DISTRIBUTION OF THE VALUES OF 
q–ADDITIVE FUNCTIONS 
ON SOME MULTIPLICATIVE SEMIGROUPS II.

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Dedicated to Dr. Bui Minh Phong on his sixtieth birthday

Communicated by K.-H. Indlekofer  
(Received November 30, 2012)

Abstract. In [1] we investigated the distribution of the values of \(q\)-additive 
functions defined on multiplicative semigroups which are generated by an 
infinite sequence of primes satisfying Wirsing’s condition. In this work we 
extend our investigations started in [1] to polynomial sequences of such 
semigroups and its subsets which contain integers with a given number of 
prime divisors.

1. Introduction

1.1.

The project is supported by the European Union and co-financed by the 
European Social Fund (grant agreement TAMOP 4.2.1/B/09/1/KMR/2010/
0003) and the second author is partly supported by the Hungarian and 
Vietnamese TET (grant agreement no. TET 10-1-2011-0645).

Mathematics Subject Classification: 11L07, 11A63
\[ N, \mathbb{R}, \mathbb{C} \text{ are the sets of natural, real, complex numbers, respectively. } N_0 = \mathbb{N} \cup \{0\}. \text{ Let } e(x) := e^{2\pi i x}; \omega(n) = \text{number of distinct prime divisor of } n; \Omega(n) = \text{number of prime power divisors of } n. \text{ Let } \{x\} = \text{fractional part of } n, ||x|| = \min(\{x\}, 1 - \{x\}). \text{ For the sake of brevity let } x_1 = \log x, x_2 = \log x_1, \text{ and in general, let } x_{k+1} = \log x_k (k = 1, 2, \ldots). \text{ Let } \gamma \text{ be the Euler’s constant, } \Gamma \text{ be the gamma function and} \]

\[ \Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} du. \]

1.2.

Let \( q \in \mathbb{N}, q \geq 2 \) be fixed, \( E = \{0, 1, \ldots, q - 1\} \). The \( q \)-ary expansion of \( n \in N_0 \) is defined by

\[ n = \sum_{j=0}^{\infty} a_j(n)q^j, \quad a_j(n) \in E. \tag{1.1} \]

A function \( f : N_0 \to \mathbb{R} \) is said to be \( q \)-additive, if \( f(0) = 0 \) and

\[ f(n) = \sum_{j=0}^{\infty} f(a_j(n)q^j), \quad a_j(n) \in E. \tag{1.2} \]

Let \( A_q \) be the set of \( q \)-additive functions. Let \( N(= N_2) = \left\lfloor \frac{\log n}{\log q} \right\rfloor \),

\[ m_k = \frac{1}{q} \sum_{b \in E} f(bq^k), \quad \sigma_k^2 = \frac{1}{q} \sum_{b \in E} f^2(bq^k) - m_k^2, \tag{1.3} \]

\[ M(x) = \sum_{k=0}^{N} m_k, \quad D^2(x) = \sum_{k=0}^{N} \sigma_k^2. \tag{1.4} \]

1.3.

Let \( \nu_x(n) := \frac{f(n) - M(x)}{D(x)}. \)

In our recent paper [1] we proved the following
Theorem A. Let $\mathcal{P}$ be an infinite sequence of primes, satisfying

\begin{equation}
\pi_P(x) := \#\{p \leq x \mid p \in \mathcal{P}\} = (\tau + o(1)) \frac{x}{\log x} \quad (x \to \infty),
\end{equation}

where $\tau > 0$ is a constant. Let $\mathcal{N}$ be the multiplicative semigroup generated by the elements of $\mathcal{P}$,

\((N_P(x) =) N(x) := \#\{n \leq x, n \in \mathcal{N}\}.
\)

Let $f \in A_q$, $f(bq^j) = O(1)$ as $b \in E$, $j = 0, 1, 2, \ldots$. Assume that $D(x)/\log^\lambda x \to \infty$ as $x$ tends to infinity for some $\lambda > 0$. Let

\begin{equation}
F_x(y) := \frac{1}{N(x)} \#\{\nu_x(n) < y, n \leq x, n \in \mathcal{N}\}.
\end{equation}

Then

\begin{equation}
\lim_{x \to \infty} F_x(y) = \Phi(y).
\end{equation}

The proof is based on a theorem of Davenport for trigonometric sums (see [2], Lemma 1) and on the method developed in [3].

