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VALUE SHARING PROBLEM AND UNIQUENESS FOR p–ADIC MEROMORPHIC FUNCTIONS

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Dedicated to Dr. Bui Minh Phong on his sixtieth anniversary

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Abstract. In this paper we discuss the uniqueness problem for p-adic meromorphic functions, and prove a version of the Hayman conjecture for p-adic meromorphic functions.

1. Introduction

The problem of determining a meromorphic (or entire) function on $\mathbb{C}$ by its single pre-images (counting or ignoring multiplicities) of finite sets is an important one and it has been studied by many mathematicians. For instance, in 1921 G. Polya showed that an entire function on $\mathbb{C}$ is determined by the inverse images, counting multiplicities, of three distinct non-omitted values. In 1926, R. Nevanlinna showed that a meromorphic function on the complex plane is uniquely determined by the inverse images, ignoring multiplicities, of 5 distinct values.

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In [15] Hayman proved the following well-known result:

Theorem A. *Let f be a meromorphic function on $\mathbb{C}$. If $f(z) \neq 0$ and $f^{(k)}(z) \neq 1$ for some fixed positive integer k and for all $z \in \mathbb{C}$, then f is constant.*

Hayman also proposed the following conjecture (see [15]).

Hayman Conjecture. *If an entire function f satisfies $f^n(z)f^{'}(z) \neq 1$ for a positive integer n and all $z \in \mathbb{C}$, then f is a constant.*

It has been verified for transcendental entire functions by Hayman himself for $n > 1$ ([15]), and by Clunie for $n \geq 1$ ([6]). These results and some related problems caused increasing attention to the value sharing problem of meromorphic functions and their derivatives (see [2], [5], [17], [18]).

In 1997 Yang and Hua [24] studied the unicity problem for meromorphic functions and differential monomials of the form $f^n f^{'}$, when they share only one value, and obtained the following theorem.

Theorem B. *Let f and g be two non-constant meromorphic functions, let $n \geq 11$ be an integer, and $a \in \mathbb{C}$ be a non-zero finite value. If $f^n f^{'}$ and $g^n g^{'}$ share the value a CM, then either $f \equiv dg$ for some $(n+1)$-th root of unity d, or $f = c_1 e^{cz}$ and $g = c_2 e^{-cz}$ for three non-zero constants c_1, c_2 and c such that $(c_1 c_2)^{n+1} c^2 = -a^2$.*

Recently, there has been an increasing interest in studying value sharing and uniqueness for meromorphic functions in a non-Archimedean field (see, for example, [4], [11-12], [19-22]). In [19] J. Ojeda proved that for a transcendental meromorphic function f in an algebraically closed field of characteristic zero, complete for a non-Archimedean absolute value $\mathbb{K}$, the function $f^{'} f^n - 1$ has infinitely many zeros, if $n \geq 2$. In [11] Ha Huy Khoai and Vu Hoai An established a similar result for a differential monomial of the form $f^n (f^{(k)})^m$, where f is a meromorphic function in $\mathbb{C}_p$. K. Boussaf, A. Escassut, J. Ojeda ([4]) studied the unicity problem for p-adic meromorphic functions $f^{'} P^{'}(f), g^{'} P^{'}(g)$ sharing a small function.

Now let $\mathbb{K}$ be an algebraically closed field of characteristic zero, complete for a non-Archimedean absolute value. We denote by $\mathcal{A}(\mathbb{K})$ the ring of entire functions in $\mathbb{K}$, by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions, i.e., the field of fractions of $\mathcal{A}(\mathbb{K})$, and $\widehat{\mathbb{K}} = \mathbb{K} \cup \{\infty\}$. In recent years, many interesting results on the value sharing problem for meromorphic functions in $\mathbb{K}$ were obtained (see, for example, [12], [16]).

Let us first recall some basic definitions. For $f \in \mathcal{M}(\mathbb{K})$ and $S \subset \widehat{\mathbb{K}}$, we define

$$E_f(S) = \bigcup_{a \in S} \{(z,m) \mid f(z) = a \text{ with multiplicity } m\}.$$

Let $\mathcal{F}$ be a nonempty subset of $\mathcal{M}(\mathbb{K})$. Two functions f, g of $\mathcal{F}$ are said to *share* S, *counting multiplicity* (share S CM), if $E_f(S) = E_g(S)$. In this paper we prove a version of the Hayman conjecture for p-adic meromorphic functions of the form $(f^n)^{(k)}$, and discuss the uniqueness problem for p-adic meromorphic functions $(f^n)^{(k)}$, $(g^n)^{(k)}$.

Namely, we prove the following theorems.

Theorem 1.1. (A version of the Hayman conjecture for p-adic meromorphic functions of the form $(f^n)^{(k)}$.) *Let f be a meromorphic function on $\mathbb{K}$, satisfying the condition $(f^n)^{(k)}(z) \neq 1$ for all $z \in \mathbb{K}$ and for some positive integers n, k. Then f is a constant function if one of the following conditions holds:*

1. f is an entire function, and $n \geq k+1$.

2. $n \geq k+2$.

From Theorem 1.1, we obtain the following corollary.

Corollary 1.2. *Let f be a meromorphic function on $\mathbb{K}$, satisfying the condition $(f^n)^{'}(z) \neq 1$ for all $z \in \mathbb{K}$ and for some positive integers n. Then f is a constant function if one of the following conditions holds:*

1. f is an entire function, and $n \geq 2$.

2. $n \geq 3$.

Remark. Indeed, in [19], Theorem 3 shows that $f^{'} + f^4$ has at least one zero that is not a zero of f, where f is a non-constant function. Hence setting $g(x) = \frac{1}{f(x)}$, we can check that $g^2 g^{'}$ takes the value 1 at least one time. By $g^2 g^{'} = \frac{1}{3}(g^3)^{'}$, we see that $(g^3)^{'}$ takes the value 1 at least one time. So the case $n = 3, k = 1$ of Theorem 1.1 has been established in [19].

