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DISTRIBUTION OF THE VALUES OF q–ADDITIVE FUNCTIONS ON SOME MULTIPLICATIVE SEMIGROUPS

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Dedicated to Dr. Bui Minh Phong on his sixtieth birthday

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Abstract. Let $\mathcal{P}$ be an infinite subset of primes,

$$\#\{p \le x \mid p \in \mathcal{P}\} = (\tau + o(1))\frac{x}{\log x} \quad (x \to \infty),$$

$\mathcal{N}$ be the multiplicative semigroup generated by $\mathcal{P}$. Distribution of the values of q-additive functions defined on $\mathcal{N}$ is investigated.

1. Introduction

1.1. Let $\mathbb{N}, \mathbb{R}, \mathbb{C}$ be the set of natural, real, complex numbers respectively, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $e(x) = e^{2\pi i x}$, $\omega(n)$ = number of distinct prime divisors of

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n, $\Omega(n)$ = number of prime power divisors of n. If x is a positive real number then let $x_1 = \log x$, $x_k = \log x_{k-1}$, $k = 2, 3, \ldots$. Let $\{x\}$ = fractional part of x, $||x|| = \min(\{x\}, 1 - \{x\})$. Let $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-u^2/2} du$.

1.2. Let $q \geq 2$ be a fixed integer, $E = \{0, 1, \ldots, q-1\}$ be the set of digits. Then every $n \in \mathbb{N}_0$ has a unique (q-ary) expansion, defined by

$$n = \sum_{j=1}^{\infty} a_j(n) q^j, \quad a_j(n) \in E. \tag{1.1}$$

The right hand side of (1.1) is clearly a finite sum, since $a_j(n) = 0$ if $q^j > n$. A function $f : \mathbb{N}_0 \to \mathbb{R}$ is said to be q-additive, if $f(0) = 0$ and

$$f(n) = \sum_{j=0}^{\infty} f(a_j(n) q^j) \tag{1.2}$$

holds for every $n \in \mathbb{N}_0$. The whole set of q-additive functions will be denoted by $\mathcal{H}$.

1.3. Let

$$N = N_x = \left[\frac{\log x}{\log q} \right], \tag{1.3}$$

$$m_k = \frac{1}{q} \sum_{b \in E} f(bq^k), \quad \sigma_k^2 = \frac{1}{q} \sum_{b \in E} f^2(bq^k) - m_k^2, \tag{1.4}$$

$$M(x) = \sum_{k=0}^{N} m_k, \quad D^2(x) = \sum_{k=0}^{N} \sigma_k^2. \tag{1.5}$$

1.4. Let $\mathcal{B} = \mathcal{B}_x$ be a set of positive integers up to x. The multiple occurrence of some numbers is allowed. Furthermore, let $B(x)$ be the number of elements in $\mathcal{B}$. For an arbitrary sequence of integers $(0 \leq)\ l_1 < \ldots < l_h$ and $b_1, \ldots, b_h \in E$, let

$$B\left(x \;\middle|\; \begin{matrix} l_1, \ldots, l_h \\ b_1, \ldots, b_h \end{matrix} \right) = \#\{n \leq x \mid n \in \mathcal{B},\ a_{l_j}(n) = b_j,\ j = 1, \ldots, h\}. \tag{1.6}$$

1.5. Let

$$\nu(n) := \frac{f(n) - M(x)}{D(x)}, \tag{1.7}$$

$$F_x(y) := \frac{1}{B(x)} \#\{n \in \mathcal{B},\ \nu(n) \le y\}. \tag{1.8}$$

Definition 1. We say that $\mathcal{B} = \mathcal{B}_x$ is a sequence of q-ary smooth sets of type α if $B(x) \gg \frac{x}{\log x}$, and

$$\sup_{\substack{N^\alpha \le l_1 < \dots < l_h < N - N^\alpha \\ b_1, \dots, b_h \in E}} \left| q^h B\left(x \;\middle|\; \begin{matrix} l_1, \dots, l_h \\ b_1, \dots, b_h \end{matrix}\right) - B(x) \right| \le c(h, \lambda) B(x) N^{-\lambda} \tag{1.9}$$

holds for every fixed $\lambda > 0$, $x \ge 2$.

