THE GENERATING FUNCTION OF ERGODIC DISTRIBUTION IN THE M/G/1 SYSTEM WITH VACATION AT THE BEGINNING OF THE BUSY PERIOD

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Dedicated to Dr. Bui Minh Phong on his sixtieth birthday

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Abstract. Many queueing systems are investigated by means of the embedded Markov chain technique deriving on such way the generating function for the probability distribution. There exists another approach, namely the stationary probabilities are obtained on the basis of regenerative processes considering the regenerative cycles and the mean values of time spent in different states during them. In the paper we get the generating function of ergodic distribution starting with these probabilities for a system with adjustment at the beginning of busy period.

1. Notations

λ - arrival rate;

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$B(x)$ - the distribution function of service time for one customer; $b(s) = \int_0^\infty e^{-sx}dB(x)$ is its Laplace-Stieltjes transform; $\tau = \int_0^\infty xdB(x)$ - its mean value;

$\rho = \lambda\tau$;

$\Gamma(s)$ - the Laplace-Stieltjes transform of the busy period's distribution function in the classical M/G/1 system, it is the solution of functional equation

$$\Gamma(s) = b(s + \lambda - \lambda\Gamma(s));$$

$\zeta = \dfrac{\tau}{1-\rho}$ - the mean value of length of a busy period;

a_i - the probability of appearance of i ones for the service of a customer in the simple M/G/1 system, $A(z) = \sum_{i=0}^\infty a_i z^i = b(\lambda(1-z))$;

ξ_i - the mean value of time spent in the i-th state for a busy period in the simple M/G/1 system, $Q(z) = \sum_{i=0}^\infty \xi_i z^i$;

$D(x)$ - the distribution function of vacation, $d(s) = \int_0^\infty e^{-sx}dD(x)$ is its Laplace-Stieltjes transform, $\eta = \int_0^\infty xdD(x)$ - its mean value;

ζ' - the mean value of length of a busy period in the system with vacation;

ξ_i' - the mean value of time spent in the i-th state for a busy period in the system with vacation, $Q^*(z) = \sum_{i=0}^\infty \xi_i' z^i$;

d_i - the probability of appearance of i customers for the vacation; $D(z) = \sum_{i=0}^\infty d_i z^i = d(\lambda(1-z))$;

$P(z) = \sum_{i=0}^\infty p_i z^i$ and $P^*(z) = \sum_{i=0}^\infty p_i^* z^i$ - the generating functions of ergodic distributions of number of customers for the simple M/G/1 system and for the system with vacation.

2. Introduction

In order to find the ergodic distribution in the M/G/1 system one usually applies the embedded Markov chain technique identifying the states of system by the number of customers at service completion epochs. It leads to the Pollaczek-Khinchin transform equation

$$P(z) = \frac{(1-\rho)(1-z)b(\lambda(1-z))}{b(\lambda(1-z)) - z},$$

the generating function of equilibrium distribution. From mathematical point of view it gives the solution of the problem, the probabilities may be found from it e.g. by means of differentiation. The direct use of differentiation gives very complicated expressions, so sometimes the FFT method can be useful.

From the end of 80's there were other approaches, too, they are mentioned in [1]. In [6] Tijms prefers to use the regenerative method, it directly leads to a numerically stable recursion scheme for the state probabilities and allows in a natural way the generalizations to more complex models. It directly computes the steady-state probabilities, the method is based on Theorems 1.3.2 and 1.3.3 of [6].

In [1] we followed another way. The functioning of M/G/1 system may be described by a regenerative process, according to [5, 6] the stationary distribution for such a process is determined by the relation of mean value of time spent in different states to the mean value of length of the regenerative cycle. The arrival rate, the mean value of service time and the probabilities of appearance of a given number of customers for the service time are the primary information about the functioning of the system, the desired probabilities were obtained from them avoiding the generating function, even we had not to know the concrete service time distribution.

In the classical case the ergodic distribution of M/G/1 system is derived by using the embedded Markov chain technique leading to the Pollaczek-Khinchin transform equation and the desired probabilities are obtained from it. In [1] we got these probabilities by using the mean values of the busy period and times spent in different states during it. In [3] we put the question on another way. By knowing the mean values (practically having the equilibrium probabilities) from them we derived the Pollaczek-Khinchin transform equation. Here we do the same for the system with adjustment at the beginning of the busy period.

3. Previous results

In [1] we proved that in the M/G/1 system the mean values of times ξ_i spent in the different states for a busy period satisfy the relations

$$\xi_0 = \tau, \qquad \xi_1 = \frac{1-a_0}{a_0}\tau, \qquad \xi_2 = \frac{1-a_0-a_1}{a_0}(\xi_0+\xi_1),$$

$$\xi_i = \sum_{k=1}^{i-2} \frac{1-a_0-a_1-\ldots-a_k}{a_0}\xi_{i-k} + \frac{1-a_0-a_1-\ldots-a_{i-1}}{a_0}(\xi_0+\xi_1), \quad k \geq 3,$$

and the stationary probabilities may be found as the relations $p_i = \xi_i/\zeta$.

In [2] we have considered a modification of the classical M/G/1 system, namely the customer arriving after free state generates an adjustment (it can also be called vacation during which the server prepares to the work). After this adjustment starts the busy period with one (if for the adjustment no new customer arrives) or more (if new customers enter) present customers at the beginning.