Theorem 1.3. (A version of Yang and Hua's Theorem B for p-adic meromorphic functions of the form $(f^n)^{(k)}$.) *Let f, g be two transcendental meromorphic functions on $\mathbb{K}$, n, k be positive integers, $n \geq 3k+8$, and let $E_{(f^n)^{(k)}}(1) = E_{(g^n)^{(k)}}(1)$. Then $f = cg$ with $c^n = 1$, $c \in \mathbb{K}$.*

The main tool of the proof is the p-adic Nevanlinna theory (see [8-13], [16]). Therefore, in the next section we first establish some properties of the characteristic functions of non-Achimedean meromorphic functions.

2. Value distribution of non-Archimedean meromorphic functions

Throughout this paper, $\mathbb{K}$ will denote an algebraically closed field of characteristic zero, complete for a non-trivial non-Archimedean absolute value denoted by $|\ .\ |$, and log be a real logarithm function of base $\rho > 1$, and ln be a real logarithm function of base e.

1. Counting functions of a non-Archimedean entire function (see [16, pp.21-23], [3], [7-13])

Let f be a non-constant entire function on $\mathbb{K}$ and $b \in \mathbb{K}$. Then we can write f in the form

$$f = \sum_{n=q}^{\infty} b_n (z-b)^n$$

with $b_q \neq 0$ and we put $\omega_f^0(b) = q$.

For a point $a \in \mathbb{K}$ we define the function $\omega_f^a :\ \mathbb{K} \to\ \mathbb{N}$ by $\omega_f^a(b) = \omega_{f-a}^0(b)$.

Fix a real number ρ_0 with $0 < \rho_0 \leq r$. Take $a \in \mathbb{K}$ and we denote the *counting function of zeroes of* $f - a$ counting multiplicity in the disk $D_r = \{z \in \in \mathbb{K} : |z| \leq r\}$, i.e. we set

$$N_f(a,r) = \frac{1}{\ln \rho} \int_{\rho_0}^{r} \frac{n_f(a,x)}{x} dx,$$

where $n_f(a,x)$ is the number of the solutions of the equation $f(z) = a$ (counting multiplicity), in the disk $D_x = \{z \in \mathbb{K} : |z| \leq x\}$. If $a = 0$, then set $N_f(r) = = N_f(0,r)$.

For l a positive integer, set

$$N_{l,f}(a,r) = \frac{1}{\ln \rho} \int_{\rho_0}^{r} \frac{n_{l,f}(a,x)}{x} dx, \text{ where } n_{l,f}(a,r) = \sum_{|z| \leq r} \min\{\omega_f^a(z), l\}.$$

Let k be a positive integer. Define the function $\omega_f^{\leq k}$ from $\mathbb{K}$ into $\mathbb{N}$ by

$$\omega_f^{\leq k}(z) = \begin{cases} 0 & \text{if } \omega_f^0(z) > k, \\ \omega_f^0(z) & \text{if } \omega_f^0(z) \leq k, \end{cases}$$

and

$$n_f^{\leq k}(r) = \sum_{|z| \leq r} \omega_f^{\leq k}(z), \quad n_f^{\leq k}(a,r) = n_{f-a}^{\leq k}(r).$$

Define

$$N_f^{\leq k}(a,r) = \frac{1}{\ln p} \int_{\rho}^{r} \frac{n_f^{\leq k}(a,x)}{x} dx.$$

If $a = 0$, then set $N_f^{\leq k}(r) = N_f^{\leq k}(0,r)$.

Set

$$N_{l,f}^{\leq k}(a,r) = \frac{1}{\ln p} \int_{\rho}^{r} \frac{n_{l,f}^{\leq k}(a,x)}{x} dx,$$

where

$$n_{l,f}^{\leq k}(a,r) = \sum_{|z| \leq r} \min\left\{v_{f-a}^{\leq k}(z), l\right\}.$$

In a like manner to used, for non-constant entire function on $\mathbb{K}$ we define

$$N_f^{<k}(a,r),\ N_{l,f}^{<k}(a,r),\ N_f^{>k}(a,r),\ N_f^{\geq k}(a,r),\ N_{l,f}^{\geq k}(a,r),\ N_{l,f}^{>k}(a,r).$$

2. Characteristic functions of a non-Achimedean meromorphic function (see [16, pp.33-46],[3], [7-13])

Recall that for a non-constant entire function $f(z)$ on $\mathbb{K}$, represented by the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

for each $r > 0$, we define $|f|_r = \max\{|a_n| r^n, 0 \leq n < \infty\}$.

Now let $f = \dfrac{f_1}{f_2}$ be a non-constant meromorphic function on $\mathbb{K}$, where f_1, f_2 are entire functions on $\mathbb{K}$ having no common zeros, we set $|f|_r = \frac{|f_1|_r}{|f_2|_r}$.

For a point $a \in \mathbb{K} \cup \{\infty\}$ we define the function $\omega_f^a : \mathbb{K} \to \mathbb{N}$ by $\omega_f^a(b) = \omega_{f_1 - af_2}^0(b)$ with $a \neq \infty$ and $\omega_f^\infty(b) = \omega_{f_2}^0(b)$.

Take $a \in \mathbb{K}$. We denote the *counting function of zeroes of* $f - a$, counting multiplicity, in the disk $D_r = \{z \in \mathbb{K} : |z| \leq r\}$, i.e. we set

$$N_f(a,r) = N_{f_1 - af_2}(r), \text{ and set } N_f(\infty, r) = N_{f_2}(r).$$

In a like manner to used, for non-constant meromorphic function on $\mathbb{K}$ we define

$$N_{l,f}(a,r),\ N_f^{\leq k}(a,r),\ N_f^{<k}(a,r),\ N_{l,f}^{<k}(a,r),$$

$$N_f^{>k}(a,r),\ N_f^{\geq k}(a,r),\ N_{l,f}^{\geq k}(a,r),\ N_{l,f}^{>k}(a,r).$$

Define the *compensation function of* f, by

$$m_f(\infty,r) = \max\left\{0, \log|f|_r\right\},$$

and set

$$m_f(a,r) = m_{\frac{1}{f-a}}(\infty,r),$$

and the *characteristic function of* f, by

$$T_f(r) = m_f(\infty,r) + N_f(\infty,r).$$

Then we have

$$N_f(a,r) + m_f(a,r) = T_f(r) + O(1) \text{ with } a \in \mathbb{K}\cup\{\infty\},$$

$$T_f(r) = T_{\frac{1}{f}}(r) + O(1),$$

$$T_f(r) = \max_{1\leq i\leq 2} \log|f_i|_r + O(1), \quad |f^{(k)}|_r \leq \frac{|f|_r}{r^k},\ m_{\frac{f^{(k)}}{f}}(\infty,r) = O(1).$$

The following lemmas were proved in [16, pp.21] (see also [10], [13]).

