

$$\sup_{(x,t)\in D}\left|\sum_{k=m}^n \frac{dX_k(x)}{dx}\left[A_k \cos\sqrt{\lambda_k}t + \frac{B_k}{\sqrt{\lambda_k}}\sin\sqrt{\lambda_k}t\right]\right| \le$$

$$\le \int\limits_0^\pi |G^*(x,s)\rho(s)|\, ds \times \sup_{(x,t)\in D}\left|\sum_{k=m}^n \lambda_k X_k(x)\left[A_k \cos\sqrt{\lambda_k}t + \frac{B_k}{\sqrt{\lambda_k}}\sin\sqrt{\lambda_k}t\right]\right|.$$

From here it follows, that if the series (3.10) converges uniformly in probability, then the series (3.11) will also converge uniformly in probability. Prove that the uniform convergence uniformly in probability series (3.12) follows from the uniform convergence in probability series (3.10). Substituting (3.14) in (3.10) we obtain for $m \ge 1,\ n \ge m$

$$\sum_{k=m}^\infty \frac{d^2X_k(x)}{dx^2}\left[A_k \cos\sqrt{\lambda_k}t + \frac{B_k}{\sqrt{\lambda_k}}\sin\sqrt{\lambda_k}t\right] =$$

$$= \int\limits_0^\pi G^{**}(x,s)\rho(s)\left[\sum_{k=m}^n \lambda_k X_k(s)\left(A_k \cos\sqrt{\lambda_k}t + \frac{B_k}{\sqrt{\lambda_k}}\sin\sqrt{\lambda_k}t\right)\right] ds +$$

$$+ \left[v'(x)u(x) - v(x)u'(x)\right]\rho(x)\times$$

$$\times \sum_{k=m}^n \lambda_k X_k(x)\left[A_k \cos\sqrt{\lambda_k}t + \frac{B_k}{\sqrt{\lambda_k}}\sin\sqrt{\lambda_k}t\right].$$

Then

$$\sup_{(x,t)\in D}\left|\sum_{k=1}^\infty \frac{d^2X_k(x)}{dx^2}\left[A_k \cos\sqrt{\lambda_k}t + \frac{B_k}{\sqrt{\lambda_k}}\sin\sqrt{\lambda_k}t\right]\right| \le$$

$$\left(\int\limits_0^\pi |G^{**}(x,s)\rho(s)|\, ds + \sup_{(x,t)\in D}|v'(x)u(x) - v(x)u'(x)|\right)\times$$

$$\times \sup_{(x,t)\in D}\left|\sum_{k=1}^\infty \frac{d^2X_k(x)}{dx^2}X_k(x)\left[A_k \cos\sqrt{\lambda_k}t + \frac{B_k}{\sqrt{\lambda_k}}\sin\sqrt{\lambda_k}t\right]\right|.$$

Lemma 3.3. *Let the initial conditions* $(\xi(x),\ x \in [0,\pi])$, *and* $(\eta(x),\ x \in \in [0,\pi])$ *be strongly Orlicz stochastic processes and assume that the hypotheses of Theorem 3.1 hold. Then also the random series (3.5) and (3.9)-(3.10) are strongly Orlicz stochastic processes.*

Proof. It follows from Theorem 2.2 that the family of random variables A_k and B_k, $k \geq 1$, is a strongly Orlicz family. According to Theorem 2.1 the random series (3.5) and (3.9)-(3.10) are strongly Orlicz stochastic processes.

For $n \geq 1$ put

$$S_n^{(0)}(x,t) = \sum_{k=1}^{n} X_k(x) \left[A_k \cos \sqrt{\lambda_k} t + \frac{B_k}{\sqrt{\lambda_k}} \sin \sqrt{\lambda_k} t \right],$$

$$S_n^{(1)}(x,t) = \sum_{k=1}^{n} \sqrt{\lambda_k} X_k(x) \left[A_k \sin \sqrt{\lambda_k} t - \frac{B_k}{\sqrt{\lambda_k}} \cos \sqrt{\lambda_k} t \right],$$

$$S_n^{(2)}(x,t) = \sum_{k=1}^{n} \lambda_k X_k(x) \left[A_k \cos \sqrt{\lambda_k} t + \frac{B_k}{\sqrt{\lambda_k}} \sin \sqrt{\lambda_k} t \right],$$

$$(x,t) \in D.$$

Theorem 3.2. *Let* $\xi(x)$ *and* $\eta(x)$ *be strongly Orlicz processes. In order that a twice continuously differentiable solution of problem (3.1)-(3.3) exist with probability one in the domain* D, *and be represented in the form of series (3.5), uniformly convergent in probability, it is sufficient that*