In [2] it also was proved that the Laplace-Stieltjes transform of the busy period's distribution function for such system is

$$d(s+\lambda-\lambda\Gamma(s))\Gamma(s)$$

and from it the mean value of a regenerative cycle is

$$\zeta' = \frac{\tau}{1-\rho} + \frac{\eta}{1-\rho},$$

the mean values of times spent in different states are given by the relations

$$\xi_0' = \tau, \quad \xi_1' = \frac{\tau}{a_0} + d_0(\eta-\tau), \quad \xi_2' + (1-d_0)(\xi_0+\xi_1) + d_1(\eta-\tau),$$

$$\xi_i' = \xi_i + \sum_{k=2}^{i-1}(1-d_0-\ldots d_{i-1-k})\xi_k + (1-d_0-\ldots-d_{i-2})(\xi_0+\xi_1) + d_{i-1}(\eta-\tau),$$

$$i \geq 3.$$

4. Theorem and proof

Theorem. *The generating function of number of customers in the M/G/1 system with vacation at the beginning of the busy period is*

$$P^*(z) = \frac{1-\rho}{\tau+\eta}\left\{\frac{A(z)[1-zD(z)]}{A(z)-z} + zD(z)\eta\right\}.$$

Proof. Let us consider the system with vacation at the beginning of the busy period. The mean values of times spent on the different levels for a busy period ξ_i' are

$$\begin{aligned}
\xi_0' =& \xi_0;\\
\xi_1' =& (\xi_1+\xi_0) + d_0(\eta-\tau);\\
\xi_2' =& \xi_2 + (1-d_0)(\xi_1+\xi_0) + d_1(\eta-\tau);\\
\xi_3' =& \xi_3 + (1-d_0)\xi_2 + (1-d_0-d_1)(\xi_1+\xi_0) + d_2(\eta-\tau);\\
\xi_4' =& \xi_4 + (1-d_0)\xi_3 + (1-d_0-d_1)\xi_2 + (1-d_0-d_1-d_2)(\xi_1+\xi_0)+\\
& + d_3(\eta-\tau);\\
\xi_5' =& \xi_5 + (1-d_0)\xi_4 + (1-d_0-d_1)\xi_3 + (1-d_0-d_1-d_2)\xi_2+\\
& + (1-d_0-d_1-d_2-d_3)(\xi_1+\xi_0) + d_4(\eta-\tau);\\
& \vdots
\end{aligned}$$

Let us multiply the expression for ξ_i' and sum up by i. We will separately deal with terms ξ_0 and $\eta-\tau$ starting from the second row. For the sum of remaining ones we have

$$\sum_{i=0}^{\infty}\xi_i z^i + (1-d_0)z\sum_{i=1}^{\infty}\xi_i z^i + (1-d_0-d_1)z^2\sum_{i=1}^{\infty}\xi_i z^i +$$

$$+(1-d_0-d_1-d_2)z^3\sum_{i=1}^{\infty}\xi_i z^i + \ldots =$$

$$= Q(z) + \left(\sum_{i=1}^{\infty}\xi_i z^i\right)[(z+z^2+z^3+\ldots) - d_0(z+z^2+z^3+\ldots)-$$

$$-d_1 z(z+z^2+z^3+\ldots) - d_2 z^2(z+z^2+z^3+\ldots) - \ldots] =$$

$$= Q(z) + \left(\sum_{i=1}^{\infty} \xi_i z^i\right) \left[\frac{z}{1-z} - d_0 \frac{z}{1-z} - d_1 z \frac{z}{1-z} - d_2 z^2 \frac{z}{1-z} - \ldots\right] =$$

$$= Q(z) + \left(\sum_{i=1}^{\infty} \xi_i z^i\right) \frac{z}{1-z}[1 - D(z)] = Q(z) + [Q(z) - \xi_0] \frac{z}{1-z}[1 - D(z)].$$

From the terms containing ξ_0 starting from the second row we obtain

$$\xi_0 z + \xi_0 z(1 - d_0)z + \xi_0 z(1 - d_0 - d_1)z^2 + \xi_0 z(1 - d_0 - d_1 - d_2)z^3 + \ldots =$$

$$= \xi_0 z + \xi_0 z(z + z^2 + z^3 + \ldots - \xi_0 z[d_0(z + z^2 + z^3 + \ldots)+$$

$$+ d_1 z(z + z^2 + z^3 + \ldots) + d_2 z^2(z + z^2 + z^3 + \ldots) + \ldots] =$$

$$= \xi_0 z + \xi_0 \frac{z}{1-z} - \xi_0 z \frac{z}{1-z}[d_0 + d_1 z + d_2 z^2 + \ldots] =$$

$$= \xi_0 z + \xi_0 z \frac{z}{1-z}[1 - D(z)].$$

The sum of terms containing $\eta - \tau$ gives $(\eta - \tau)zD(z)$.

Collecting these terms for the generating function of times spent in different states for a busy period $Q^*(z)$ we get

$$Q^*(z) = Q(z) + [Q(z) - \xi_0] \frac{z}{1-z}[1 - D(z)]+$$

$$+ \xi_0 z + \xi_0 z \frac{z}{1-z}[1 - D(z)] + (\eta - \tau)zD(z)$$

or

$$Q^*(z) = \frac{Q(z)[1 - zD(z)]\tau + z\eta D(z)(1-z)}{1-z},$$

where

$$Q(z) = \frac{(1-z)A(z)}{A(z) - z}\tau.$$

Dividing it by the mean value of a busy period $\frac{\tau + \eta}{1 - \rho}$ for the generating function of number of customers we finally obtain

$$P^*(z) = \frac{1-\rho}{\tau + \eta} \frac{Q(z)[1 - zD(z)] + z\eta D(z)(1-z)}{1-z}.$$

This proves the theorem.

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