Lemma 2.1. *Let f be a non-constant entire function on $\mathbb{K}$. Then*

$$T_f(r) - T_f(\rho_0) = N_f(0,r),$$

where $0 < \rho_0 \leq r$.

Notice that $N_f(r)$ depends on fixed ρ_0.

Lemma 2.2. *Let f be a non-constant meromorphic function on $\mathbb{K}$ and let $a_1, a_2, ..., a_q$ be distinct points of $\mathbb{K}$. Then*

$$(q-1)T_f(r) \leq N_{1,f}(\infty,r) + \sum_{i=1}^{q} N_{1,f}(a_i,r) - \log r + O(1).$$

Let f be a meromorphic function on $\mathbb{K}$. By $S_f(r)$ we denote an arbitrary term of the form $o(T_f(r))$ for $r \to \infty$. A meromorphic function f on $\mathbb{K}$ is called a *transcendental meromorphic function* if

$$\lim_{r\to\infty} \frac{T_f(r)}{\log r} = \infty.$$

For two non-constant meromorphic functions f, g on $\mathbb{K}$ we denote by $N_f(0, r; g \neq \neq 0)$ the counting function of those zeros of f which are not the zeros of g, where a zero of f is counted according to its multiplicity.

3. The Hayman-conjecture and the uniqueness problem for p-adic meromorphic functions of the form $(f^n)^{(k)}$

We are going to prove Theorem 1.1, Theorem 1.3. We need the following lemmas.

Lemma 3.1. *Let f be a non-constant meromorphic function on $\mathbb{K}$, n, k be positive integers, $n > k$, and let a be a pole of f. Then*

$$(f^n)^{(k)}(z) = \frac{\varphi_k(z)}{(z-a)^{np+k}},$$

where $\varphi_k(z)$ is a holomorphic function in a neighborhood of a,

$$p = \omega_f^\infty(a), \quad \varphi_k(a) \neq 0.$$

Proof. Since a is a pole of f we obtain

$$f^n(z) = \frac{\varphi(z)}{(z-a)^{np}}, \quad p = \omega_f^\infty(a), \quad \varphi(a) \neq 0.$$

Now we prove by induction. With $k = 1$ we have

$$(f^n)^{(1)}(z) = \left(\frac{\varphi(z)}{(z-a)^{np}}\right)' = \frac{\varphi'(z-a) - np\varphi}{(z-a)^{np+1}}.$$

Set $\varphi_1(z) = \varphi^{'}(z-a) - np\varphi$. Then

$$(f^n)^{(1)}(z) = \frac{\varphi_1(z)}{(z-a)^{np+1}}, \quad \varphi_1(a) \neq 0.$$

Assume

$$(f^n)^{(k)}(z) = \frac{\varphi_k(z)}{(z-a)^{np+k}}, \quad \varphi_k(a) \neq 0.$$

We have

$$(f^n)^{(k+1)}(z) = ((f^n)^{(k)})'(z) = \left(\frac{\varphi_k(z)}{(z-a)^{np+k}}\right)' = \frac{\varphi_k^{'}(z-a) - (np+k)\varphi_k(z)}{(z-a)^{np+k+1}}.$$

Set

$$\varphi_{k+1}(z) = \varphi_k^{'}(z-a) - (np+k)\varphi_k(z).$$

Then

$$(f^n)^{(k+1)}(z) = \frac{\varphi_{k+1}(z)}{(z-a)^{np+k+1}}, \quad \varphi_{k+1}(a) \neq 0.$$

Lemma 3.1 is proved.

Lemma 3.2. *Let f be a non-constant meromorphic function on $\mathbb{K}$, n, k be positive integers, $n > k$, and let a, b be a pole and a zero of f, respectively. Then*

1. $\dfrac{(f^n)^{(k)}(z)}{f^{n-k}(z)} = \dfrac{h_k(z)}{(z-a)^{pk+k}}$, *where* $p = \omega_f^{\infty}(a), h_k(a) \neq 0;$
2. $\dfrac{(f^n)^{(k)}(z)}{f^{n-k}(z)} = (z-b)^{(m-1)k} S_k(z)$, *where* $m = \omega_f^0(b), \quad S_k(b) \neq 0.$

Proof. 1. Since a is a pole of f we obtain

$$f(z) = \frac{h(z)}{(z-a)^p}, \quad h(a) \neq 0, \quad (f^n)^{(k)}(z) = \frac{\varphi_k(z)}{(z-a)^{np+k}}, \quad \varphi_k(a) \neq 0,$$

$$f^{n-k}(z) = \frac{h^{n-k}(z)}{(z-a)^{p(n-k)}}.$$

Thus

$$\frac{(f^n)^{(k)}(z)}{f^{n-k}(z)} = \frac{h_k(z)}{(z-a)^{pk+k}}, \quad h_k(z) = \frac{\varphi_k(z)}{h^{n-k}(z)}, \quad h_k(a) \neq 0.$$

2. Since b is a zero of f we obtain

$$f(z) = (z-b)^m l(z), \quad l(b) \neq 0, \quad f^n(z) = (z-b)^{mn} l^n(z),$$

$$(f^n)^{(k)}(z) = (z-b)^{mn-k} l_k(z), \quad l_k(b) \neq 0, \quad f^{n-k}(z) = (z-b)^{m(n-k)} l^{n-k}(z).$$

So

$$\frac{(f^n)^{(k)}(z)}{f^{n-k}(z)} = (z-b)^{(m-1).k} S_k(z), \quad S_k(z) = \frac{l_k(z)}{l^{n-k}(z)}, \quad S_k(b) \neq 0.$$

Lemma 3.2 is proved.