Theorem 1. *Let $f \in \mathcal{A}_q, f(bq^j) = \mathcal{O}(1)$ as $b \in E$, $j = 0, 1, \dots$ Assume that $\frac{D(x)}{\log^\delta x} \to \infty$ as x tends to infinity is satisfied for some $\delta > 0$. Let $\mathcal{B}_x$ be a q-ary smooth sequence of type $\alpha < \delta/2$. Then*

$$\lim_{x \to \infty} F_x(y) = \Phi(y)$$

holds for every y. Here

$$\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-u^2/2} du.$$

Proof. Let $n \le x$

$$f_\alpha(n) := \sum_{N^\alpha \le j \le N - N^\alpha} f(a_j(n) q^j).$$

Since $f(bq^j)$ is bounded,

$$|f_\alpha(n) - f(n)| \le c N^\alpha$$

holds. Let

$$M_\alpha(x) = \sum_{N^\alpha \le j \le N - N^\alpha} m_j, \quad D_\alpha^2(x) = \sum_{N^\alpha \le j \le N - N^\alpha} \sigma_j^2.$$

We have $|M(x) - M_\alpha(x)| \le cN^\alpha$, $|D_\alpha^2(x) - D^2(x)| \le cN^\alpha$. Let

$$\nu_\alpha(n) = \frac{f_\alpha(n) - M_\alpha(x)}{D_\alpha(x)}.$$

We already defined $\nu(n)$ in (1.7). From the assumption we obtain that

$$\max_{n \le x} |\nu_\alpha(n) - \nu(n)| \to 0$$

as $x \to \infty$. From the assumption (1.9) we deduce easily that

$$\frac{1}{B(x)} \sum_{\substack{n \le x \\ n \in \mathcal{B}_x}} \nu_\alpha(n)^k - \frac{1}{x} \sum_{n \le x} \nu_\alpha(n)^k \to 0 \quad \text{as} \quad x \to \infty,$$

and so

$$\frac{1}{B(x)} \sum_{\substack{n \le x \\ n \in \mathcal{B}_x}} \nu(n)^k - \frac{1}{x} \sum_{n \le x} \nu(n)^k \to 0 \quad \text{as} \quad x \to \infty \tag{1.10}$$

for each $k \in \mathbb{N}_0$. One can prove easily that for $k \in \mathbb{N}_0$

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} \nu(n)^k = \int_{-\infty}^{\infty} x^k d\Phi.$$

(1.10) implies that

$$\lim_{x \to \infty} \frac{1}{B(x)} \sum_{\substack{n \le x \\ n \in \mathcal{B}}} \nu(n)^k = \int_{-\infty}^{\infty} x^k d\Phi$$

holds for every k. Therefore, our theorem directly follows from the Frechet-Shohat theorem. A more detailed argument can be found in [1].

2. Some auxiliary results

2.1.

Lemma 1 (Theorem of Davenport [2]). *Let x be a positive integer, $1 < U_0 < U_1 < x$, $1 \le Q \le x$, $(a,Q) = 1$. Let $\Theta_1(n,x)$, $\Theta_2(r,x)$ be arbitrary functions, each of which is absolutely bounded. Then*

$$\sum_{U_0<n\le U_1} \Theta_1(n,x) \sum_{1\le r\le x/n} \Theta_2(r,n) e\left(\frac{axr}{Q}\right) =$$

$$= \mathcal{O}\left(x \log^2 x \sqrt{\frac{1}{U_0} + \frac{U_1}{x} + \frac{1}{Q} + \frac{Q}{x}} \right).$$

2.2.

Lemma 2. *Let $0 < \Delta < \frac{1}{2q}$, $\chi_0(x) = \sum_{m=-\infty}^{\infty} c_m e(mx)$ be a mod 1 periodic function such that $0 \le \chi_0(x) \le 1$,*

$$\chi_0(x) = \begin{cases} 1, & \textit{if } \Delta < \{x\} < \frac{1}{q} - \Delta, \\ 0, & \textit{if } \frac{1}{q} + \Delta < \{x\} < 1 - \Delta, \end{cases}$$

$c_0 = \frac{1}{q}$, $c_{jq} = 0$ *when* $j = \pm 1, \pm 2, \ldots,$

$$|c_m| \le \min\left(\frac{1}{q},\ \frac{1}{\pi|m|},\ \frac{1}{\Delta \pi^2 m^2}\right).$$

Let $\chi_b(x) = \chi_0(x - \frac{b}{q}) = \sum c_m^{(b)} e(mx)$. *Then* $\chi_m^{(b)} = c_m e(-\frac{mb}{q})$, *thus* $|c_m^{(b)}| = |c_m|$.