(i) the derivatives

$$\frac{d^2\xi(x)}{dx^2}, \quad \frac{d\eta(x)}{dx}, \quad 0 \leq x \leq \pi$$

exist and are continuous with probability one;

(ii) for all $(x,t) \in D$ *the series*

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} X_k(x) X_l(x) \Big[EA_kA_l \cos \sqrt{\lambda_k} t \cos \sqrt{\lambda_l} t +$$

$$+ \frac{EB_kB_l}{\sqrt{\lambda_k}\sqrt{\lambda_l}} \sin \sqrt{\lambda_k} t \sin \sqrt{\lambda_l} t + \frac{2EA_kB_l}{\sqrt{\lambda_l}} \cos \sqrt{\lambda_k} t \sin \sqrt{\lambda_l} t \Big],$$

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sqrt{\lambda_k}\sqrt{\lambda_l} X_k(x) X_l(x) \Big[EA_kA_l \sin \sqrt{\lambda_k} t \sin \sqrt{\lambda_l} t +$$

$$+\frac{EB_kB_l}{\sqrt{\lambda_k}\sqrt{\lambda_l}}\cos\sqrt{\lambda_k}t\cos\sqrt{\lambda_l}t-2\frac{EA_kB_l}{\sqrt{\lambda_l}}\cos\sqrt{\lambda_k}t\sin\sqrt{\lambda_l}t\Big],$$

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\lambda_k\lambda_l X_k(x)X_l(x)\Big[EA_kA_l\cos\sqrt{\lambda_k}t\cos\sqrt{\lambda_l}t+$$

$$+\frac{EB_kB_l}{\sqrt{\lambda_k}\sqrt{\lambda_l}}\sin\sqrt{\lambda_k}t\sin\sqrt{\lambda_l}t+2\ \frac{EA_kB_l}{\sqrt{\lambda_l}}\cos\sqrt{\lambda_k}t\sin\sqrt{\lambda_l}t\Big]$$

converge;

(iii) for $n \geq 1$ and $k = 0, 1, 2$

$$\sup_{\substack{|x-y|\leq h\\ |t-s|\leq h}}\left(E\left|S_n^{(k)}(x,t)-S_n^{(k)}(y,s)\right|^2\right)^{\frac{1}{2}}\leq\sigma_k(h),$$

where $\sigma_k(h)$ is a monotononically increasing continuous function such that $\sigma_k(h) \to 0$ as $h \to 0$ moreover

$$\int_{0+}^{\varepsilon}u^{(-1)}\left(\left(\frac{\pi}{2\sigma^{(-1)}(u)}+1\right)\left(\frac{T}{2\sigma^{(-1)}(u)}+1\right)\right)du<\infty$$

where $\sigma_k^{-1}(\varepsilon)$ is the inverse function of $\sigma_k(\varepsilon)$.

Proof. Conditions (ii) imply that the series (3.5) and (3.9)-(3.10) converge in the mean square sense. According to Theorem 2.3 and Lemma 3.3 the series (3.5), (3.9)-(3.10) converge in probability in the space $C(D)$.

Now Theorem 3.2 follows from Theorem 3.1.

When using introduction from Lemma 3.1 we denote

$$Z_n^{(0)}=\sum_{k=1}^{n}A_k\sin(k\gamma(x))\cos kt+\frac{B_k}{k}\sin(k\gamma(x))\sin kt,$$

$$Z_n^{(1)}=\sum_{k=1}^{n}kA_k\sin(k\gamma(x))\sin kt-B_k\sin(k\gamma(x))\cos kt,$$

$$Z_n^{(2)}=\sum_{k=1}^{n}k^2A_k\sin(k\gamma(x))\cos kt+kB_k\sin(k\gamma(x))\sin kt,$$

$$(x,\ t)\in D,$$

where

$$\gamma(x) = \int\limits_0^x \left(\frac{\rho(u)}{p(u)}\right)^{\frac{1}{2}} du, \quad x \in [0, \pi].$$

Then Theorem 3.2 can be formulated as follows.