Lemma 3.3. *Let f be a non-constant meromorphic function on $\mathbb{K}$ and n, k be positive integers, $n \geq k+1$. Then*

$$T_f(r) \leq T_{(f^n)^{(k)}}(r) + O(1),$$

in particular, $(f^n)^{(k)}$ is non-constant.

Proof. Set $A = (f^n)^{(k)} - 1$. Then we have

$$A + 1 = (f^n)^{(k)} = f^{n-k} P,$$

$$N_f(0,r) \leq N_{A+1}(0,r), \qquad \frac{1}{f^n} = \frac{1}{f^{n-k}} \frac{1}{f^k} = \frac{1}{A+1} \frac{P}{f^k}.$$

Moreover,

$$m_{\frac{P}{f^k}}(\infty, r) = m_{\frac{f^{n-k}P}{f^n}}(\infty, r) = m_{\frac{(f^n)^{(k)}}{f^n}}(\infty, r) = O(1).$$

Therefore,

$$m_f(0,r) \leq n m_f(0,r) = m_{f^n}(0,r) =$$
$$= m_{\frac{1}{f^n}}(\infty, r) \leq m_{\frac{1}{A+1}}(\infty, r) + O(1) = m_{A+1}(0,r) + O(1).$$

Thus,

$$T_f(r) = N_f(0,r) + m_f(0,r) \leq N_{A+1}(0,r) + m_{A+1}(0,r) = T_{(f^n)^{(k)}} + O(1).$$

From this, and because f is non-constant, it follows that $(f^n)^{(k)}$ is non-constant. Lemma 3.3 is proved.

Lemma 3.4. *Let f be a non-constant meromorphic function on $\mathbb{K}$ and n, k be positive integers, $n \geq k+2$, $a \in \mathbb{K}$, $a \neq 0$. Then*

$$\frac{n-k-2}{n+k} T_f(r) \leq N_{1,(f^n)^{(k)}}(a, r) - \log r + O(1).$$

Proof. Since $n \geq k+2$ we have $\frac{n-k-2}{n+k} \geq 0$. Because $n \geq k+2$, from Lemma 3.3 it follows that $(f^n)^{(k)}$ is not constant.

Applying Lemma 2.2 to $(f^n)^{(k)}$ with the values ∞, 0 and a, we obtain

$$T_{(f^n)^{(k)}}(r) \leq N_{1,(f^n)^{(k)}}(\infty, r) + N_{1,(f^n)^{(k)}}(0, r) + N_{1,(f^n)^{(k)}}(a, r) - \log r + O(1).$$

Denote by $N_{f^{(k)}}(0, r; f \neq 0)$ the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity. Write $(f^n)^{(k)} = f^{n-k}P$. Then

$$\frac{P}{f^k} = \frac{(f^n)^{(k)}}{f^n}.$$

We see that any pole of $\frac{P}{f^k}$ can occur only at poles of $\frac{(f^n)^{(k)}}{f^n}$, and if z_0 is a pole of $\frac{(f^n)^{(k)}}{f^n}$, then z_0 is either a pole of f or a zero of f. By Lemma 3.1, Lemma 3.2 we see that if a, b are a pole and a zero of f, respectively, then $\frac{(f^n)^{(k)}}{f^n} = \frac{B}{(z-a)^k}$ and $\frac{(f^n)^{(k)}}{f^n} = \frac{C}{(z-b)^k}$. From this it follows that

$$\begin{aligned} N_P(0, r; f \neq 0) = N_{\frac{P}{f^k}}(0, r) \leq T_{\frac{P}{f^k}} + O(1) \leq \\ \leq N_{\frac{P}{f^k}}(\infty, r) + m_{\frac{P}{f^k}}(\infty, r) + O(1) \leq \\ \leq kN_{1,f}(\infty, r) + kN_{1,f}(0, r) + O(1). \end{aligned}$$

Therefore,

$$N_P(0, r; f \neq 0) \leq kN_{1,f}(\infty, r) + kN_{1,f}(0, r) + O(1).$$

From this it follows

$$\begin{aligned} (3.1) \qquad N_{1,(f^n)^{(k)}}(0, r) = N_{1,f^{n-k}P}(0, r) \leq \\ \leq N_{1,f}(0, r) + N_P(0, r; f \neq 0) \leq \\ \leq N_{1,f}(0, r) + kN_{1,f}(\infty, r) + kN_{1,f}(0, r) + O(1) \leq \\ \leq (k+1)N_{1,f}(0, r) + kN_{1,f}(\infty, r) + O(1). \end{aligned}$$

By Lemma 3.1, Lemma 3.2 , if a, b are a pole and a zero of f, respectively, then

$$(f^n)^{(k)} = \frac{B_k}{(z-a)^{np+k}}, \quad B_k(a) \neq 0 \quad \text{and}$$

$$(f^n)^{(k)} = C_k(z-b)^{mn-k}, \quad C_k(b) \neq 0.$$

Therefore we see that

$$N_{(f^n)^{(k)}}(0,r) - N_{1,(f^n)^{(k)}}(0,r) \geq \tag{3.2}$$

$$\geq \big((n-1)(k+1)\big) N_{1,f}^{\geq (k+1)}(0,r) + (n-k-1) N_{1,f}^{\leq k}(0,r).$$

