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THE DISTRIBUTION OF ADDITIVE FUNCTIONS IN SHORT INTERVALS ON THE SET OF SHIFTED INTEGERS HAVING A FIXED NUMBER OF PRIME FACTORS

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Dedicated to Dr. Bui Minh Phong on his sixtieth birthday

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Abstract. Given a strongly additive function f, we establish short interval estimates for f on the set of shifted primes. We also consider similar sums, but running on sets of integers $m+1$, where each integer m has a fixed number of prime factors.

1. Introduction

Given integers $q \geq 2$ and $a \geq 0$, let

$$\psi(x;q,a) = \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \Lambda(n),$$

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where $\Lambda(n)$ stands for the von Mangoldt function. Let also ϕ stand for the Euler function. The well known Bombieri-Vinogradov theorem (see Bombieri [3] and Vinogradov [16]) provides an estimate for the error term in the Prime Number Theorem for arithmetic progressions, averaged over the moduli $q \le Q$; it can be stated as follows.

Bombieri-Vinogradov theorem. *Given an arbitrary number $A > 0$, there exists $B = B(A) > 0$ such that*

$$\max_{\substack{1\le q\le Q\\(a,q)=1}} \left|\psi(x;q,a) - \frac{x}{\phi(q)}\right| = O\left(\frac{x}{\log^A x}\right),$$

where $Q = \dfrac{\sqrt{x}}{\log^B x}$.

The problem of finding an estimate similar to the Bombieri-Vinogradov theorem for short intervals was first studied by Jutila [9] who obtained an estimate of the form

$$\sum_{q\le Q} \max_{\substack{1\le a\le q\\(a,q)=1}} \max_{h\le y} \max_{\frac{x}{2}\le z\le x} \left|\psi(z+h;q,a) - \psi(z;q,a) - \frac{h}{\phi(q)}\right| \ll \frac{y}{\log^A x}, \tag{1.1}$$

where, if we set $y = x^\theta$ and $Q = x^\eta/\log^B x$, the exponent η is bounded by a certain value which depends on θ and on

$$\inf\left\{\xi : \zeta\left(\frac{1}{2}+it\right) \ll t^\xi\right\},$$

where ζ is the Riemann zeta function. This estimate was later improved by various authors, namely Huxley & Iwaniec [8], Ricci [14], Perelli, Pintz & Salerno [12], [13], Zhan [17] and Timofeev [15]. Using the estimate obtained by Perelli, Pintz & Salerno [13], one can replace $\psi(x;q,a)$ by

$$\pi(x;q,a) := \#\{p \le x : p \equiv a \pmod q\}$$

in order to obtain the following version of the Bombieri-Vinogradov theorem for short intervals.

Theorem A.

$$\sum_{q\le Q} \max_{\substack{1\le a\le q\\(a,q)=1}} \max_{2\le h\le y} \max_{\frac{x}{2}\le z\le x} \left|\pi(z+h;q,a) - \pi(z;q,a) - \frac{\mathrm{li}(h)}{\varphi(q)}\right| \ll \frac{y}{\log^A x}, \tag{1.2}$$

where $y = x^{\frac{7}{12}+\varepsilon}$, $Q = x^{1/40}$ *and* $\mathrm{li}(x) := \int_0^x \frac{dt}{\log t}$. *Here* $A > 0$ *and* $\varepsilon > 0$ *are arbitrary constants, with the implied constants in* $\ll$ *depending only on* A *and* ε.

Recall the well known Erdős-Kac and Erdős-Wintner theorems.

Erdős-Kac theorem. *Let* $f(n)$ *be a strongly additive function and let* $\Phi(z) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\, dt$ *stand for the normal Gaussian distribution. Further set*

$$A(x) := \sum_{p \le x} \frac{f(p)}{p} \quad \textit{and} \quad B(x) := \sqrt{\sum_{p \le x} \frac{f^2(p)}{p}}$$

and assume that $B(x) \to \infty$ *as* $x \to \infty$. *Then,*

$$\lim_{x \to \infty} \frac{1}{x} \# \left\{ n \le x : \frac{f(n) - A(x)}{B(x)} \le z \right\} = \Phi(z).$$

The above result was established by Erdős and Kac in 1939 [5].

Erdős-Wintner theorem. *Let* $f(n)$ *be an additive function. Then,* f *possesses a distribution function if and only if each of the three series*

$$\sum_{|f(p)|>1} \frac{1}{p}, \qquad \sum_{|f(p)| \le 1} \frac{f(p)}{p}, \qquad \sum_{|f(p)| \le 1} \frac{f^2(p)}{p}$$

are convergent.

This result was established by Erdős and Wintner in 1939 [6].

2. First series of main results

Let $\varepsilon > 0$ be a fixed small number. Let $\pi(x)$ stand for the number of prime numbers not exceeding x. Let $I_{x,y} = [x, x+y]$, where $x^{\frac{7}{12}+\varepsilon} \le y \le x$, and let $\pi(I_{x,y}) := \sum_{p \in I_{x,y}} 1$. By using standard techniques, we can prove the following theorems.