Theorem 3.3 *Let $\xi(x)$ and $\eta(x)$ be strongly Orlicz processes. In order that a twice continuously differentiable solution of problem (3.1)-(3.3) exist with probability one in the domain D, and be represented in the form of series (3.5), uniformly convergent in probability, it is sufficient that*

(i) the derivatives

$$\frac{d^2\xi(x)}{dx^2}, \quad \frac{d\eta(x)}{dx}, \quad 0 \le x \le \pi,$$

exist and are continuous with probability one;

(ii) for all $(x,t) \in D$ the series

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\Big[EA_kA_l\sin(k\gamma(x))\cos kt\sin(l\gamma(x))\cos lt + \frac{EB_kB_l}{kl}\sin(k\gamma(x))\times$$

$$\times \sin kt\sin(l\gamma(x))\sin lt + \frac{2EA_kB_l}{l}\sin(k\gamma(x))\cos kt\sin(l\gamma(x))\sin lt\Big],$$

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left[klEA_kA_l\sin(k\gamma(x))\sin kt\sin(l\gamma(x))\sin lt + EB_kB_l\sin(k\gamma(x)) \times\right.$$

$$\left.\times \cos kt\sin(l\gamma(x))\cos lt - 2kEA_kB_l\sin(k\gamma(x))\sin kt\sin(l\gamma(x))\cos lt\right],$$

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\big[k^2l^2EA_kA_l\sin(k\gamma(x))\cos kt\sin(l\gamma(x))\cos lt+$$

$$+klEB_kB_l\sin(k\gamma(x) \times \sin(k\gamma(x))\sin kt\sin(l\gamma(x))\sin lt+$$

$$+2k^2lEA_kB_l\sin(k\gamma(x))\cos kt\sin(l\gamma(x))\sin lt\big]$$

converge;

(iii) for $n \ge 1$ and $k = 0, 1, 2$

$$\sup_{\substack{|x-y|\le h\\ |t-s|\le h}} \left(E\left|S_n^{(k)}(x,t) - S_n^{(k)}(y,s)\right|^2\right)^{\frac{1}{2}} \le \sigma_k(h),$$

where $\sigma_k(h)$ is a monotonically increasing continuous function such that $\sigma_k(h) \to 0$ as $h \to 0$, moreover

$$\int\limits_{0+}^{\varepsilon} u^{(-1)} \left(\left(\frac{\pi}{2\sigma_k^{(-1)}(u)} + 1 \right) \left(\frac{T}{2\sigma_k^{(-1)}(u)} + 1 \right) \right) du < \infty$$

where $\sigma_k^{-1}(\varepsilon)$ is the inverse function of $\sigma_k(\varepsilon)$.

Proof. Let us show, that if the condition (iii) of this theorem is fulfilled, then series (3.9) will converge uniformly in probability. For the series (3.5) and (3.8) the arguments will be analogous. According to Lemma 3.1

$$\sqrt{\lambda_k} = k + O\left(\frac{1}{k}\right),$$

$$X_k(x) = \sqrt{\frac{2}{\pi}} \sin(\gamma(x)) + \frac{\beta_k(x)}{k},$$

where

$$|\beta_k(x)| < C, \quad \gamma(x) = \int\limits_0^x \left(\frac{\rho(u)}{p(u)}\right)^{\frac{1}{2}} du, \quad x \in [0, \pi].$$

Therefore

$$\sum_{k=1}^{\infty} \lambda_k X_k(x) \left[A_k \cos\sqrt{\lambda_k} t + \frac{B_k}{\sqrt{\lambda_k}} \sin\sqrt{\lambda_k} t \right] =$$

$$= \sum_{k=1}^{\infty} \lambda_k \sqrt{\frac{2}{\pi}} \sin k(\gamma(x)) A_k \cos\sqrt{\lambda_k} t + \sum_{k=1}^{\infty} \sqrt{\lambda_k} \sqrt{\frac{2}{\pi}} \sin k(\gamma(x)) B_k \cos\sqrt{\lambda_k} t +$$

$$+ \sum_{k=1}^{\infty} \frac{\lambda_k}{k} A_k \beta^*(x) + \sum_{k=1}^{\infty} \frac{\sqrt{\lambda_k}}{k} B_k \beta^{**}(x) = \sum_{k=1}^{\infty} k^2 \sqrt{\frac{2}{\pi}} \sin k(\gamma(x)) A_k \cos\sqrt{\lambda_k} t +$$