On the other hand,

$$N_{1,f}(0,r) = N_{1,f}^{\leq k}(0,r) + N_{1,f}^{\geq (k+1)}(0,r).$$

From this and (3.1), (3.2) we obtain

$$N_{1,f}^{\leq k}(0,r) \leq$$

$$\leq \frac{1}{n-k-1}\big(N_{(f^n)^{(k)}}(0,r) - N_{1,(f^n)^{(k)}}(0,r) - (n-1)(k+1)N_{1,f}^{\geq (k+1)}(0,r)\big),$$

$$N_{1,(f^n)^{(k)}}(0,r) \leq$$

$$\begin{aligned}
&\leq (k+1)N_{1,f}(0,r) + kN_{1,f}(\infty,r) + O(1) \leq \\
&\leq (k+1)N_{1,f}^{\leq k}(0,r) + (k+1)N_{1,f}^{\geq (k+1)}(0,r) + kN_{1,f}(\infty,r) \leq \\
&\leq (k+1)N_{1,f}^{\geq (k+1)}(0,r) + kN_{1,f}(\infty,r) + \frac{k+1}{n-k-1}\big(N_{(f^n)^{(k)}}(0,r) \leq \\
&\leq -N_{1,(f^n)^{(k)}}(0,r) - (n-1)(k+1)N_{1,f}^{\geq (k+1)}(0,r)\big) + O(1).
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{n}{n-k-1}N_{1,(f^n)^{(k)}}(0,r) \leq \frac{k+1}{n-k-1}N_{(f^n)^{(k)}}(0,r) + kN_{1,f}(\infty,r) + \\
+ \left(k+1 - \frac{(k+1)^2(n-1)}{n-k-1}\right) N_{1,f}^{\geq (k+1)}(0,r) + O(1).
\end{aligned}$$

Note that

$$k+1 - \frac{(k+1)^2(n-1)}{n-k-1} < 0,$$

we have

$$N_{1,(f^n)^{(k)}}(0,r) \leq \frac{k+1}{n}N_{(f^n)^{(k)}}(0,r) + \frac{k(n-k-1)}{n}N_{1,f}(\infty,r) + O(1).$$

Moreover, if a is a pole of f with multiplicity p, then a is a pole of $(f^n)^{(k)}$ with multiplicity $np+k \geq n+k$. Thus

$$\frac{1}{n+k} N_{(f^n)^{(k)}}(\infty, r) \geq N_{1,f}(\infty, r), \quad N_{1,(f^n)^{(k)}}(\infty, r) = N_{1,f}(\infty, r).$$

Therefore,

$$\begin{aligned} T_{(f^n)^{(k)}}(r) \leq & \frac{k+1}{n} N_{(f^n)^{(k)}}(0, r) + \left(1 + \frac{k(n-k-1)}{n}\right) N_{1,(f^n)^{(k)}}(\infty, r) + \\ & + N_{1,(f^n)^{(k)}}(a, r) - \log r + O(1), \end{aligned}$$

$$\begin{aligned} T_{(f^n)^{(k)}}(r) \leq & \frac{k+1}{n} N_{(f^n)^{(k)}}(0, r) + \frac{n+k(n-k-1)}{(n+k)n} N_{(f^n)^{(k)}}(\infty, r) + \\ & + N_{1,(f^n)^{(k)}}(a, r) - \log r + O(1). \end{aligned}$$

From this and by Lemma 2.1, we have

$$T_{(f^n)^{(k)}}(r) \leq$$

$$\leq \left(\frac{k+1}{n} + \frac{n+k(n-k-1)}{(n+k)n}\right) T_{(f^n)^{(k)}}(r) + N_{1,(f^n)^{(k)}}(a, r) - \log r + O(1),$$

$$\left(1 - \frac{n+(n+k)(k+1)+k(n-k-1)}{n(n+k)}\right) T_{(f^n)^{(k)}}(r) \leq$$

$$\leq N_{1,(f^n)^{(k)}}(a, r) - \log r + O(1),$$

$$\left(1 - \frac{2(k+1)}{n+k}\right) T_{(f^n)^{(k)}}(r) \leq N_{1,(f^n)^{(k)}}(a, r) - \log r + O(1),$$

$$\frac{n-k-2}{n+k} T_f(r) \leq N_{1,(f^n)^{(k)}}(a, r) - \log r + O(1).$$

Lemma 3.4 is proved.

Lemma 3.5. *Let f and g be non-constant meromorphic functions on $\mathbb{K}$. If $E_f(1) = E_g(1)$, then one of the following three cases holds:*

$$\begin{aligned} 1) \quad T_f(r) \leq & N_{1,f}(\infty, r) + N_{1,f}^{\geq 2}(\infty, r) + N_{1,f}(0, r) + N_{1,f}^{\geq 2}(0, r) + \\ & + N_{1,g}(\infty, r) + N_{1,g}^{\geq 2}(\infty, r) + N_{1,g}(0, r) + N_{1,g}^{\geq 2}(0, r) - \\ & - \log r + O(1), \end{aligned}$$

and the same inequality holds for $T_g(r)$;

2) $f \equiv g$;
3) $fg \equiv 1$.

Proof. Set

$$F = \frac{1}{f-1}, \quad G = \frac{1}{g-1},$$

(3.3)
$$L = \frac{f''}{f'} - 2\frac{f'}{f-1} - \frac{g''}{g'} + 2\frac{g'}{g-1}.$$

Then

(3.4)
$$L = \frac{F''}{F'} - \frac{G''}{G'}.$$

Next we consider the following two cases:

Case 1. $L \not\equiv 0$. Since $E_f(1) = E_g(1)$, if $f(a) = 1, g(a) = 1$ and $\omega_f^1(a) = = \omega_g^1(a)$, then $L(a) = 0$. We now consider the poles of L. It is clear that all poles of L are of order 1. We can easily see from (3.3) that any simple pole of f and g is not a pole of L and the poles of L only occur at the zeros of f' and g', and the multiple poles of f and g.

From (3.3) we have

$$m_L(\infty, r) = O(1),$$

and

(3.5)
$$N_f^{\leq 1}(1, r) = N_g^{\leq 1}(1, r) \leq N_L(0, r) \leq T_L(r) + O(1) \leq N_L(\infty, r) + O(1).$$

On the other hand, by Lemma 2.2,

$$T_f(r) \leq N_{1,f}(\infty, r) + N_{1,f}(0, r) + N_{1,f}(1, r) - N_{0,f'}(r) - \log r + O(1),$$

where $N_{0,f'}(r)$ denotes the counting function of those zeros of f' but not that of $f(f-1)$. Also, $N_{1,0,f'}(r)$ is defined similarly, where each zero of f' is counted with multiplicity 1. From (3.3), (3.4) and (3.5) we deduce that

