DISTRIBUTION OF THE VALUES OF q–ADDITIVE FUNCTIONS ON SOME MULTIPLICATIVE SEMIGROUPS II.

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Dedicated to Dr. Bui Minh Phong on his sixtieth birthday

Communicated by K.-H. Indlekofer

(Received November 30, 2012)

Abstract. In [1] we investigated the distribution of the values of q-additive functions defined on multiplicative semigroups which are generated by an infinite sequence of primes satisfying Wirsing's condition. In this work we extend our investigations started in [1] to polynomial sequences of such semigroups and its subsets which contain integers with a given number of prime divisors.

1. Introduction

1.1.

The project is supported by the European Union and co-financed by the European Social Fund (grant agreement TAMOP 4.2.1/B/09/1/KMR/2010/0003) and the second author is partly supported by the Hungarian and Vietnamese TET (grant agreement no. TET 10-1-2011-0645).

Mathematics Subject Classification: 11L07, 11A63

$\mathbb{N}, \mathbb{R}, \mathbb{C}$ are the sets of natural, real, complex numbers, respectively. $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $e(x) := e^{2\pi i x}$; $\omega(n)$ = number of distinct prime divisor of n; $\Omega(n)$ = number of prime power divisors of n. Let $\{x\}$ = fractional part of n, $||x|| = \min(\{x\}, 1-\{x\})$. For the sake of brevity let $x_1 = \log x$, $x_2 = \log x_1$, and in general, let $x_{k+1} = \log x_k$ $(k = 1, 2, \ldots)$. Let γ be the Euler's constant, Γ be the gamma function and

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du.$$

1.2.

Let $q \in \mathbb{N}$, $q \geq 2$ be fixed, $E = \{0, 1, \ldots, q-1\}$. The q-ary expansion of $n \in \mathbb{N}_0$ is defined by

$$n = \sum_{j=0}^{\infty} a_j(n) q^j, \quad a_j(n) \in E. \tag{1.1}$$

A function $f : \mathbb{N}_0 \to \mathbb{R}$ is said to be q-additive, if $f(0) = 0$ and

$$f(n) = \sum_{j=0}^{\infty} f(a_j(n) q^j), \quad a_j(n) \in E. \tag{1.2}$$

Let $\mathcal{A}_q$ be the set of q-additive functions. Let $N (= N_x) = \left[\frac{\log x}{\log q}\right]$,

$$m_k = \frac{1}{q} \sum_{b \in E} f(bq^k), \quad \sigma_k^2 = \frac{1}{q} \sum_{b \in E} f^2(bq^k) - m_k^2, \tag{1.3}$$

$$M(x) = \sum_{k=0}^{N} m_k, \quad D^2(x) = \sum_{k=0}^{N} \sigma_k^2. \tag{1.4}$$

1.3.

Let

$$\nu_x(n) := \frac{f(n) - M(x)}{D(x)}. \tag{1.5}$$

In our recent paper [1] we proved the following

Theorem A. *Let $\mathcal{P}$ be an infinite sequence of primes, satisfying*

$$\pi_{\mathcal{P}}(x) := \#\{p \le x \mid p \in \mathcal{P}\} = (\tau + o(1))\frac{x}{\log x} \quad (x \to \infty), \tag{1.6}$$

where $\tau > 0$ is a constant. Let $\mathcal{N}$ be the multiplicative semigroup generated by the elements of $\mathcal{P}$,

$$(N_{\mathcal{P}}(x) =)N(x) := \#\{n \le x,\ n \in \mathcal{N}\}.$$

Let $f \in \mathcal{A}_q$, $f(bq^j) = \mathcal{O}(1)$ as $b \in E$, $j = 0, 1, 2 \ldots$. Assume that $D(x)/\log^{\lambda} x \to \infty$ as x tends to infinity for some $\lambda > 0$. Let

$$F_x(y) := \frac{1}{N(x)}\#\{\nu_x(n) < y,\ n \le x,\ n \in \mathcal{N}\}. \tag{1.7}$$

Then

$$\lim_{x\to\infty} F_x(y) = \Phi(y). \tag{1.8}$$

The proof is based on a theorem of Davenport for trigonometric sums (see [2], Lemma 1) and on the method developed in [3].