$$+ \sum_{k=1}^{\infty} k \sqrt{\frac{2}{\pi}} \sin k(\gamma(x)) B_k \cos\sqrt{\lambda_k} t + \sum_{k=1}^{\infty} O\left(\frac{1}{k^2}\right) \sqrt{\frac{2}{\pi}} \sin k(\gamma(x)) A_k \cos\sqrt{\lambda_k} t +$$

$$+ \sum_{k=1}^{\infty} O\left(\frac{1}{k}\right) \sqrt{\frac{2}{\pi}} \sin k(\gamma(x)) B_k \cos\sqrt{\lambda_k} t + \sum_{k=1}^{\infty} \frac{\lambda_k}{k} A_k \beta^*(x) + \sum_{k=1}^{\infty} \frac{\sqrt{\lambda_k}}{k} B_k \beta^{**}(x),$$

$$|\beta_k^*(x)| \le C_1, \quad |\beta_k^{**}(x)| \le C_2.$$

Since $\xi(x)$ has continuous second derivative and $\eta(x)$ has continuous derivative, thus from [12] (p.462, p.463) it follows the series converge with probability one $\sum\limits_{k=1}^{\infty}|C_k|<\infty$, $\sum\limits_{k=1}^{\infty}\left|C_k\sqrt{|\lambda_k|}\right|<\infty$. Thus, the series

$$\sum_{k=1}^{\infty}\frac{\lambda_k}{k}A_k\beta^*(x),\quad \sum_{k=1}^{\infty}\frac{\lambda_k}{k}A_k\beta^*(x),$$

$$\sum_{k=1}^{\infty}O\left(\frac{1}{k^2}\right)\sqrt{\frac{2}{\pi}}\sin k(\gamma(x))A_k\cos\sqrt{\lambda_k}t,$$

$$\sum_{k=1}^{\infty}O\left(\frac{1}{k}\right)\sqrt{\frac{2}{\pi}}\sin k(\gamma(x))B_k\cos\sqrt{\lambda_k}t.$$

converge with probability one.

From here it follows, that for uniform convergence in probability series (3.9) it is sufficient the series to converge

$$\sum_{k=1}^{\infty}k^2A_k\sin(k\gamma(x))\cos kt+kB_k\sin(k\gamma(x))\sin kt,$$

where

$$\gamma(x)=\int\limits_0^x\left(\frac{\rho(u)}{p(u)}\right)^{\frac{1}{2}}du,\quad x\in[0,\pi].$$

Now the theorem follows from Theorem 3.2.

3.1. The conditions of existence with probability one of twice continuously differentiable solution of the boundary-value problem of homogeneous string vibration in a partial case

Example 3.4. Assume that $\xi(x)$, $\eta(x)$ are strongly Orlicz processes $L_u\Omega$. Let $u(x)$ be a function such that $u(x)=|x|^p$ for some $p>1$ and all $|x|>1$. Then conditions (iii) of Theorem 3.2 holds for the function $\sigma_k(h)=C_k|h|^\delta$ with $0<\delta\le 1$. Indeed, for $\varepsilon>0$

$$I=\int\limits_0^{\varepsilon}u^{-1}\left(\left(\frac{\pi}{2\sigma_k^{(-1)}}+1\right)\left(\frac{T}{2\sigma_k^{(-1)}}+1\right)\right)du<\infty,$$

$$I \leq \int\limits_0^{\varepsilon} \left(\frac{\pi C_k^{\frac{1}{\delta}}}{2u^{\frac{1}{\delta}}} \cdot \frac{T C_k^{\frac{1}{\delta}}}{2u^{\frac{1}{\delta}}} \right)^{\frac{1}{p}} du \leq D \int\limits_0^{\varepsilon} \frac{1}{u^{\frac{2}{p\delta}}} du.$$

The latter integral converges under $\delta > \frac{2}{\pi}$.