(3.6)
$$\begin{aligned} N_f^{\leq 1}(1, r) \leq & N_{1,f}^{\geq 2}(\infty, r) + N_{1,g}^{\geq 2}(\infty, r) + \\ & + N_{1,0,f'}(r) + N_{1,0,g'}(r) + N_{1,f}^{\geq 2}(0, r) + N_{1,g}^{\geq 2}(0, r) + O(1). \end{aligned}$$

Since $E_f(1) = E_g(1)$,

$$N_{1,f}(1, r) = N_f^{\leq 1}(1, r) + N_{1,g}^{\geq 2}(1, r).$$

Then
(3.7)
$T_f(r) \leq N_{1,f}(\infty, r) + N_{1,f}(0,r) + N_f^{\leq 1}(1,r) + N_{1,g}^{\geq 2}(1,r) - N_{0,f'}(r) - \log r + O(1).$

Now we consider $N_{1,g}^{\geq 2}(1,r)$.
By Lemma 2.1,

$$\begin{aligned} N_{g'}(0,r) - N_g(0,r) + N_{1,g}(0,r) &= N_{\frac{g'}{g}}(0,r) \leq T_{\frac{g'}{g}}(r) + O(1) = \\ &= N_{\frac{g'}{g}}(\infty, r) + m_{\frac{g'}{g}}(\infty, r) + O(1) = \\ &= N_{1,g}(\infty, r) + N_{1,g}(0,r) + O(1). \end{aligned}$$

Therefore

$$N_{g'}(0,r) \leq N_{1,g}(\infty, r) + N_g(0,r) + O(1).$$

Moreover

$$N_{0,g'}(r) + N_{1,g}^{\geq 2}(1,r) + N_g^{\geq 2}(0,r) - N_{1,g}^{\geq 2}(0,r) \leq N_{g'}(0,r).$$

The above two inequalities yield

$$N_{0,g'}(r) + N_{1,g}^{\geq 2}(1,r) \leq N_{1,g}(\infty, r) + N_{1,g}(0,r) + O(1).$$

Combining this inequality and (3.6) and (3.7), we obtain 1).

Case 2. $L \equiv 0$. Then

$$\frac{f''}{f'} - 2\frac{f'}{f-1} \equiv \frac{g''}{g'} - 2\frac{g'}{g-1}. \tag{3.8}$$

By (3.8) we have

$$\frac{F''}{F'} \equiv \frac{G''}{G'}.$$

Thus

$$f \equiv \frac{ag+b}{cg+d},$$

where $a, b, c, d \in \mathbb{K}$ satisfying $ad - bc \neq 0$. Then $T_f(r) = T_g(r) + O(1)$. Next we consider the following subcases:

Subcase 1. $ac \neq 0$. Then

$$f - \frac{a}{c} \equiv \frac{b - \frac{ad}{c}}{cg + d}.$$

By Lemma 2.3

$$T_f(r) \leq N_{1,f}(\infty, r) + N_{1,f-\frac{a}{c}}(0, r) + N_{1,f}(0, r) + O(1) =$$

$$= N_{1,f}(\infty, r) + N_{1,g}(\infty, r) + N_{1,f}(0, r) + O(1).$$

We get 1).

Subcase 2. $a \neq 0, c = 0$. Then $f \equiv \frac{ag+b}{d}$. If $b \neq 0$, by Lemma 2.2,

$$T_f(r) \leq N_{1,f}(\infty, r) + N_{1,f-\frac{b}{d}}(0, r) + N_{1,f}(0, r) + O(1) =$$

$$= N_{1,f}(\infty, r) + N_{1,g}(0, r) + N_{1,f}(0, r) + O(1).$$

We get 1).

If $b = 0$, then $f \equiv \frac{ag}{d}$. If $\frac{a}{d} = 1$, then $f \equiv g$. We obtain 2). If $\frac{a}{d} \neq 1$, then by $E_f(1) = E_g(1)$ and Lemma 2.2

$$f \neq 1, \quad f \neq \frac{a}{d},$$

$$T_f(r) \leq N_{1,f}(\infty, r) + N_{1,f}\left(\frac{a}{d}, r\right) + N_{1,f}(1, r) + O(1) =$$

$$= N_{1,f}(\infty, r) + O(1).$$

We get 1).

Subcase 3. $a = 0, c \neq 0$. Then $f \equiv \frac{b}{cg+d}$. If $d \neq 0$, by Lemma 2.2,

$$T_f(r) \leq N_{1,f}(\infty, r) + N_{1,f-\frac{b}{d}}(0, r) + N_{1,f}(0, r) + O(1) =$$

$$= N_{1,f}(\infty, r) + N_{1,g}(0, r) + N_{1,f}(0, r) + O(1).$$

We obtain 1).

If $d = 0$, then $f \equiv \frac{b}{cg}$. If $\frac{b}{c} = 1$, then $fg \equiv 1$. We obtain 3).

If $\frac{b}{c} \neq 1$, then by $E_f(1) = E_g(1)$ and Lemma 2.2,

$$f \neq 1, \quad f \neq \frac{b}{c},$$

$$T_f(r) \leq N_{1,f}(\infty, r) + N_{1,f}\left(\frac{b}{c}, r\right) + N_{1,f}(1, r) + O(1) =$$

$$= N_{1,f}(\infty, r) + O(1).$$

We get 1).

The proof of Lemma 3.5 is complete.

Lemma 3.6. *Let f be a non-constant meromorphic function on $\mathbb{K}$ and n, k be positive integers, $n > 2k$. Then*

1. $(n-2k)T_f(r) + kN_f(\infty, r) + N_{\frac{(f^n)^{(k)}}{f^{n-k}}}(0, r) \leq T_{(f^n)^{(k)}}(r) + O(1)$;
2. $N_{\frac{(f^n)^{(k)}}{f^{n-k}}}(0, r) \leq kT_f(r) + kN_{1,f}(\infty, r) + O(1)$.