We observed that by using a theorem of L.K. Hua ([3], see Lemma 6.3), by using the method used by N.L. Bassily and I. Kátai [5] one can prove

**Theorem 1.** Let $f \in A_q$, $f(bq^j) = O(1)$ as $b \in E$, $j = 0, 1, 2, \ldots$, $D(x)/\log^\delta x \to \infty$ as $x$ tends to infinity with a suitable $\delta > 0$. Assume that $\mathcal{P}$ satisfies the condition (1.6). Let $P \in \mathbb{Z}[x]$ be a polynomial of degree $t$, with positive leading coefficient. Let

\begin{equation}
G_x(y) := \frac{1}{N(x)} \#\{n \leq x, n \in \mathcal{N}, \nu_x(P(n)) < y\}.
\end{equation}

Then

\begin{equation}
\lim_{x \to \infty} G_x(y) = \Phi(y)
\end{equation}

holds for every $y$.

1.4.

Let $P \in \mathbb{Z}[x]$ be a polynomial of degree $t$ taking positive integer values on $\mathbb{N}$. Let $q, E$ be as in 1.2. If $n \in \mathbb{N}$, $n = \epsilon_0(n) + \epsilon_1(n)q + \cdots + \epsilon_{r-1}(n)q^{r-1}$, then
write \( \pi = \epsilon_0(n) \cdots \epsilon_{r-1}(n) (\in E^r) \), \( \epsilon_{r-1} \neq 0 \). Let \( \mathcal{P}, \mathcal{N} \) be as in Theorem A. Let \( n_1 < n_2 < \ldots \) be the whole sequence of the integers in \( \mathcal{N} \), and let

\[
(1.11) \quad \eta = 0, P(n_1), P(n_2) \ldots
\]

where the right hand side of (1.11) is the \( q \)-ary expansion of \( \eta \).

**Theorem 2.** We have that \( \{q^m\eta\} (m = 1, 2, \ldots) \) is a sequence uniformly distributed mod 1.

This assertion can be derived from Theorem 3, formulated in 1.5.

**1.5.**

Let \( \mathcal{P}, \mathcal{N}, \mathcal{P} \) as earlier. Let \( \beta = b_0 b_1 \ldots b_{k-1} \) be a typical element of \( E^k \). We write \( \Phi_k(n) = \epsilon_j(n) \cdots \epsilon_{j+k-1}(n) \). Let \( F_k : E^k \rightarrow \mathbb{R} \) be a function such that \( F(0, \ldots, 0) = 0 \). Let

\[
\begin{align*}
\alpha_n &:= \sum_{j=0}^{\infty} F_k(\Phi_j(n)), \quad \kappa_1 := \sum_{j=0}^{\infty} F_k(\Phi_j^1(n)), \\
M &:= q^{-k} \sum_{b_1 \ldots b_k \in E^k} F_k(b_1 \ldots b_k), \\
\sigma_h^2 &= q^{-(k+h)} \sum_{b_0 \ldots b_{h+k-1} \in E^{k+h}} (F_k(b_0 \ldots b_{k-1}) - M)(F_k(b_h \ldots b_{h+k-1}) - M)
\end{align*}
\]

for \( h = 0, 1, \ldots, k-1 \). Let

\[
\sigma^2 = \sigma_0^2 + \sum_{h=1}^{k-1} \sigma_h^2.
\]

**Theorem 3.** Assume that \( \sigma \neq 0 \). Then

\[
\lim_{x \to \infty} \# \left\{ n \leq x, n \in \mathcal{N} \mid \frac{\alpha_n - MNr}{\sigma \sqrt{Nr}} < y \right\} = \Phi(y)
\]

holds for every \( y \in \mathbb{R} \).