This lemma is proved in [3].

2.3.

The Erdős-Turán inequality for the discrepancy of sequences mod 1

The discrepancy D_M of the real numbers $x_1, \ldots, x_M$ mod 1 is defined by

$$\sup \left| \frac{1}{M} \#\{n \le M \mid \{x_n\} \in [\alpha, \beta)\} - (\beta - \alpha) \right| \tag{2.1}$$

where the supremum is taken for all intervals $[\alpha, \beta) \subset [0, 1]$.

Lemma 3 ([4]). *Let* $\psi_m := \sum_{e=1}^{M} e(mx_l)$. *We have*

$$D_M \le c\left(\sum_{0<h\le K}\frac{|\psi_h|}{h}+\frac{M}{K}\right) \tag{2.2}$$

for any positive integer K. c *is an absolute constant.*

2.4.

The theorem of E. Wirsing

Lemma 4 ([5]). *Let* F *be a multiplicative function satisfying the following conditions:* $F(n) \ge 0$ $(n \in \mathbb{N})$; $F(p^\alpha) \le c_1 c_2^\alpha$, $c_2 < 2$ *for every prime* p *and* $\alpha = 2, 3, \ldots$. *Assume that*

$$\sum_{p\le x} F(p) = (\tau + o(1))\frac{x}{\log x} \quad (x \to \infty) \tag{2.3}$$

where $\tau > 0$ *is a constant. Then, for* $x \to \infty$,

$$\sum_{n\le x} F(n) = \left(\frac{e^{-\gamma\tau}}{\Gamma(\tau)} + o(1)\right)\frac{x}{\log x}\prod_{p\le x}\left(1+\frac{F(p)}{p}+\frac{F(p^2)}{p^2}+\cdots\right). \tag{2.4}$$

Here Γ *is the Euler's gamma function, and* γ *is the Euler's constant.*

Analyzing the proof, one can see that the following variant of Wirsing's theorem remains true.

Lemma 5. *Let* F_λ *be a family of multiplicative functions, satisfying the following conditions:* $F_\lambda(n) \ge 0$ $(n \in \mathbb{N})$; $F_\lambda(p^\alpha) \le c_1 c_2^\alpha$, $c_2 < 2$ *for every prime* p *and* $\alpha = 2, 3, \ldots$.

Let $\epsilon(x) \to 0$ $(x \to \infty)$. *Assume that*

$$\left|\sum_{p\le x} F_\lambda(p) - \tau_\lambda\frac{x}{\log x}\right| \le \epsilon(x)\frac{x}{\log x} \tag{2.5}$$

where $0 < c_3 < \tau_\lambda < c_4$, *with* c_3, c_4 *suitable positive constants. Then there exists a function* $\epsilon_1(x) \to 0$ $(x \to \infty)$ *such that*

$$\left| \sum_{n \le x} F_\lambda(n) - \frac{e^{-\gamma\tau_\lambda}}{\Gamma(\tau)} \frac{x}{\log x} \prod_{p \le x} \left(1 + \frac{F_\lambda(p)}{p} + \frac{F_\lambda(p^2)}{p^2} + \cdots \right) \right| \le \tag{2.6}$$
$$\le \epsilon_1(x) \frac{x}{\log x} \prod_{p \le x} \left(1 + \frac{F_\lambda(p)}{p} + \frac{F_\lambda(p^2)}{p^2} + \cdots \right).$$

2.5.