Proof. 1. Set $A = (f^n)^{(k)}$. Then $A = f^{n-k}P$. By Lemma 3.1 we have

$$\begin{aligned} N_A(\infty, r) &= nN_f(\infty, r) + kN_{1,f}(\infty, r), \\ nN_f(\infty, r) &= N_A(\infty, r) - kN_{1,f}(\infty, r) \end{aligned} \tag{3.9}$$

From this and by Lemma 3.2 we see that

(3.10)

$$(n-k)m_f(\infty, r) = m_{f^{n-k}}(\infty, r) + O(1) = m_{\frac{A}{P}}(\infty, r) + O(1) \leq$$

$$\leq m_A(\infty, r) + m_{\frac{1}{P}}(\infty, r) + O(1) =$$

$$= m_A(\infty, r) + m_P(0, r) + O(1) = m_A(\infty, r) + T_P(r) - N_P(0, r) + O(1) =$$

$$= m_A(\infty, r) + N_P(\infty, r) + m_{\frac{P}{f^k}f^k}(\infty, r) - N_P(0, r) + O(1) \leq$$

$$\leq m_A(\infty, r) + kN_f(\infty, r) + km_f(\infty, r) + kN_{1,f}(\infty, r) - N_P(0, r) + O(1) =$$

$$= m_A(\infty, r) + kT_f(r) + kN_{1,f}(\infty, r) - N_P(0, r) + O(1).$$

From (3.9) and (3.10) we obtain

$$nN_f(\infty, r) + (n-k)m_f(\infty, r) =$$

$$= (n-k)(N_f(\infty, r) + m_f(\infty, r)) + kN_f(\infty, r) =$$

$$= (n-k)T_f(r) + kN_f(\infty, r) + O(1) \leq$$

$$= N_A(\infty, r) + m_A(\infty, r) - kN_{1,f}(\infty, r) + kT_f(r) + kN_{1,f}(\infty, r) -$$

$$- N_P(0, r) + O(1) =$$

$$= T_{(f^n)^{(k)}}(r) - N_P(0, r) + kT_f(r) + O(1).$$

Thus

$$(n-2k)T_f(r)+kN_f(\infty,r)+N_P(0,r)\le T_{(f^n)^{(k)}}(r)+O(1).$$

2. By Lemma 2.2, Lemma 3.1, Lemma 3.2, we get

$$\begin{aligned}
&N_P(0,r)\le T_P(r)+O(1)=\\
=&m_P(\infty,r)+N_P(\infty,r)+O(1)=\\
=&m_{\frac{P}{f^k}f^k}(\infty,r)+N_P(\infty,r)+O(1)\le\\
\le&m_{\frac{P}{f^k}}(\infty,r)+m_{f^k}(\infty,r)+N_P(\infty,r)+O(1)\le\\
\le&km_f(\infty,r)+N_P(\infty,r)+O(1)=\\
=&k(T_f(r)-N_f(\infty,r))+kN_{1,f}(\infty,r)+kN_f(\infty,r)+O(1)=\\
=&kT_f(r)+kN_{1,f}(\infty,r)+O(1).
\end{aligned}$$

So

$$N_{\frac{(f^n)^{(k)}}{f^{n-k}}}(0,r)\le kT_f(r)+kN_{1,f}(\infty,r)+O(1).$$

Now we use the above Lemmas to prove the main results of the paper.

Proof of Theorem 1.1. 1. Let f be an entire function, and $n\ge k+1$. Assume that f is non-constant. Then $T_f(r)\to\infty$ when $r\to\infty$. By Lemma 3.3 we see that $T_{(f^n)^{(k)}-1}(r)\to\infty$ when $r\to\infty$. By Lemma 2.1 we obtain

$$T_{(f^n)^{(k)}-1}(r)-T_{(f^n)^{(k)}-1}(\rho_0)=N_{(f^n)^{(k)}-1}(0,r),\ \text{where}\ \ 0<\rho_0\le r.$$

Therefore $N_{(f^n)^{(k)}-1}(0,r)\to\infty$ when $r\to\infty$, and $(f^n)^{(k)}-1$ must have a zero, a contradiction. So f is constant.

2. Let f be a meromorphic function, $n\ge k+2$. Assume that f is non-constant. Applying Lemma 3.4 to $(f^n)^{(k)}$ with the value 1, we conclude that

$$\frac{n-k-2}{n+k}T_f(r)+\log r+O(1)\le N_{1,(f^n)^{(k)}}(1,r).$$

Since f is non-constant, we see that $T_f(r)\to\infty$ when $r\to\infty$. From this and $n\ge k+2$, we have $\dfrac{n-k-2}{n+k}T_f(r)+\log r\to\infty$ when $r\to\infty$. Thus

$N_{1,(f^n)^{(k)}}(1,r) \to \infty$ when $r \to \infty$. Therefore, $(f^n)^{(k)} - 1$ must have a zero, a contradiction. So f is constant.

Proof of Theorem 1.2. Set

$$A = (f^n)^{(k)}, \quad B = (g^n)^{(k)}, \quad P = \frac{A}{f^{n-k}}, \quad Q = \frac{B}{g^{n-k}}.$$

Next we are applying Lemma 3.5 to $(f^n)^{(k)}$, $(g^n)^{(k)}$ with the following cases:

Case 1.

$$T_A(r) \le N_{1,A}(\infty,r) + N^{\ge 2}_{1,A}(\infty,r) + N_{1,A}(0,r) + N^{\ge 2}_{1,A}(0,r) + N_{1,B}(\infty,r)+$$

$$+N^{\ge 2}_{1,B}(\infty,r) + N_{1,B}(0,r) + N^{\ge 2}_{1,B}(0,r) - \log r + O(1),$$

$$T_B(r) \le N_{1,B}(\infty,r) + N^{\ge 2}_{1,B}(\infty,r) + N_{1,B}(0,r) + N^{\ge 2}_{1,B}(0,r) + N_{1,A}(\infty,r)+$$

$$+N^{\ge 2}_{1,A}(\infty,r) + N_{1,A}(0,r) + N^{\ge 2}_{1,B}(0,r) - \log r + O(1). \tag{3.11}$$