We observed that by using a theorem of L.K. Hua ([3], see Lemma 6.3), by using the method used by N.L. Bassily and I. Kátai [5] one can prove

Theorem 1. *Let $f \in \mathcal{A}_q$, $f(bq^j) = \mathcal{O}(1)$ $(b \in E,\ j = 0, 1, 2, \ldots)$, $D(x)/\log^{\delta} x \to \infty$ as x tends to infinity with a suitable $\delta > 0$. Assume that $\mathcal{P}$ satisfies the condition (1.6). Let $P \in \mathbb{Z}[x]$ be a polynomial of degree t, with positive leading coefficient. Let*

$$G_x(y) := \frac{1}{N(x)}\#\{n \le x,\ n \in \mathcal{N},\ \nu_{x^t}(P(n)) < y\}. \tag{1.9}$$

Then

$$\lim_{x\to\infty} G_x(y) = \Phi(y) \tag{1.10}$$

holds for every y.

1.4.

Let $P \in \mathbb{Z}[x]$ be a polynomial of degree t taking positive integer values on $\mathbb{N}$. Let q, E be as in 1.2. If $n \in \mathbb{N}$, $n = \epsilon_0(n) + \epsilon_1(n)q + \cdots + \epsilon_{r-1}(n)q^{r-1}$, then write $\overline{n} = \epsilon_0(n) \cdots \epsilon_{r-1}(n)$ $(\in E^r)$, $\epsilon_{r-1} \neq 0$. Let $\mathcal{P}, \mathcal{N}$ be as in Theorem A. Let $n_1 < n_2 < \dots$ be the whole sequence of the integers in $\mathcal{N}$, and let

$$\eta = 0, \overline{P(n_1)}\ \overline{P(n_2)} \dots \tag{1.11}$$

where the right hand side of (1.11) is the q-ary expansion of η.

Theorem 2. *We have that* $\{q^m \eta\}$ $(m = 1, 2, \dots)$ *is a sequence uniformly distributed mod 1.*

This assertion can be derived from Theorem 3, formulated in 1.5.

1.5.

Let $\mathcal{P}, \mathcal{N}, P$ as earlier. Let $\beta = b_0 b_1 \dots b_{k-1}$ be a typical element of E^k. We write $\Phi_1^{(k)}(n) = \epsilon_j(n) \dots \epsilon_{j+k-1}(n)$. Let $F_k : E_1^k \to \mathbb{R}$ be a function such that $F(0, \dots, 0) = 0$. Let

$$\alpha_n := \sum_{j=0}^{\infty} F_k(\Phi_j^k(\overline{P(n)})), \quad \kappa_1 := \sum_{j=0}^{\infty} F_k(\Phi_j^k(n)),$$

$$M := q^{-k} \sum_{b_1 \dots b_k \in E^k} F_k(b_1 \dots b_k),$$

$$\sigma_h^2 = q^{-(k+h)} \sum_{b_0 \dots b_{k+h-1} \in E^{k+h}} (F_k(b_0 \dots b_{k-1}) - M)(F_k(b_h \dots b_{h+k-1}) - M)$$

for $h = 0, 1, \dots, k-1$. Let

$$\sigma^2 = \sigma_0^2 + \sum_{h=1}^{k-1} \sigma_h^2.$$

Theorem 3. *Assume that* $\sigma \neq 0$. *Then*

$$\lim_{x \to \infty} \# \left\{ n \le x,\ n \in \mathcal{N} \ \middle|\ \frac{\alpha_n - MNr}{\sigma \sqrt{Nr}} < y \right\} = \Phi(y)$$

holds for every $y \in \mathbb{R}$.

We can prove also

Theorem 4. *Let* $\mathcal{P}, \mathcal{N}, P, f$ *be as in Theorem 1. Let*

$$G_{x,k}(y) := \frac{1}{\pi_k(x)} \#\{n \le x,\ n \in \mathcal{N},\ \omega(n) = k,\ \nu_{x^t}(P(n)) < y\}.$$

Then, if $k_0(x) \to \infty$, then

$$\sup_y \sup_{k_0(x) \le k \le o_x(1)\frac{x_2}{x_3}} |G_{x,k} - \Phi(y)| = 0.$$

Remark. Unfortunately, we cannot prove that

$$\lim_{x\to\infty} G_{x,1}(y) = \Phi(y).$$

2. Auxiliary results

2.1.

The Erdős-Turán inequality ([6]):

The discrepancy D_M of the real numbers $x_1, \dots, x_M$ (mod 1) is defined by

$$\sup \left| \frac{1}{M} \sum_{\substack{n=1 \\ \{x_n\} \in [\alpha,\beta)}}^{M} 1 - (\beta - \alpha) \right| \tag{2.1}$$

where the supremum is taken for all intervals $[\alpha, \beta) \subset [0, 1)$.