Let $\mathcal{P}$ be an infinite sequence of primes, $\mathcal{N}$ be the multiplicative semigroup generated by $\mathcal{P}$. Let

$$\pi_{\mathcal{P}}(x) = \#\{p \le x \mid p \in \mathcal{P}\}; \quad N_{\mathcal{P}}(x) = \#\{n \le x \mid n \in \mathcal{N}\}.$$

Assume that

$$\pi_{\mathcal{P}}(x) = \tau \frac{x}{\log x} + o\left(\frac{x}{\log x} \right) \quad (x \to \infty) \tag{2.7}$$

where $0 < \tau \le 1$. Then, from the theorem of Wirsing we obtain that

$$N_{\mathcal{P}}(x) = \left(\frac{e^{-\gamma\tau}}{\Gamma(\tau)} + o(1) \right) \frac{x}{\log x} \prod_{\substack{p \le x \\ p \in \mathcal{P}}} \frac{1}{1 - 1/p} \quad (x \to \infty). \tag{2.8}$$

Let

$$R_x := \sum_{\substack{p \le x \\ p \in \mathcal{P}}} \frac{1}{p}. \tag{2.9}$$

Then

$$R_x = (\tau + o(1)) \log\log x \qquad (x \to \infty). \tag{2.10}$$

Lemma 6. *Let* $\mathcal{P}$ *satisfy the condition (2.7). Then, there is a suitable sequence* $\delta_x \to 0$ $(x \to \infty)$ *such that*

$$\frac{1}{N(x) R_x} \sum_{\substack{|\omega(n) - R_x| > \delta_x R_x \\ n \le x, n \in \mathcal{N}}} \omega(n) \to 0 \quad (x \to \infty). \tag{2.11}$$

Proof. Let F_κ be a family of multiplicative functions, defined on prime powers p^α as follows:

$$F_\kappa(p^\alpha) = \begin{cases} e^\kappa, & \text{if } p \in \mathcal{P}, \\ 0, & \text{if } p \notin \mathcal{P}. \end{cases}$$

First we assume that κ is a small positive, later that it is a small negative number. Since

$$\sum_{p \le x} F_\kappa(p) = (e^\kappa \tau + o(1)) \frac{x}{\log x} \quad (x \to \infty)$$

holds uniformly as κ varies in a bounded interval, furthermore

$$F_\kappa(n)\omega(n) \le 2 \sum_{\substack{mp=n \\ p\in\mathcal{P}, p<\sqrt{x}}} F_\kappa(m) e^\kappa,$$

by Lemma 5 we obtain that

$$\begin{aligned} \sum_{\substack{n\le x \\ n\in\mathcal{N}}} e^{\kappa\omega(n)}\omega(n) \le & 2 \sum_{\substack{p\le\sqrt{x} \\ p\in\mathcal{P}}} e^\kappa \sum_{n\le x/p} e^{\kappa\omega(n)} \le \\ \le & 2e^\kappa \frac{e^{-\gamma e^\kappa \tau}}{\Gamma(e^\kappa\tau)} \frac{x}{\log x} R_x \prod_{\substack{p\le x \\ p\in\mathcal{P}}} \left(1 + \frac{e^\kappa}{p-1}\right) = \\ = & \frac{e^{-\gamma e^\kappa \tau}}{\Gamma(e^\kappa\tau)} \frac{x}{\log x} R_x \exp(e^\kappa R_x + b_x), \end{aligned} \tag{2.12}$$

where b_x is bounded uniformly as $0 \le \kappa \le 1/10$, say. Since

$$\sum_{\substack{\omega(n)>(1+\delta_x)R_x \\ n\le x,\ n\in\mathcal{N}}} \omega(n) \le e^{-\kappa\delta_x R_x} \sum_{\substack{n\le x \\ n\in\mathcal{N}}} e^{\kappa(\omega(n)-R_x)}\omega(n), \tag{2.13}$$

and

$$N(x) = (1+o(1)) \frac{e^{-\gamma\tau}}{\Gamma(\tau)} \frac{x}{\log x} \exp(R_x + \mathcal{O}(1)) \quad (x \to \infty), \tag{2.14}$$

from (2.12), (2.13) we have that

$$\frac{1}{N(x)R_x} \sum_{\substack{\omega(n)>(1+\delta_x)R_x \\ n\le x,\ n\in\mathcal{N}}} \omega(n) \le c \exp((-\kappa\delta_x - \kappa + e^\kappa - 1)R_x). \tag{2.14}$$

c may depend on τ. Choose $\kappa = x_4^{-1}$, $\delta_x = 2\kappa$. We obtain, that (2.14) tends to zero.