Theorem 3.5. *Let $\xi(x)$, and $\eta(x)$, $x \in [0, \pi]$ be strongly Orlicz processes $L_u\Omega$ where $u(x)$ is a function such that $u(x) = |x|^p$ for some $p > 1$ and all $|x| > 1$. Set*

$$B_\xi(x, y) = E\xi(x)\xi(y),$$

$$B_\eta(x, y) = E\eta(x)\eta(y).$$

In order that a twice continuously differentiable solution of the problem (3.1)-(3.3) exist with probability one in the domain D and be represented in the form of series (3.5), uniformly convergent in probability it is sufficient that

(i) the partial derivatives

$$B_\xi^{**}(x, y) = \frac{\partial^4 B(x, y)}{\partial x^2 \partial y^2}, \quad B_\eta^{*}(x, y) = \frac{\partial^2 B(x, y)}{\partial x \partial y},$$

exist for $x, y \in [0, \pi]$ and are continuous, and

$$\sup_{|x-y| \leq h} \left(B_\xi^{**}(x, x) + B_\xi^{**}(y, y) - 2B_\xi^{**}(x, y) \right) \leq C_{**} \left| h \right|^{\delta},$$

$$\sup_{|x-y| \leq h} \left(B_\eta^{*}(x, x) + B_\eta^{*}(y, y) - 2B_\eta^{*}(x, y) \right) \leq C_{1*} \left| h \right|^{\delta},$$

for sufficiently small h, where $\delta > \frac{2}{p}$;

(ii) the series

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} k^2 l^2 \left[|EA_k A_l| + \frac{|EB_k B_l|}{kl} + 2\frac{EA_k B_l}{l} \right]$$

converge;

(iii)

$$\sum_{k=1}^{\infty} \left(k^2 \left(EA_k^2 \right)^{\frac{1}{2}} + \frac{\left(EB_k^2 \right)^{\frac{1}{2}}}{k} \right) (k)^{\delta} < \infty$$

for arbitrary $\delta > \frac{2}{p}$.

Proof. Condition (i) of Theorem 3.5 implies condition (i) of Theorem 3.2. According to Example 3.4 the conditions of Theorem 2.5 hold for processes $\xi(x)$ and $\eta(x)$ if

$$\sigma_k(h) = C_k|h|^\delta, \quad \delta > \frac{2}{p}.$$

It is clear that the series in condition (ii) of Theorem 3.2 converge if so do the series in condition (ii) of this theorem.

Example 3.4 and Lemma 2.3 imply that conditions (iii) of Theorem 3.2 follow from condition (iii) of Theorem 3.5. It is clear that

$$\left(E\left|Z_n^{(0)}(x,t) - Z_n^{(0)}(y,s)\right|^2\right)^{\frac{1}{2}} =$$

$$= \left(E\left|\sum_{k=1}^{n}\left(A_k \sin(k\gamma(x))\cos kt + \frac{B_k}{k}\sin(k\gamma(x))\cos kt\right) -\right.\right.$$

$$\left.\left.- \sum_{k=1}^{n}\left(A_k \sin(k\gamma(x_1))\cos kt +_1 \frac{B_k}{k}\sin(k\gamma(x_1))\cos kt_1\right)\right|^2\right)^{\frac{1}{2}} \leq$$

$$\leq \left(\sum_{k}^{n}\sum_{l=1}^{n}[|EA_kA_l|\,|\sin(k\gamma(x))\cos kt - \sin(k\gamma(x_1))\cos kt_1| \;\times\right.$$

$$\times\,|\sin(l\gamma(x))\cos lt - \sin(l\gamma(x_1))\cos lt_1| +$$

$$+\left|\frac{EB_kB_l}{kl}\right|\,|\sin(k\gamma(x))\sin kt - \sin(k\gamma(x_1))\sin kt_1| \times$$

$$\times\,|\sin(l\gamma(x))\sin lt - \sin(l\gamma(x_1))\sin lt_1| +$$

$$+\left|2\frac{EA_kB_l}{l}\right|\,|\sin(k\gamma(x))\cos kt - \sin(k\gamma(x_1))\cos kt_1| \times$$

$$\left.\times\;|\sin(l\gamma(x))\sin lt - \sin(l\gamma(x_1))\sin lt_1|]\right)^{\frac{1}{2}} \leq$$

$$\leq \left(\left(\sum_{k=1}^{n}|EA_k|\,|\sin(k\gamma(x))\cos kt - \sin(k\gamma(x_1))\cos kt_1|\right)^2 +\right.$$