Theorem 1. *Let g be a strongly multiplicative function such that $|g(p)| \leq \leq 1$ and $g(p) \to 1$ as $p \to \infty$. Assume that the infinite sum $\sum_p \frac{1-g(p)}{p}$ converges. Letting*

$$M(g) := \prod_p \left(1 + \frac{g(p)-1}{p-1}\right).$$

Then,

$$\max_{x^{7/12+\varepsilon} \leq y \leq x} \left| \frac{1}{\pi(I_{x,y})} \sum_{p \in I_{x,y}} g(p+1) - M(g) \right| \to 0 \quad as x \to \infty.$$

Theorem 2. *Let f be a strongly additive function such that $f(p) \neq 0$ for all primes p and such that $f(p) \to 0$ as $p \to \infty$. Let $A(x) = \sum_{p \leq x} \frac{f(p)}{p-1}$ and assume that $\sum_p \frac{f^2(p)}{p} < \infty$. Moreover, let*

$$\varphi(\tau) := \prod_p \left(1 + \frac{e^{i\tau f(p)} - 1}{p-1}\right) e^{-i\tau f(p)/(p-1)}$$

and let $F(u)$ be the distribution function whose characteristic function is $\varphi(\tau)$. Finally, let

$$F_{I_{x,y}}(u) := \frac{1}{\pi(I_{x,y})} \# \left\{ p \in I_{x,y} : f(p+1) - A(x) < y \right\}.$$

Then,

$$\lim_{x \to \infty} \max_{x^{7/12+\varepsilon} \leq y \leq x} \max_{u \in \mathbb{R}} \left| F_{I_{x,y}}(u) - F(u) \right| = 0.$$

Theorem 3. *Let f be a strongly additive function and set $A(x) = \sum_{p \leq x} \frac{f(p)}{p-1}$ and $B(x) = \sqrt{\sum_{p \leq x} \frac{f^2(p)}{p-1}}$. Assume that $B(x) \to \infty$ and that $\max_{p \leq x} \frac{|f(p)|}{B(x)} \to 0$ as $x \to \infty$. Then,*

$$\lim_{x \to \infty} \max_{x^{7/12+\varepsilon} \leq y \leq x} \max_{u \in \mathbb{R}} \left| \frac{1}{\pi(I_{x,y})} \# \left\{ p \in I_{x,y} : \frac{f(p+1) - A(x)}{B(x)} < u \right\} - \Phi(u) \right| = 0.$$

The second author proved [10] that if g is a multiplicative function satisfying $|g(n)| \le 1$ for all $n \ge 1$, and

$$\sum_{p\in\mathcal{P}} \frac{1-g(p)}{p} \quad \text{is convergent,} \tag{2.1}$$

and if $N(g)$ is the product

$$N(g) = \prod_p \left(1 - \frac{1}{p-1} + \sum_{k=1}^{\infty} \frac{g(p^k)}{p^k}\right),$$

then

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\le x} g(p+1) = N(g). \tag{2.2}$$

Hence, he deduced that if f is additive and satisfies the 3-series conditions (1.3), then the limit

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \#\{p \le x : f(p+1) < z\} = F(z)$$

exists for almost all $z \in \mathbb{R}$, meaning in other words that f has a limiting distribution on the set of shifted primes.

In the proof of this result, the Bombieri-Vinogradov inequality was used. However, the full strength of the inequality was not necessary. In fact, the inequality due to Barban [1] was sufficient, namely the following:

For a certain constant $\delta > 0$ and every fixed $A > 0$,

$$\sum_{k\le x^\delta} \mu^2(k) \max_{(\ell,k)=1} \left|\pi(x;k,\ell) - \frac{\mathrm{li}(x)}{\phi(k)}\right| < \frac{x}{\log^A x}.$$

In fact, any positive $\delta < \frac{3}{23}$ is admissible.

Now, the natural question is "can we deduce from Theorem A a short interval version of the Erdős-Wintner theorem for shifted primes or not?"

In fact, one could easily construct a strongly additive function f which is 0 on a set of primes $\wp_0$ such that $\sum_{p\in\wp_0} 1/p < \infty$ and such that $f(p) \in \{-1, 1\}$ for all $p \in \wp \setminus \wp_0$, while the short interval version of the Erdős-Wintner theorem for shifted primes does not hold.

Let us now assume that the condition (2.1) is complemented by the fact that $g(p) \to 1$ as $p \to \infty$. Then, the short interval version of (2.2) can be proved by the method of the second author applying Theorem A (see Theorem 1). Hence, we can deduce the following assertion:

Let f is an additive function such that $f(p) \to 0$ as $p \to \infty$. Further assume that both series

$$\sum_p \frac{f(p)}{p} \qquad \text{and} \qquad \sum_p \frac{f^2(p)}{p} \quad \text{converge.}$$

Then, the function $F_{I_{x,y}}(u) := \frac{1}{\pi(I_{x,y})}\#\{p \in I_{x,y} : f(p+1) < u\}$ has a limit distribution $F(u)$. Moreover, the characteristic function of F is given by

$$\varphi_F(\tau) = \prod_{p\in\mathcal{P}} \left(1 - \frac{1}{p-1} + \sum_{k=1}^{\infty} \frac{e^{i\tau f(p^k)}}{p^k}\right).$$