Let $\psi_m := \sum_{l=1}^{M} e(mx_l)$. We have

$$D_M \le c \left(\sum_{0<h\le K} \frac{|\Psi_h|}{h} + \frac{M}{K} \right) \tag{2.2}$$

for any positive integer K. c is an absolute constant.

2.2.

Lemma 6.3 of L.K. Hua ([4]):

Let l be a positive integer ($\le x_1^{\sigma_3}$), and

$$\Omega = \sum_d \sum_m e(f(ldm)),$$
$$f(z) = \frac{h}{Q} z^t + \alpha_1 z^{t-1} + \dots + \alpha_t,$$

where $(h, Q) = 1$, all α being real, and $x_1^{\sigma} < Q < x^t \cdot x_1^{-\sigma}$. The index d in Ω runs through a set of positive integers satisfying the conditions

$$D < d \le D', \quad 1 < D < \frac{x}{l}, \quad D' \le 2D.$$

Further, for a fixed d, the index m runs through a set of positive integers satisfying the inequality

$$P'/d < m \le \frac{x}{Dl},$$

where P' is a positive number. Hence, for $x_1^{\sigma_5} < D < x \cdot x_1^{-\sigma_6}$, subject to the conditions

$$\sigma \ge 2t\sigma_3 + 2^{2t+1}\sigma_6 + 2^{3(2t-1)}$$

we have

$$\Omega \ll \frac{x}{l} x_1^{-\sigma_6}.$$

2.3.

Theorem of E. Wirsing ([7]):

Let F be a multiplicative function, satisfying the conditions: $F(n) \ge 0$ ($n \in \mathbb{N}$); $F(p^\alpha) \le c_1 c_2^\alpha$, $c_2 < 2$ for every prime p and $\alpha = 2, 3 \dots$. Assume that

$$\sum_{p \le x} F(p) = (\tau + o(1)) \frac{x}{\log x} \quad (x \to \infty),$$

where $\tau > 0$ is a constant. Then, for $x \to \infty$,

$$\sum_{n \le x} F(n) = \left(\frac{e^{-\gamma\tau}}{\Gamma(\tau)} + o_x(1) \right) \frac{x}{\log x} \prod_{p \le x} \left(1 + \frac{F(p)}{p} + \frac{F(p^2)}{p^2} + \cdots \right).$$

Analyzing the proof, one can see easily that the following version of the theorem of E. Wirsing is true.

Lemma 1. *Let F_λ be a family of multiplicative functions satisfying the following conditions: $F_\lambda(n) \geq 0$ $(n \in \mathbb{N})$; $F_\lambda(p^\alpha) \leq c_1 c_2^\alpha$, $c_2 < 2$ for every prime p and $\alpha = 2, 3 \ldots$. Assume that*

$$\Big| \sum_{p \leq x} F_\lambda(p) - \tau_\lambda \frac{x}{\log x} \Big| \leq \epsilon(x) \frac{x}{\log x},$$

where $0 < c_3 < \tau_\lambda$, c_3 is a suitable constant, $\epsilon(x) \to 0$ as x tends to infinity. Then there exists a function $\epsilon_1(x) \to 0$ $(x \to \infty)$ such that

$$\Big| \sum_{n \leq x} F_\lambda(n) - \frac{e^{-\gamma \tau_\lambda}}{\Gamma(\tau_\lambda)} \frac{x}{\log x} \prod_{p \leq x} \left(1 + \frac{F_\lambda(p)}{p} + \frac{F_\lambda(p^2)}{p^2} + \cdots \right) \Big| \leq$$
$$\leq \epsilon_1(x) \frac{x}{\log x} \prod_{p \leq x} \left(1 + \frac{F_\lambda(p)}{p} + \frac{F_\lambda(p^2)}{p^2} + \cdots \right).$$

Let $\mathcal{P}$, $\mathcal{N}$ be as defined in Theorem A. Defining the multiplicative function F on prime powers p^α by

$$F(p^\alpha) = \begin{cases} 1, & \text{if } p \in \mathcal{P}, \\ 0, & \text{if } p \notin \mathcal{P}, \end{cases}$$

from Wirsing's theorem we obtain that

$$N_{\mathcal{P}}(x) = \left(\frac{e^{-\gamma\tau}}{\Gamma(\tau)} + o(1) \right) \frac{x}{\log x} \prod_{\substack{p \leq x \\ p \in \mathcal{P}}} \frac{1}{1 - 1/p}.$$

2.4.