Instead of proving that

$$\frac{1}{N(x)R_x} \sum_{\substack{\omega(n)<(1-\delta_x)R_x \\ n\in\mathcal{N},\ n\le x}} \omega(n) \to 0 \quad (x\to\infty) \tag{2.15}$$

we shall show that

$$\frac{1}{N(x)} \#\{n \le x \mid \omega(n) < (1-\delta_x)R_x,\ n \in \mathcal{N}\} \to 0 \quad (x \to \infty).$$

To prove this we choose $F_{-\kappa}$ instead of F_κ, and argue as earlier. We have (2.16)

$$\sum_{\substack{n\le x\\ n\in\mathcal{N}}} F_{-\kappa}(n) = \sum_{\substack{n\le x\\ n\in\mathcal{N}}} e^{-\kappa\omega(n)} = \left(\frac{e^{-\gamma\tau e^{-\kappa}}}{\Gamma(\tau e^{-\kappa})} + o(1)\right) \frac{x}{\log x} \prod_{\substack{p\le x\\ p\in\mathcal{P}}} \left(1 + \frac{e^{-\kappa}}{p-1}\right).$$

Since $e^{-\kappa(\omega(n)-(1-\delta_x)R_x)} \ge 1$ if $\omega(n) < (1-\delta_x)R_x$, therefore

$$\sum_{\substack{n\le x\\ n\in\mathcal{N}\\ \omega(n)<(1-\delta_x)R_x}} 1 \le e^{(1-\delta_x)R_x\kappa} \sum_{\substack{n\le x\\ n\in\mathcal{N}}} F_{-\kappa}(n).$$

Arguing as earlier, by using (2.16) we can get the relation (2.15).

2.6.

Lemma 7. *Let $\mathcal{P}$, $\mathcal{N}$ be as in Section 2.5. For every K let $p_1 < \ldots < p_T$ be a finite sequence of primes from $\mathcal{P}$. Let $\mathcal{P}_K = \{p_1,\ldots,p_T\}$, and let*

$$\omega_{\mathcal{P}_K}(n) = \sum_{\substack{p\mid n\\ p\in\mathcal{P}_K}} 1, \quad A_K = \sum_{j=1}^{T} \frac{1}{p_j}, \quad A_K > K.$$

Then

$$\limsup_{x\to\infty} \frac{1}{N(x)} \sum_{\substack{n\le x\\ n\in\mathcal{N}}} |\omega_{\mathcal{P}_K}(n) - A_K| \le \sqrt{A_K}. \tag{2.17}$$

Proof. Since

$$N\left(\frac{x}{p}\right) = \#\{n \le x \mid n \in \mathcal{N},\ p|n\}$$

and from the theorem of E. Wirsing (Lemma 4) one can get easily that

$$N\left(\frac{x}{p}\right) = \frac{1}{p}N(x) + o(N(x)) \quad (x\to\infty),$$

we obtain that

$$\sum_{\substack{n\le x\\ n\in\mathcal{N}}} \omega_{\mathcal{P}_K}(n) = A_K N(x) + o(N(x)) \quad (x\to\infty),$$

$$\sum_{\substack{n\le x\\ n\in\mathcal{N}}} \omega^2_{\mathcal{P}_K}(n) = \left(A_K^2 + A_K - \sum_{p\in\mathcal{P}_K}\frac{1}{p^2}\right)N(x) + o(N(x)) \quad (x\to\infty).$$

Thus

$$\sum_{\substack{n\le x\\ n\in\mathcal{N}}} (\omega_{\mathcal{P}_K}(n) - A_K)^2 = \left(A_K^2 + A_K - \sum_{p\in\mathcal{P}_K}\frac{1}{p^2} - 2A_K^2 + A_K^2\right)N(x) + \\ + o(N(x)) \quad (x\to\infty),$$

whence

$$\frac{1}{N(x)}\sum_{\substack{n\le x\\ n\in\mathcal{N}}} |\omega_{\mathcal{P}_K}(n) - A_K| \le \frac{1}{\sqrt{N(x)}}\left\{\sum_{\substack{n\le x\\ n\in\mathcal{N}}} |\omega_{\mathcal{P}_K}(n) - A_K|^2\right\}^{1/2} \le \\ \le \sqrt{A_K} + o(1) \quad (x\to\infty),$$

and so our assertion holds.

2.7.