Note that

$$N_{1,A}(\infty,r) = N_{1,f}(\infty,r) = N^{\ge 2}_{1,A}(\infty,r),$$

$$N_{1,B}(\infty,r) = N_{1,g}(\infty,r) = N^{\ge 2}_{1,B}(\infty,r);$$

$$N_{1,A}(0,r) + N^{\ge 2}_{1,A}(0,r) \le 2N_{1,f}(0,r) + N_P(0,r),$$

$$N_{1,B}(0,r) + N^{\ge 2}_{1,B}(0,r) \le 2N_{1,g}(0,r) + N_Q(0,r).$$

From this and (3.11) we get

$$(n-2k)T_f(r) + kN_f(\infty,r) + N_P(0,r) \le 2N_{1,f}(\infty,r) + 2N_{1,f}(0,r) + N_P(0,r)+$$

$$+2N_{1,g}(\infty,r) + 2N_{1,g}(0,r) + N_Q(0,r) - \log r + O(1),$$

$$(n-2k)T_g(r) + kN_g(\infty,r) + N_Q(0,r)) \le 2N_{1,g}(\infty,r) + 2N_{1,g}(0,r) + N_P(0,r)+$$

$$+2N_{1,f}(\infty,r) + 2N_{1,f}(0,r) + N_Q(0,r) - \log r + O(1).$$

Combining the above inequalities we have

$$(n-2k)(T_f(r) + T_g(r)) + k(N_f(\infty,r) + N_g(\infty,r)) \le 4(N_{1,f}(\infty,r)+$$

$$+N_{1,f}(0,r) + N_{1,g}(\infty,r) + N_{1,g}(0,r)) + N_P(0,r) + N_Q(0,r) - 2\log r + O(1).$$

Note that

$$N_P(0,r) \le kT_f(r) + kN_{1,f}(\infty,r) + O(1),$$
$$N_Q(0,r) \le kT_g(r) + kN_{1,g}(\infty,r) + O(1),$$
$$N_{1,f}(\infty,r) \le N_f(\infty,r) \le T_f(r) + O(1),$$
$$N_{1,g}(\infty,r) \le N_g(\infty,r) \le T_g(r) + O(1),$$
$$N_{1,f}(0,r) \le N_f(0,r) \le T_f(r) + O(1),$$
$$N_{1,g}(0,r) \le N_g(0,r) \le T_g(r) + O(1).$$

So

$$\begin{aligned}&(n-2k)(T_f(r)+T_g(r)) + k(N_f(\infty,r)+N_g(\infty,r)) \le \\ \le& 4(N_{1,f}(\infty,r)+N_{1,f}(0,r)+N_{1,g}(\infty,r)+N_{1,g}(0,r)) + \\ &+ k(T_f(r)+T_g(r)) + k(N_{1,f}(\infty,r)+N_{1,g}(\infty,r)) - 2\log r + O(1),\end{aligned}$$

$$\begin{aligned}&(n-2k)(T_f(r)+T_g(r) \le \\ \le& k(T_f(r)+T_g(r)) + 4(N_{1,f}(\infty,r)+N_{1,g}(\infty,r)) + 4(N_{1,f}(0,r)+ \\ &+ N_{1,g}(0,r)) - 2\log r + O(1)) \le \\ \le& (k+4)(T_f(r)+T_g(r)) + 4(T_f(r)+T_g(r)) - 2\log r + O(1) \le \\ \le& (k+8)(T_f(r)+T_g(r)) - 2\log r + O(1).\end{aligned}$$

Therefore

$$(n-3k-8)(T_f(r)+T_g(r)) + 2\log r + O(1) \le 0.$$

As $n \ge 3k+8$, we obtain a contradiction.

Case 2. $(f^n)^{(k)}(g^n)^{(k)} = 1$. We prove $f \ne 0$, $f \ne \infty$, $g \ne 0$, $g \ne \infty$. Assume f has zeros. Let a be a zero of f with $\omega_f^0(a) = p$, $p \ge 1$. Then a is a pole of g with $\omega_g^\infty(a) = q$, $q \ge 1$, such that $np - k = nq + k$ and $n(p-q) = 2k$. From this and by $n \ge 3k+8$ we have a contradiction. By a similar argument we have $g \ne 0$, $f \ne \infty$, $g \ne \infty$. As f, g are non-constant we obtain a contradiction.

Case 3. $(f^n)^{(k)} = (g^n)^{(k)}$. Then $f^n = g^n + p$, where p is a polynomial of degree $< k$. We prove $p \equiv 0$. Assume $p \not\equiv 0$. Set $F = \frac{f^n}{p}$, $G = \frac{g^n}{p}$. Since f, g are transcendental, and p is a polynomial, we have

$$T_F(r) = T_{f^n}(r) + S_f(r), \quad T_G(r) = T_{g^n}(r) + S_g(r),$$

$$N_{1,F}(0,r) = N_{1,f^n}(0,r) + S_f(r);$$

$$N_{1,G}(0,r) = N_{1,g^n}(0,r) + S_g(r), \quad N_{1,F}(\infty,r) = N_{1,f^n}(\infty,r) + S_f(r),$$

$$N_{1,G}(\infty, r) = N_{1,g^n}(\infty, r) + S_g(r).$$

Then $F - 1 = G$. Note that by $f^n = g^n + p$ we have

$$T_f(r) = T_g(r) + S_g(r), \quad T_g(r) = T_f(r) + S_f(r), \quad S_f(r) = S_g(r).$$

By Lemma 2.2

$$T_F(r) \leq N_{1,F}(0, r) + N_{1,F}(\infty, r) + N_{1,F}(1, r) - \log r + O(1),$$

$$\begin{aligned} & T_{f^n}(r) = \\ =& nT_f(r) + O(1) \leq \\ \leq& N_{1,f^n}(0, r) + N_{1,f^n}(\infty, r) + N_{1,f^n}(1, r) + S_f(r) = \\ =& N_{1,f}(0, r) + N_{1,f}(\infty, r) + N_{1,g}(0, r) + S_f(r) \leq \\ \leq& 2T_f(r) + T_g(r) + S_f(r). \end{aligned}$$

Thus $nT_f(r) \leq 3T_f(r) + S_f(r)$, $(n-3)T_f(r) \leq S_f(r)$. From this and $n \geq 3k+8$ we obtain a contradiction. So $p = 0$. Therefore $f^n = g^n$ and $f = cg$ with $c^n = 1$.

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