Lemma 2. *Let $0 < \Delta < \frac{1}{2q}$, $\chi_0(x) = \sum c_m e(mx)$ be a $\pmod 1$ periodic function such that $0 \leq \chi_0(x) \leq 1$,*

$$\chi_0(x) = \begin{cases} 1 & \textit{if} \quad \Delta < \{x\} < \frac{1}{q} - \Delta, \\ 0 & \textit{if} \quad \frac{1}{q} + \Delta < \{x\} < 1 - \Delta, \end{cases}$$

$c_0 = \frac{1}{q}$, $c_{jq} = 0$ when $j \neq 0$,

$$|c_m| \leq \min\left(\frac{1}{q}, \frac{1}{\pi |m|}, \frac{1}{\Delta \pi^2 m^2} \right).$$

Let $\chi_b(x) = \chi_0\left(x - \frac{b}{q}\right) = \sum c_m^{(b)} e(mx)$. *Then* $c_m^{(b)} = c_m e\left(-\frac{mb}{q}\right)$, *thus* $|c_m^{(b)}| = |c_m|$. *See in [5].*

2.5.

Let $\mathcal{P}$, $\mathcal{N}$ be as earlier,

$$\pi_k(x) = \#\{n \le x \mid n \in \mathcal{N},\ \omega(n) = k\},\ N_k(x) = \#\{n \le x \mid\ n \in \mathcal{N},\ \Omega(n) = k\}.$$

Let

$$T(x) := \sum_{\substack{p^\nu \le x \\ p \in \mathcal{P}}} \frac{1}{p^\nu}.$$

Lemma 3. *There is a function* $\epsilon(x) \to 0$ $(x \to \infty)$ *and positive constants* c_1, c_2 *such that*

$$\frac{c_2(\tau - \epsilon(x))x}{\log x} \frac{T\left(x^{\frac{1}{2(k-1)}}\right)^{k-1}}{(k-1)!} - (\log x)\sqrt{x} \le \pi_k(x) \le \le \frac{c_1 x}{\log x} \frac{T(x)^{k-1}}{(k-1)!} \tag{2.3}$$

holds for every k, *and*

$$N_k(x) \le \frac{c_3 x}{\log x} \frac{T(x)^{k-1}}{(k-1)!} \tag{2.4}$$

holds for $1 \le k \le (1-\delta)p_0 T(x)$, *where* p_0 *is the smallest prime in* $\mathcal{P}$, δ *is an arbitrary constant,* $0 < \delta < 1$, *and* $c_3 = c_3(\delta)$ *is a suitable constant.*

Proof of Lemma 3. We have

$$\sum_{\substack{n \le x \\ n \in \mathcal{P}_k}} \log n \le \sum_{\substack{p^\nu m \le x \\ m \in \mathcal{P}_{k-1}}} \log p^\nu = \sum_{\substack{m \le x \\ m \in \mathcal{P}_{k-1}}} \sum_{p^\nu \le \frac{x}{m}} \log p^\nu \le$$

$$\le 2x \sum_{\substack{m \le x \\ m \in \mathcal{P}_{k-1}}} \frac{1}{m} \le \frac{2xT(x)^{k-1}}{(k-1)!}.$$

Thus

$$(\pi_k(x) - \pi_k(\sqrt{x}))\frac{1}{2}\log x \le 2x\frac{T(x)^{k-1}}{(k-1)!},$$

$$\pi_k(x) \le \pi_k(\sqrt{x}) + \frac{4x}{\log x}\frac{T(x)^{k-1}}{(k-1)!}.$$

Iterating this, we obtain that the right hand side of (2.3) is true. Furthermore,

$$\pi_k(x)\log x \ge \sum_{\substack{p^\nu m \le x \\ m\in\mathcal{P}_{k-1} \\ p\in\mathcal{P},\ p\nmid m \\ m\le\sqrt{x}}} \log p^\nu \ge \sum_{\substack{m\le\sqrt{x} \\ m\in\mathcal{P}_{k-1}}} \left\{ \sum_{\substack{p^\nu \le \frac{x}{m} \\ p\in\mathcal{P}}} \log p^\nu - \sum_{p\mid m} \log p^\nu \right\} \ge$$

$$\ge (\tau - \epsilon(x)) \sum_{\substack{m\le\sqrt{x} \\ m\in\mathcal{P}_{k-1}}} \frac{x}{m} - (\log x) \sum_{\substack{m\le\sqrt{x} \\ m\in\mathcal{P}_{k-1}}} \sum_{p\mid m} 1,$$

and so

$$\pi_k(x) \ge (\tau - \epsilon(x)) x \frac{T\left(x^{\frac{1}{2(k-1)}}\right)^{k-1}}{(k-1)!} - \sqrt{x}\log x.$$