Let $\mathcal{N}$ be as in 2.5. From the theorem of Wirsing (see Lemma 4) we obtain that

$$N\left(\frac{x}{y}\right) \le \frac{cN(x)}{y}$$

holds for $1\le y\le\sqrt{x}$. Let

(2.18) $$\omega_1(n) := \sum_{\substack{p\mid n\\ p\in\mathcal{P}\\ p<\exp((\log x)^{\epsilon_x})}} 1$$

where $\epsilon_x \to 0$ as $x \to \infty$. Hence we obtain that

$$\sum_{\substack{n\le x\\ n\in\mathcal{N}}} \omega_1(n) < c\epsilon_x R_x N(x).$$

For some $n \in \mathcal{N}$ consider all possible representations $n = pm$, where $p \in \mathcal{P}$. Let

$$\omega_2(n) = \sum_{\substack{n=pm\\ m\le x^{\epsilon_x}}} 1.$$

Then

$$\sum_{\substack{n\le x\\ n\in\mathcal{N}}} \omega_2(n) \le \sum_{\substack{m\le x^{\epsilon_x}\\ m\in\mathcal{N}}} \pi_P\left(\frac{x}{m}\right) \le \frac{c\tau x}{\log x} \sum_{\substack{m\le x^{\epsilon_x}\\ m\in\mathcal{N}}} \frac{1}{m} \le$$

$$\le \frac{c\tau x}{\log x} \prod_{\substack{p\le x^{\epsilon_x}\\ p\in\mathcal{P}}} \frac{1}{1-1/p} \le \frac{c\tau x}{\log x} \exp\left(\sum_{\substack{p< x^{\epsilon_x}\\ p\in\mathcal{P}}} \frac{1}{p}\right).$$

Hence we have that

$$\frac{1}{N(x)} \sum_{\substack{n\le x\\ n\in\mathcal{N}}} \omega_2(n) \to 0 \quad (x\to\infty). \tag{2.19}$$

3. Formulation and proof of Theorem 2

3.1.

Theorem 2. *Let $\mathcal{N}$ be as in 2.5. Assume that $f \in \mathcal{A}_q$, $f(bq^j) = \mathcal{O}(1)$ $(b \in E,\ j = 0, 1, \ldots)$. Assume furthermore that there is a constant $\lambda > 0$ for which $D(x)/\log^{\lambda} x \to \infty$ $(x \to \infty)$. Let*

$$F_x(y) = \frac{1}{N(x)} \#\{\nu(n) < y,\ n \in \mathcal{N},\ n \le x\}.$$

Then

$$\lim_{x\to\infty} F_x(y) = \Phi(y).$$

3.2.

Proof of Theorem 2

Let $y \in \mathbb{R}$ be fixed. Let $n_1 < \ldots < n_s$ ($\leq x$) be the set all of the integers in $\mathcal{N}$ up to x, for which $\nu(n) < y$. Thus $F_x(y) = s/N(x)$. Let

$$\mathcal{H}_x = \mathcal{H} = \#\{\{m,p\},\ p \in \mathcal{P},\ m \in \mathcal{N},\ m > x^{\epsilon_x},\ p > e^{(\log x)^{\epsilon_x}},\ mp < x\}.$$

Let Z be the number of those $\{m,p\} \in \mathcal{H}$ for which $\nu(mp) < y$. It is clear that

$$Z \leq \omega(n_1) + \cdots + \omega(n_s) \leq (1+\delta_x)R_x s + \sum_{\substack{n \leq x \\ n \in \mathcal{N} \\ \omega(n) > (1+\delta_x)R_x}} \omega(n).$$

From Lemma 6 we obtain that

$$\frac{Z}{R_x} \leq (1+\delta_x)s + o(N(x)) \quad (x \to \infty).$$

Similarly

$$Z \geq (1-\delta_x)R_x s - \sum_{\omega(n) < (1-\delta_x)R_x} \omega(n) - \Sigma_1 - \Sigma_2,$$

where in Σ_1 we sum over those $\{m,p\}$ for which $m < x^{\epsilon_x}$, $m \in \mathcal{N}$, $p \in \mathcal{P}$ and in Σ_2 over those for which $p < e^{(\log x)^{\epsilon_x}}$, $p \in \mathcal{P}$ and $m \in \mathcal{N}$. As we have seen in 2.7.