$$+\left(\sum_{k=1}^{n}\sum_{l=1}^{n}\left|2\frac{EA_kB_l}{l}\right|\,|\sin(k\gamma(x))\cos kt - \sin(k\gamma(x_1))\cos kt_1| \times\right.$$

$$\times\,|\sin(l\gamma(x))\sin lt - \sin(l\gamma(x_1))\sin lt_1|\Big) +$$

$$+\left(\sum_{k=1}^{n}\left|\frac{EB_k}{k}\right|\left|\sin(k\gamma(x))\sin kt-\sin(k\gamma(x_1))\sin kt_1\right|^2\right)^{\frac{1}{2}}\leq$$

$$\leq\sum_{k=1}^{n}\left[\left(EA_k^2\right)^{\frac{1}{2}}\left|\sin(k\gamma(x))\cos kt-\sin(k\gamma(x_1))\cos kt_1\right|+\right.$$

$$\left.+\frac{\left(EB_k^2\right)^{\frac{1}{2}}}{k}\left|\sin(k\gamma(x))\sin kt-\sin(k\gamma(x_1))\sin kt_1\right|\right].$$

Furthermore,

$$|\sin(k\gamma(x))\cos kt-\sin(k\gamma(x_1))\cos kt_1|\leq$$

$$\leq|\sin(k\gamma(x))-\sin(k\gamma(x_1))|+|\cos kt-\cos kt_1|\leq$$

$$\leq 2\left(\left|\sin\frac{k(\gamma(x)-\gamma(x_1))}{2}\right|+\left|\sin\frac{k(t-t_1))}{2}\right|\right). \tag{3.15}$$

$$\gamma(x+h)-\gamma(x)=\int_{x}^{x+h}\left(\frac{\rho(u)}{p(u)}\right)^{\frac{1}{2}}du\leq c_0h,\quad c_0=\max_{u\in[0,\pi]}\left(\frac{\rho(u)}{p(u)}\right)^{\frac{1}{2}}>0.$$

The inequality

$$|\sin\alpha|\leq|\alpha|^{\delta},\quad 0<\delta\leq 1$$

together with (3.15) imply that

$$|\sin(k\gamma(x))\cos kt-\sin(k\gamma(x_1))\cos kt_1|\leq$$

$$\leq 2\left(\left|\frac{k(\gamma(x)-\gamma(x_1))}{2}\right|^{\delta}+\left|\frac{k(t-t_1)}{2}\right|^{\delta}\right)\leq$$

$$\leq 2\left(\frac{k^{\delta}c_0^{\delta}h^{\delta}}{2^{\delta}}+\frac{k^{\delta}h^{\delta}}{2^{\delta}}\right)=\frac{1}{2^{\delta-1}}(c_0+1))k^{\delta}h^{\delta}.$$

Similarly,

$$|\sin(k\gamma(x))\sin kt-\sin(k\gamma(x_1))\sin kt_1|\leq\frac{1}{2^{\delta-1}}(c_0+1))k^{\delta}h^{\delta}.$$

One can easily obtain that

$$\left|E\left(Z_n^{(0)}(x,t)-Z_n^{(0)}(y,s)\right)^2\right|^{\frac{1}{2}}\leq C_1\,|h|^{\delta},$$

where

$$C_1 = \frac{1}{2^{\delta-1}}(c_0+1)\sum_{k=1}^{\infty}\left(\left(EA_k^2\right)^{\frac{1}{2}} + \frac{\left(EB_k^2\right)^{\frac{1}{2}}}{k}\right)(k)^{\delta}.$$

Similarly,

$$\left|E\left(Z_n^{(1)}(x,t) - Z_n^{(1)}(y,s)\right)^2\right|^{\frac{1}{2}} \leq$$

$$\leq \sum_{k=1}^{n}\Big[k\left(EA_{k^2}\right)^{\frac{1}{2}}\left|\sin(k\gamma(x))\cos kt - \sin(k\gamma(x_1))\cos kt_1\right| +$$

$$+\frac{\left(EB_k^2\right)^{\frac{1}{2}}}{k}\left|\sin(k\gamma(x))\sin kt - \sin(k\gamma(x_1))\sin kt_1\right|\Bigg] \leq C_2\left|h\right|^{\delta},$$

where

$$C_2 = \frac{1}{2^{\delta-1}}(c_0+1)\sum_{k=1}^{\infty}\left(k\left(EA_k^2\right)^{\frac{1}{2}} + \frac{\left(EB_k^2\right)^{\frac{1}{2}}}{k}\right)(k)^{\delta}.$$