To prove (2.4), write $n \in \mathcal{N}_k$ in the form $n = Km$, where K is the squareful part and m is the squarefree part of n.

The size of those $n \le x$ for which $K > x^{1/2}$ is

$$\le \sum_{K>\sqrt{x}} \frac{x}{K} \le cx^{3/4}.$$

Thus,

$$N_k(x) \le \sum_{K\le\sqrt{x}} \pi_{k-\Omega(K)}\left(\frac{x}{K}\right) + cx^{3/4}.$$

From inequality (2.3) we have

$$N_k(x) \le \frac{c_1 x}{\log\sqrt{x}} \sum_{K\le\sqrt{x}} \frac{T(x)^{k-\Omega(K)-1}}{K(k-\Omega(K)-1)!} + cx^{3/4}.$$

Furthermore,

$$\sum_{K<\sqrt{x}} \frac{T(x)^{k-\Omega(K)-1}}{K(k-\Omega(K)-1)!} \le \frac{T(x)^{k-1}}{(k-1)!} \sum_{K\le\sqrt{x}} \left(\frac{k}{T(x)}\right)^{\Omega(K)} \frac{1}{K}.$$

Since $\frac{k}{T(x)} \le (1-\delta)p_0$,

$$\sum_{K\le\sqrt{x}} \left(\frac{k}{T(x)}\right)^{\Omega(K)} \frac{1}{K} \le \prod_{p\in\mathcal{P}} \left(1+\left(\frac{k}{T(x)}\right)\frac{1}{p^2}\frac{1}{1-\left(\frac{k}{T(x)}\right)\frac{1}{p}}\right).$$

Since $cx^{3/4}$ is clearly smaller than $c\frac{x}{\log x}\frac{T(x)^{k-1}}{(k-1)!}$, our inequality holds.

3. Proof of Theorem 1

Let $y \in \mathbb{R}$ be fixed. Let $n_1 < \ldots < n_s$ ($\le x$) be the set all of the integers in $\mathcal{N}$ up to x, for which $\nu_{x^t}(P(n)) < y$. Then $s = G_x(y)\cdot N(x)$. Let $\mathcal{H}_x = \mathcal{H} = \{\{m,p\},\ p\in P,\ m\in\mathcal{N},\ m > x^{\epsilon_x},\ p > e^{(\log x)^{\epsilon_x}},\ mp\le x\}$. Here we assume that $\epsilon_x \to 0$ ($x\to\infty$) (slowly).

Let $R_x = \sum_{p\le x} 1/p$. Let Z be the number of those $\{m,p\}\in\mathcal{H}_x$ for which $\nu_{x^t}(P(mp)) < y$. Repeating the argument, used in [1], we obtain that

$$\frac{1}{N(x)}\left|\frac{Z}{R_x} - s\right| \to 0 \quad (x\to\infty).$$

Let $H(x) = \#\mathcal{H}_x$. Let $(1\le)\ l_1 < \ldots < l_h \le tN$, $b_1,\ldots,b_h \in E$ and

$$H\left(x \;\middle|\; \begin{matrix} l_1,\ldots,l_h \\ b_1,\ldots,b_h\end{matrix}\right) = \#\{\{m,p\}\in\mathcal{H}_x,\ \epsilon_{l_j}(P(mp)) = b_j,\ j=1,\ldots,h\}.$$

By using the method developed in [3, 5, 1] we can prove that

$$\max_{\substack{N^\alpha\le l_1<\ldots<l_h<tN-N^\alpha\\ b_1,\ldots,b_h\in E}} \left|q^h H\left(x \;\middle|\; \begin{matrix} l_1,\ldots,l_h \\ b_1,\ldots,b_h\end{matrix}\right) - H(x)\right| \le c(h,\lambda)H(x)N^{-\lambda} \tag{3.1}$$

holds for every fixed h, every $\alpha > 0$, and every $\lambda > 0$.