$$\Sigma_1 + \Sigma_2 = o(R_x N(x)) \quad (x \to \infty)$$

and Lemma 6 implies that

$$\sum_{\substack{\omega(n) < (1-\delta_x)R_x \\ n \leq x \\ n \in \mathcal{N}}} \omega(n) = o(R_x N(x)) \quad (x \to \infty).$$

Thus we have

$$\frac{Z}{R_x} \geq s(1-\delta_x) + o_x(N(x)) \quad (x \to \infty).$$

Let $H(x) = \#\mathcal{H}_x$. Let $(1 \leq)\ l_1 < \ldots < l_h \leq N$, $\quad b_1, \ldots, b_h \in E$ and

$$H\left(x \;\middle|\; \begin{matrix} l_1, \ldots, l_h \\ b_1, \ldots, b_h \end{matrix}\right) := \#\{\{m,p\} \in \mathcal{H}_x,\ \varepsilon_{l_j}(mp) = b_j,\ j = 1, \ldots, h\}.$$

We can prove that for every fixed h, and every $\alpha > 0$

$$\max_{\substack{N^\alpha \leq l_1 < \ldots < l_h < N - N^\alpha \\ b_1, \ldots, b_h \in E}} \left| q^h H\left(x \;\middle|\; \begin{matrix} l_1, \ldots, l_h \\ b_1, \ldots, b_h \end{matrix}\right) - H(x) \right| \leq c(h,\lambda) H(x) N^{-\lambda} \tag{3.1}$$

holds for every fixed λ.

The proof is very similar to that of the theorem in [1]. Let

$$U := [1-\Delta, 1] \cup \bigcup_{b=1}^{q-1} \left[\frac{b}{q} - \Delta,\ \frac{b}{q} + \Delta\right] \cup [0, \Delta],$$
$$E_j := \# \left\{ \{m, p\} \in \mathcal{H}_x, \quad \left\{\frac{mp}{q^{j+1}}\right\} \in U \right\},$$

further

$$F(x_1, \dots, x_h) := \phi_{b_1}(x_1) \cdots \phi_{b_h}(x_h),$$
$$t(y) := F\left(\frac{y}{q^{l_1+1}}, \dots, \frac{y}{q^{l_h+1}}\right).$$

Let

$$V = \left[\frac{1}{q^{l_1+1}}, \dots, \frac{1}{q^{l_h+1}}\right],$$

$\mathcal{M}$ the whole set of vectors

$$M = [m_1, \dots, m_h]$$

with integer entries. Let

$$VM = \frac{A_M}{H_M}, \quad (A_M, H_M) = 1.$$

It is clear that

$$t(y) = \sum_{M \in \mathcal{M}} T_M e(MVy),$$

where $|T_M| = |c_{m_1}| \cdots |c_{m_h}|$, $T[0, \dots, 0] = \frac{1}{q^h}$.

We have

$$\left| H\left(x \;\middle|\; \begin{matrix} l_1, \dots, l_h \\ b_1, \dots, b_h \end{matrix}\right) - \frac{1}{q^h} H(x) \right| \le \tag{3.2}$$
$$\le \sum_{M \ne 0} |T_M| \left| \sum_{\{m,p\} \in \mathcal{H}_x} e\left(\frac{A_M}{H_M} mp\right) \right| + E_{l_1} + \dots + E_{l_h}.$$

If M is such a vector for which $q | m_j$ for some j, then $T_M = 0$. Let $M = [m_1, \dots, m_h]$, $q \nmid m_h$. Then

$$H_M(m_h + q^{l_h - l_{h-1}} m_{h-1} + \dots + m_1 q^{l_h - l_1}) = A_M q^{l_h+1}.$$

Let $q = p_1^{e_1} \cdots p_s^{e_s}$ be the prime decomposition of q. Since $q \not| m_h$, there exists a p_t for which $p_t^{e_t} \not| m_h$. Thus there exists an $\eta > 0$ depending only on q such that $H_M \geq q^{\eta l_h} \geq q^{\eta N^\alpha}$. On the other hand $H_M \leq q^{l_h+1} < cxq^{-N^\alpha}$.