$$\left|E\left(Z_n^{(2)}(x,t) - Z_n^{(2)}(y,s)\right)^2\right|^{\frac{1}{2}}$$

$$\sum_{k=1}^{n}\Big[k^2\left(EA_{k^2}\right)^{\frac{1}{2}}\left|\sin(k\gamma(x))\cos kt - \sin(k\gamma(x_1))\cos kt_1\right|$$

$$\frac{\left(EB_k^2\right)^{\frac{1}{2}}}{k}\left|\sin(k\gamma(x))\sin kt - \sin(k\gamma(x_1))\sin kt_1\right|\Bigg] \leq C_3\left|h\right|^{\delta},$$

where

$$C_3 = \frac{1}{2^{\delta-1}}(c_0+1)\sum_{k=1}^{\infty}\left(k^2\left(EA_k^2\right)^{\frac{1}{2}} + \left(EB_k^2\right)^{\frac{1}{2}}k\right)(k)^{\delta}.$$

Convergence of series C_3 implies the convergence of series C_1 and C_2.

References

[1] **Barrasa de La Krus, E. and Kozachenko, Yu.V.,** Boundary-value problems for equations of mathematical physics with strictly Orlics random initial conditions, *Random Oper. and Stoch. Equ.,* **3** (3) (1995), 201-220.

[2] **Buldygin, V.V. and Kozachenko, Yu.V.,** *Metric characterization of random variables and random processes*, American Mathematical Society, Providence, Rhode Island, 2000.

[3] **Gladkaya, O.N.,** Conditions for directional differentiability of the sample functions of random fields, *Teor. Imovirnost. Matem. Stat.,* **17** (1977), 33-40. (English transl. in *Theory Probab. Mathem. Statist.,* **17** (1977), 35-43.)

[4] **Dovgay, B.V. and Kozachenko, Yu.V.,** The condition for application of Fourier method to the solution of nonhomogeneous string oscillation equation with φ-subgaussian right hand side, *Random Oper. and Stoch. Equ.,* **13** (3) (2005), 281-296.

[5] **Dovgay, B.V., Kozachenko, Yu.V. and Slyvka-Tylyshchak, G.I.,** *The boundary-value problems of mathematical physics with random factors,* "Kyiv university", Kyiv, 2008. (in Ukrainian)

[6] **Kozachenko, Yu.V. and Kovalchuk, Ya.A.,** Boundary value problems with random initial conditions and series of functions of $Sub_{\varphi}(\Omega)$, *Ukr. Matem. Zh.,* **50** (4) (1998), 504-515.

[7] **Kozachenko, Yu.V. and Slyvka, G.I.,** Justification of the Fourier method for hyperbpolic equations with random initial conditions, *Teor. Imovirnost. Matem. Stat.,* **69** (2003), 63-78. (English transl. in *Theory Probab. Mathem. Statist.,* **69** (2004), 67-83.

[8] **Kozachenko, Yu.V. and Slyvka, G.I.,** Boundary-value problems for equations of mathematical physics with strictly $Sub_{\varphi}(\Omega)$ random initial conditions, *Theory of Stochastic Processes,* **10 (26)** (2004), 60-71.

[9] **Kozachenko, Yu.V. and Veresh, K.J.,** Boundary-value problems for a nonhomogeneous parabolic equation with Orlicz right side, *Random Oper. and Stoch. Equ.,* **18** (2010), 97-119.

[10] **Slyvka, G.I.,** A boundary-value problem of the mathematical physics with random initial conditions, *Visn. Kyiv Univ. Ser. Fiz.-Mat. Nauk,* **5** (2002), 172-178.

[11] **Slyvka, G.I. and Veresh, K.J.,** Justifications of the Fourier method for hyperbolic equations with random initials conditions from Orlicz Spaces, *Visn. Uzh. Univ. Ser. Math. and Inform.,* **16** (2008), 174-183.

[12] **Polozhiy, G.N.,** *Equations of mathematical physics,* Vysshaya shkola, Moscow, 1964. (in Russian).

A. Slyvka-Tylyshchak
Taras Shevchenko National University of Kyiv
Kyiv, Ukraine
aslyvka@tn.uz.ua