By using the theorem of L.K. Hua ([4]) we can obtain that

$$\sum_{\{m,p\}\in\mathcal{H}_x} e\left(\frac{A_M}{H_M}P(mp)\right) \ll H(x)\log^{-B} x$$

holds for every fixed B, where

$$\frac{A_M}{H_M} = \frac{m_h}{q^{l_h+1}} + \cdots + \frac{m_1}{q^{l_1+1}}, \qquad q \nmid m_j \quad (j = 1, \ldots, h),$$

$N^\alpha \le l_1 < \ldots < l_h < tN - N^\alpha$. Continuing as in [1], by using the Frechet-Shohat theorem, we obtain Theorem 1.

4. Proofs of Theorems 2 and 3

These can be done by the method used in [9].

5. Proof of Theorem 4

Let

$$\pi_k(x) = \#\{n \le x \mid n \in \mathcal{N},\ \omega(n) = k\}$$

and

$$\mathcal{H}_{x,k} =$$

$$= \{\{m, p\}, m \in N,\ p \in \mathcal{P},\ \omega(m) = k-1,\ p > e^{(\log x)^{\epsilon_x}},\ m > x^{\epsilon'_x},\ mp \le x\},$$

where $\epsilon'_x \to 0$ $(x \to \infty)$. Since

$$\Sigma_1 := \sum_{\substack{m \le x^{\epsilon'_x} \\ m \in \mathcal{N},\ \omega(m)=k-1}} \sum_{\substack{p \le \frac{x}{m} \\ p \in \mathcal{P}}} 1 \ll \frac{x}{\log x} \sum_{\substack{m \le x^{\epsilon'_x} \\ m \in \mathcal{N},\ \omega(m)=k-1}} \frac{1}{m} \ll \tag{5.1}$$

$$\ll \frac{x}{\log x} \frac{T^{k-1}(x^{\epsilon'_x})}{(k-1)!},$$

we obtain from the left hand side of (2.3) that the right hand side of (5.1) is at most $o_x(1) k \pi_k(x)$ uniformly for $2 \le k \ll \frac{x_2}{x_3}$. Furthermore, from (2.3) we deduce that

$$\Sigma_2 := \sum_{\substack{p\le e^{(\log x)^{\epsilon_x}}\\ p\in\mathcal{P}}} \sum_{\substack{m\le\frac{x}{p}\\ m\in\mathcal{N},\ \omega(m)=k-1}} 1 \ll \sum_{\substack{p\le e^{(\log x)^{\epsilon_x}}\\ p\in\mathcal{P}}} \pi_{k-1}\left(\frac{x}{p}\right) \ll$$
$$\ll \frac{x}{\log x}\frac{T^{k-2}(x)}{(k-2)!} \sum_{\substack{p\le e^{(\log x)^{\epsilon_x}}\\ p\in\mathcal{P}}} \frac{1}{p} \ll$$
$$\ll \epsilon_x k\pi_k(x).$$

Thus, by the right hand side of (2.3),

$$\#\mathcal{H}_{x,k} = k\pi_k(x) + \Sigma_1 + \Sigma_2 + \mathcal{O}((k-1)\pi_{k-1}(x)) =$$
$$= k\pi_k(x) + o_x(1)k\pi_k(x).$$

Let $H_k(x) = \#\mathcal{H}_{x,k}$. Let $(1\le)\ l_1 < \ldots < l_h \le tN$, $b_1,\ldots,b_h\in E$ and

$$H_k\left(x \;\middle|\; \begin{matrix} l_1,\ldots,l_h\\ b_1,\ldots,b_h\end{matrix}\right) = \#\left\{\{m,p\}\in\mathcal{H}_{x,k},\ \epsilon_{l_j}(P(mp)) = b_j,\ j=1,\ldots,h\right\}.$$

In the same way as we have seen by (3.1)

$$\max_{\substack{N^\alpha\le l_1<\ldots<l_h<tN-N^\alpha\\ b_1,\ldots,b_h\in E}} \left| q^h H_k\left(x \;\middle|\; \begin{matrix} l_1,\ldots,l_h\\ b_1,\ldots,b_h\end{matrix}\right) - H_k(x)\right| \le c(h,\lambda)H_k(x)N^{-\lambda}$$

holds for every fixed h, every $\alpha > 0$, and every $\lambda > 0$ uniformly for $2 \le k \ll \frac{x_2}{x_3}$. Arguing as in [5], the proof is finished.

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