By using the Davenport theorem (Lemma 4) we obtain that

$$\sum_{\{m,p\}\in\mathcal{H}_x} e\left(\frac{A_M}{H_M} mp\right) \ll H(x) \log^{-B} x$$

holds for every fixed B. The constant implied by $\ll$ on the right hand side does not depend on M. One can observe also that (see [1])

$$\sum |T_M| \leq \left(2 + 2\log\frac{1}{\Delta}\right)^h.$$

Finally we can estimate E_j by using the Erdős-Turán inequality (Lemma 3) for the discrepancy. Let

$$\psi_k := \sum_{\{m,p\}\in\mathcal{H}_x} e\left(kmp\frac{1}{q^{l_j+1}}\right).$$

Then

$$|E_j| \leq (2q\Delta)H(x) + c\sum_{k=1}^{T} \frac{|\psi_k|}{k} + \frac{cH(x)}{T},$$

where c is an absolute constant, T is arbitrary. Let K be an arbitrary large constant,

$$T = [\log^K x], \quad \Delta = \frac{1}{T}.$$

By the theorem of Davenport we obtain that $\max_{1\leq k\leq T} |\psi_k| \leq H(x)\log^{-K} x$ say. Hence we obtain (3.2). Our sequence $\mathcal{H}_x$ is q-ary smooth of type α for every $\alpha > 0$, therefore Theorem 1 can be applied for every α. The proof of Theorem 2 is complete.

4. A remark to a theorem of H. Daboussi

4.1.

The famous theorem of H. Daboussi [7, 8] asserts that if α is an irrational number, $\mathcal{M}_1$ be the set of complex valued multiplicative functions f satisfying the condition $|f(n)| \le 1$ $(n \in \mathbb{N})$, then

$$\sup_{f\in\mathcal{M}_1} \frac{1}{x}\left|\sum_{n\le x} f(n)e(n\alpha)\right| \to 0 \quad \text{as } x\to\infty.$$

There are a lot of generalizations of this theorem, see e.g. [9, 11].

Theorem 3. *Let $\mathcal{P}, \mathcal{N}$ be as in 2.5. Let α be an irrational number for which*

$$\min_{1\le k\le \log^B x} ||k\alpha|| > \frac{\log^B x}{x}$$

holds for every B and $x > x_0(B)$. Then

$$\sup_{f\in M_1} \frac{1}{N(x)}\left|\sum_{\substack{n\le x\\ n\in\mathcal{N}}} f(n)e(n\alpha)\right| \to 0 \quad \textit{as } x\to\infty. \tag{4.1}$$

4.2.

Proof of Theorem 3

We shall prove only that

$$\lim_{x\to\infty} \frac{1}{N(x)}\left|\sum_{\substack{n\le x\\ n\in\mathcal{N}}} e(\alpha n k)\right| = 0 \tag{4.2}$$

for every $k\in\mathbb{N}$, $k\ne 0$. The deduction of (4.1) from (4.2) can be done in the same way as which was used in [10].

Let $\tau = \frac{x}{\log^B x}$. Then there is an integer Q such that $Q \le \tau$, and $||Q\alpha|| < \frac{1}{\tau}$. Due to the the condition of the theorem $Q \ge \log^{2B} x$, consequently for a suitable integer A,

$$\left|\alpha - \frac{A}{Q}\right| < \frac{1}{Q\tau} \le \frac{1}{x\log^B x},$$

$(A, Q) = 1$ and so

$$\sum_{\substack{n\le x\\ n\in\mathcal{N}}} e(\alpha n k) = \sum_{\substack{n\le x\\ n\in\mathcal{N}}} e\left(\frac{Ak}{Q} n\right) + \mathcal{O}\left(\frac{kN(x)}{\log^B x}\right).$$

To prove (4.2) we shall estimate

$$S = \sum_{\substack{n\le x \\ n\in\mathcal{N}}} e\left(\frac{Akn}{Q}\right).$$

By using Lemma 6, it is enough to prove that

$$\frac{1}{R_x N(x)} \sum_{\substack{n\le x \\ n\in\mathcal{N}}} e\left(\frac{Akn}{Q}\right)\omega(n) \to 0 \quad (x\to\infty)$$

and by repeating the argument used in 2.7 that

$$\frac{1}{\#\mathcal{H}_x} \sum_{\{m,p\}\in\mathcal{H}_x} e\left(\frac{Akmp}{Q}\right) \to 0 \quad (x\to\infty). \tag{4.3}$$

(4.3) follows from the theorem of Davenport.

We note that Lemma 7 is a tool to deduce the theorem from (4.3).

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