We can prove also

**Theorem 4.** Let \( \mathcal{P}, \mathcal{N}, \mathcal{P}, f \) be as in Theorem 1. Let

\[
G_{x,k}(y) := \frac{1}{\pi_k(x)} \# \{ n \leq x, n \in \mathcal{N}, \omega(n) = k, \nu_x(P(n)) < y \}.
\]
Then, if \( k_0(x) \to \infty \), then

\[
\sup_y \sup_{k_0(x) \leq k \leq \alpha_x(1)} |G_{x,k} - \Phi(y)| = 0.
\]

**Remark.** Unfortunately, we cannot prove that

\[
\lim_{x \to \infty} G_{x,1}(y) = \Phi(y).
\]

### 2. Auxiliary results

#### 2.1. The Erdős–Turán inequality ([6]):

The discrepancy \( D_M \) of the real numbers \( x_1, \ldots, x_M \) (mod 1) is defined by

\[
(2.1) \quad D_M \leq c \left( \sum_{0 < k \leq K} \frac{|\Psi_k|}{h} + \frac{M}{K} \right)
\]

for any positive integer \( K \). \( c \) is an absolute constant.

#### 2.2. Lemma 6.3 of L.K. Hua ([4]):
Let \( l \) be a positive integer \((\leq x^\sigma_3)\), and
\[
\Omega = \sum_d \sum_m e(f(ldm)),
\]
\[
f(z) = \frac{h}{Q} z^t + \alpha_1 z^{t-1} + \cdots + \alpha_t,
\]
where \((h, Q) = 1\), all \( \alpha \) being real, and \( x_1^\sigma < Q < x^t \cdot x_1^{-\sigma} \). The index \( d \) in \( \Omega \) runs through a set of positive integers satisfying the conditions
\[
D < d \leq D', \quad 1 < D < \frac{x}{l}, \quad D' \leq 2D.
\]
Further, for a fixed \( d \), the index \( m \) runs through a set of positive integers satisfying the inequality
\[
P'/d < m \leq \frac{x}{Dl},
\]
where \( P' \) is a positive number. Hence, for \( x_1^{\sigma_5} < D < x \cdot x_1^{-\sigma} \), subject to the conditions
\[
\sigma \geq 2t\sigma_3 + 2^{2t+1}\sigma_6 + 2^{3(2t-1)}
\]
we have
\[
\Omega \ll \frac{x}{l} x_1^{-\sigma}.
\]

2.3.
Theorem of E. Wirsing ([7]):

Let \( F \) be a multiplicative function, satisfying the conditions: \( F(n) \geq 0 \) (\( n \in \mathbb{N} \)); \( F(p^\alpha) \leq c_1 c_2^\alpha \), \( c_2 < 2 \) for every prime \( p \) and \( \alpha = 2, 3, \ldots \). Assume that
\[
\sum_{p \leq x} F(p) = (\tau + o(1)) \frac{x}{\log x} \quad (x \to \infty),
\]
where \( \tau > 0 \) is a constant. Then, for \( x \to \infty \),
\[
\sum_{n \leq x} F(n) = \left( e^{-\gamma \tau} + o_x(1) \right) \frac{x}{\log x} \prod_{p \leq x} \left( 1 + \frac{F(p)}{p} + \frac{F(p^2)}{p^2} + \cdots \right).
\]

Analyzing the proof, one can see easily that the following version of the theorem of E. Wirsing is true.
Lemma 1. Let $F_\lambda$ be a family of multiplicative functions satisfying the following conditions: $F_\lambda(n) \geq 0$ ($n \in \mathbb{N}$); $F_\lambda(p^\alpha) \leq c_1 e_2^\alpha$, $c_2 < 2$ for every prime $p$ and $\alpha = 2, 3, \ldots$. Assume that

$$\left| \sum_{p \leq x} F_\lambda(p) - \tau_\lambda \frac{x}{\log x} \right| \leq \epsilon(x) \frac{x}{\log x},$$

where $0 < c_3 < \tau_\lambda$, $c_3$ is a suitable constant, $\epsilon(x) \to 0$ as $x$ tends to infinity.

Then there exists a function $\epsilon_1(x) \to 0$ ($x \to \infty$) such that

$$\left| \sum_{n \leq x} F_\lambda(n) - \frac{e^{-\gamma \tau_\lambda}}{\Gamma(\tau_\lambda)} \frac{x}{\log x} \prod_{p \leq x} \left( 1 + \frac{F_\lambda(p)}{p} + \frac{F_\lambda(p^2)}{p^2} + \cdots \right) \right| \leq \epsilon_1(x) \frac{x}{\log x} \prod_{p \leq x} \left( 1 + \frac{F_\lambda(p)}{p} + \frac{F_\lambda(p^2)}{p^2} + \cdots \right).$$

Let $\mathcal{P}, \mathcal{N}$ be as defined in Theorem A. Defining the multiplicative function $F$ on prime powers $p^\alpha$ by

$$F(p^\alpha) = \begin{cases} 1, & \text{if } p \in \mathcal{P}, \\ 0, & \text{if } p \notin \mathcal{P}, \end{cases}$$

from Wirsing’s theorem we obtain that

$$N_\mathcal{P}(x) = \left( e^{-\gamma} + o(1) \right) \frac{x}{\log x} \prod_{p \in \mathcal{P}} \frac{1}{1 - 1/p}.$$

2.4.

Lemma 2. Let $0 < \Delta < \frac{1}{2q}$, $\chi_0(x) = \sum c_m e(mx)$ be a $(\mod 1)$ periodic function such that $0 \leq \chi_0(x) \leq 1$,

$$\chi_0(x) = \begin{cases} 1 & \text{if } \Delta < \{x\} < \frac{1}{q} - \Delta, \\ 0 & \text{if } \frac{1}{q} + \Delta < \{x\} < 1 - \Delta, \end{cases}$$

$c_0 = \frac{1}{q}$, $c_jq = 0$ when $j \neq 0$,

$$|c_m| \leq \min \left( \frac{1}{q}, \frac{1}{\pi |m|}, \frac{1}{\Delta^2 m^2} \right).$$
Let $\chi_b(x) = \chi_0(x - \frac{b}{q}) = \sum c_m^{(b)} e(mx)$. Then $c_m^{(b)} = c_m e\left(-\frac{mb}{q}\right)$, thus $|c_m^{(b)}| = |c_m|$. See in [5].

2.5.

Let $P$, $N$ be as earlier, $\pi_k(x) = \#\{n \leq x \mid n \in N, \omega(n) = k\}$, $N_k(x) = \#\{n \leq x \mid n \in N, \Omega(n) = k\}$.

Let

$$T(x) := \sum_{p^\nu \leq x \atop p \in P} \frac{1}{p^\nu}.\$$

**Lemma 3.** There is a function $\epsilon(x) \to 0 \ (x \to \infty)$ and positive constants $c_1, c_2$ such that

$$\frac{c_2(\tau - \epsilon(x))x}{\log x} T\left(\frac{x}{\pi_k(x)}\right)^{k-1} (k-1)! - (\log x) \sqrt{x} \leq \pi_k(x) \leq c_1 x \frac{T(x)^{k-1}}{\log x (k-1)!}$$

holds for every $k$, and

$$N_k(x) \leq \frac{c_3 x T(x)^{k-1}}{\log x (k-1)!}$$

holds for $1 \leq k \leq (1 - \delta)p_0 T(x)$, where $p_0$ is the smallest prime in $P$, $\delta$ is an arbitrary constant, $0 < \delta < 1$, and $c_3 = c_3(\delta)$ is a suitable constant.

**Proof of Lemma 3.** We have

$$\sum_{n \leq x \atop n \in \mathcal{P}_k} \log\nu \leq \sum_{p^\nu \leq x \atop m \in \mathcal{P}_{k-1}} \log p^\nu \leq \sum_{m \in \mathcal{P}_{k-1}} \sum_{p^\nu \leq \frac{x}{m}} \log p^\nu \leq 2x \sum_{m \in \mathcal{P}_{k-1}} \frac{1}{m} \leq \frac{2x T(x)^{k-1}}{(k-1)!}.\$$

Thus

$$(\pi_k(x) - \pi_k(\sqrt{x})) \frac{1}{2} \log x \leq \frac{2x T(x)^{k-1}}{(k-1)!},$$
\[ \pi_k(x) \leq \pi_k(\sqrt{x}) + \frac{4x}{\log x} \frac{T(x)^{k-1}}{(k-1)!}. \]

Iterating this, we obtain that the right hand side of (2.3) is true. Furthermore,

\[ \pi_k(x) \log x \geq \sum_{\substack{p^m \leq x \\ m \in \mathbb{P}_{k-1} \cap \sqrt{x}}} \log p^m \geq \sum_{\substack{p \leq \sqrt{x} \\ p \mid m}} \log p - \sum_{p \mid m} \log p^m \]

\[ \geq (\tau - \epsilon(x)) \sum_{m \leq \sqrt{x}} \frac{x}{m} - (\log x) \sum_{m \leq \sqrt{x}} \sum_{p \mid m} 1, \]

and so

\[ \pi_k(x) \geq (\tau - \epsilon(x))x \frac{T(\sqrt{x})^{k-1}}{(k-1)!} - \sqrt{x} \log x. \]

To prove (2.4), write \( n \in \mathcal{N}_k \) in the form \( n = Km \), where \( K \) is the squareful part and \( m \) is the squarefree part of \( n \).

The size of those \( n \leq x \) for which \( K > x^{1/2} \) is

\[ \leq \sum_{K > \sqrt{x}} \frac{x}{K} \leq cx^{3/4}. \]

Thus,

\[ \mathcal{N}_k(x) \leq \sum_{K \leq \sqrt{x}} \pi_{k-\Omega(K)} \left( \frac{x}{K} \right) + cx^{3/4}. \]

From inequality (2.3) we have

\[ \mathcal{N}_k(x) \leq \frac{c_1 x}{\log \sqrt{x}} \sum_{K \leq \sqrt{x}} \frac{T(x)^{k-\Omega(K)-1}}{K(k-\Omega(K)-1)!} + cx^{3/4}. \]

Furthermore,

\[ \sum_{K < \sqrt{x}} \frac{T(x)^{k-\Omega(K)-1}}{K(k-\Omega(K)-1)!} \leq \frac{T(x)^{k-1}}{(k-1)!} \sum_{K \leq \sqrt{x}} \left( \frac{k}{T(x)} \right)^{\Omega(K)} \frac{1}{K}. \]
Since $\frac{k}{T(x)} \leq (1 - \delta)p_0,$

$$\sum_{K \leq \sqrt{x}} \left( \frac{k}{T(x)} \right)^{\alpha(K)} \frac{1}{K} \leq \prod_{p \in P} \left( 1 + \left( \frac{k}{T(x)} \right) \frac{1}{p^2} - \frac{1}{\left( \frac{k}{T(x)} \right) \frac{1}{p}} \right).$$

Since $cx^{3/4}$ is clearly smaller than $c\frac{x}{\log x} \frac{T(x)^{k-1}}{(k-1)!},$ our inequality holds.

3. Proof of Theorem 1

Let $y \in \mathbb{R}$ be fixed. Let $n_1 < \ldots < n_s (\leq x)$ be the set all of the integers in $\mathbb{N}$ up to $x,$ for which $\nu_{x^*}(P(n)) < y.$ Then $s = G_x(y) \cdot N(x).$ Let $\mathcal{H}_x = \{m,p\}, p \in P, m \in N, m > x^{\epsilon_x}, p > e^{(\log x)^{\epsilon_x}}, mp \leq x\}.$ Here we assume that $\epsilon_x \to 0 \ (x \to \infty) \ (slowly).$

Let $R_x = \sum_{p \leq x} 1/p.$ Let $Z$ be the number of those $\{m, p\} \in \mathcal{H}_x$ for which $\nu_{x^*}(P(mp)) < y.$ Repeating the argument, used in [1], we obtain that

$$\frac{1}{N(x)} \left| \frac{Z}{R_x} - s \right| \to 0 \ (x \to \infty).$$

Let $H(x) = \# \mathcal{H}_x.$ Let $(1 \leq l_1 < \ldots < l_h \leq tN, b_1, \ldots, b_h \in E$ and

$$H \left( x \left| l_1, \ldots, l_h \right. \left. b_1, \ldots, b_h \right) = \#\{m,p\} \in \mathcal{H}_x, \epsilon_j(P(mp)) = b_j, j = 1, \ldots, h \right).$$

By using the method developed in [3, 5, 1] we can prove that

$$(3.1) \max_{N^{\alpha} \leq l_1, \ldots, l_h \leq tN - N^{\alpha}} \left| q^h H \left( x \left| l_1, \ldots, l_h \right. \left. b_1, \ldots, b_h \right) - H(x) \right| \leq c(h, \lambda)H(x)N^{-\lambda}$$

holds for every fixed $h,$ every $\alpha > 0,$ and every $\lambda > 0.$

By using the theorem of L.K. Hua ([4]) we can obtain that

$$\sum_{(m,p) \in \mathcal{H}_x} e \left( \frac{A_M}{H_M} P(mp) \right) \ll H(x) \log^{-B} x$$
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holds for every fixed $B$, where
\[
\frac{A_M}{H_M} = \frac{m_h}{q^{h+1}} + \cdots + \frac{m_1}{q^{1+1}}, \quad q|m_j \ (j = 1, \ldots, h),
\]

$N^\alpha \leq l_1 < \ldots < l_h < tN - N^\alpha$. Continuing as in [1], by using the Frechet-Shohat theorem, we obtain Theorem 1.

4. Proofs of Theorems 2 and 3

These can be done by the method used in [9].

5. Proof of Theorem 4

Let
\[
\pi_k(x) = \# \{ n \leq x \mid n \in \mathcal{N}, \ \omega(n) = k \}
\]
and
\[
\mathcal{H}_{x, k} = \{ (m, p), m \in \mathcal{N}, p \in \mathcal{P}, \ \omega(m) = k - 1, \ p > e^{(\log x)^{x^*}}, \ m > x^{x^*}, \ mp \leq x \},
\]
where $e_x^* \to 0 \ (x \to \infty)$. Since
\[
\Sigma_1 := \sum_{m \leq x^{x^*}} \sum_{p \leq x^*} \frac{1}{m} \ll \frac{x}{\log x} \sum_{m \leq x^{x^*}} \frac{1}{m} \ll \sum_{m \leq x^{x^*}} \frac{x}{\log x} (x^*)^{-1} (k - 1)!
\]
we obtain from the left hand side of (2.3) that the right hand side of (5.1) is at most $\alpha_k(1)k\pi_k(x)$ uniformly for $2 \leq k \ll \frac{2x}{x^*}$. Furthermore, from (2.3) we
deduce that

\[ \Sigma_2 := \sum_{p \leq e^{(\log x)^\epsilon x}} \sum_{m \leq \frac{x}{p}, \omega(m) = k-1} \pi_{k-1} \left( \frac{x}{p} \right) \ll \]

\[ \ll \frac{x}{\log x} \frac{T^{k-2}(x)}{(k-2)!} \sum_{p \leq e^{(\log x)^\epsilon x}} \frac{1}{p} \ll \epsilon x k \pi_k(x). \]

Thus, by the right hand side of (2.3),

\[ \#H_{x,k} = k \pi_k(x) + \Sigma_1 + \Sigma_2 + \mathcal{O}((k-1)\pi_{k-1}(x)) = \]

\[ = k \pi_k(x) + o_x(1)k \pi_k(x). \]

Let \( H_k(x) = \#H_{x,k}. \) Let \((1 \leq) l_1 < \ldots < l_h \leq tN, b_1, \ldots, b_h \in E) \text{ and} \]

\[ H_k \left( x \mid l_1, \ldots, l_h \right) = \# \{m, p \in \mathcal{H}_{x,k}, \epsilon_{ij}(P(mp)) = b_j, j = 1, \ldots, h\}. \]

In the same way as we have seen by (3.1)

\[ \max_{N^\alpha \leq l_1 \leq \ldots \leq l_h \leq tN - N^\alpha} \left| q^h H_k \left( x \mid l_1, \ldots, l_h \right) - H_k(x) \right| \leq c(h, \lambda) H_k(x) N^{-\lambda} \]

holds for every fixed \( h, \) every \( \alpha > 0, \) and every \( \lambda > 0 \) uniformly for \( 2 \leq k \ll \frac{2x}{x^3}. \)

Arguing as in [5], the proof is finished.

References

[1] Germán, L. and Káta, I., Distribution of the values of \( q \)-additive functions on some multiplicative semigroups (submitted)


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