Theorem 4. *Let f be a strongly additive function such that $|f(p)| \le 1$ for all primes p. Let $h \in \mathbb{Z}[x]$. For each integer $d \ge 1$, let $\eta(d)$ denote the number of residue classes $r \pmod d$ which are coprime with d and which satisfy $h(r) \equiv 0 \pmod d$. Let also*

$$A(x) = \sum_{p\le x} \eta(p)\frac{f(p)}{p-1} \qquad \text{and} \qquad B(x) = \sqrt{\sum_{p\le x} \eta(p)\frac{f^2(p)}{p-1}}. \tag{2.3}$$

Assume that $B(x) \to \infty$ as $x \to \infty$. Then,

$$\lim_{x\to\infty} \max_{x^{7/12+\varepsilon}\le y\le x} \max_{u\in\mathbb{R}} \left| \frac{1}{\pi(I_{x,y})}\# \left\{ p \in I_{x,y} : \frac{f(|h(p)|) - A(x)}{B(x)} < u \right\} - \Phi(u) \right| = 0.$$

Theorem 5. *Let f be a strongly additive function such that $f(p) \to 0$ as $p \to \infty$. Let h and η be as in Theorem 4. Assume also that the two series*

$$\sum_p \eta(p)\frac{f(p)}{p-1} \qquad \textit{and} \qquad \sum_p \eta(p)\frac{f^2(p)}{p-1}$$

are convergent. It is known that the limit distribution

$$F(z) := \lim_{x\to\infty} \frac{1}{\pi(x)} \#\{p \le x : f(|h(p)|) < z\}$$

exists (see Theorem 12.14 in the book of Elliott [4]). Then,

$$\lim_{x\to\infty} \max_{x^{7/12+\varepsilon}\le y\le x} \max_{z\in\mathbb{R}} \left| \frac{1}{\pi(I_{x,y})} \#\left\{p \in I_{x,y} : f(|h(p)|) < z\right\} - F(z) \right| = 0.$$

Since the proofs of Theorems 1-5 can be obtained on the same way as their non short versions, we shall omit them.

Remark 1. *The condition* $f(p) \to 0$ *as* $p \to \infty$ *in Theorems 1, 2 and 5 and the condition* $\frac{1}{B(x)} \max_{p\le x} f(p) \to 0$ *as* $x \to \infty$ *in Theorems 3 and 4 allows us to evaluate* f *and* g*. We can prove a Turan-Kubilius type inequality and proceed in the usual way.*

Remark 2. *Observe that it can be shown that the above theorems remain true if we consider the values over the set* $ap+1$ *with* $p \in I_{x,y}$*, where* $1 \le a \le$ $\le x^{\varepsilon/2}$*, say.*

Given an integer $n \ge 2$, let $\omega(n)$ stand for the number of distinct prime factors of n and $\Omega(n)$ for the number of prime factors of n counting their multiplicity, and further set $\omega(1) = \Omega(1) = 0$. We now define

$$\begin{aligned} \wp_k &:= \{n \in \mathbb{N} : \omega(n) = k\}, \\ \mathcal{N}_k &:= \{n \in \mathbb{N} : \Omega(n) = k\}, \\ \Pi_k(I_{x,y}) &:= \#\{n \in I_{x,y} : \omega(n) = k\}, \\ N_k(I_{x,y}) &:= \#\{n \in I_{x,y} : \Omega(n) = k\}. \end{aligned}$$

In Kátai [11], it was proved that

$$\Pi_k(I_{x,y}) = (1 + o(1)) \frac{y}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!} \qquad (x \to \infty)$$

uniformly for positive integers $k \le \log\log x + c_x\sqrt{\log\log x}$, where c_x is a function which tends to infinity, but sufficiently slowly. Using essentially the same method, it was later proved by Bassily and Kátai [2] that, in the same range of k,

$$N_k(I_{x,y}) = (1 + o(1)) \frac{y}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!} \qquad (x \to \infty).$$

It is highly probable that the analogues of Theorems 1 through 5 hold if we replace the set of primes by the set of integers in $\wp_k$ uniformly for $k < < B \log\log x$ for any arbitrary fixed number B or by the set of integers $n \in \mathcal{N}_k$ uniformly for $k < (2-\varepsilon)\log\log x$ for any arbitrary small number $\varepsilon > 0$. Such a result would follow if we could prove the analogue of the short interval version of the Bombieri-Vinogradov theorem as in Theorem A, substituting p by $m \in \wp_k$ or $m \in \mathcal{N}_k$. But as of today, we cannot prove this. Nevertheless, we can prove that the analogues of Theorems 1-5 hold uniformly for $m \in \wp_k$ and $m \in \mathcal{N}_k$ uniformly for $k \le k_x$, provided $k_x^2/\log\log x \to 0$ as $x \to \infty$. To prove this, we need the following lemma, where $P(n)$ (resp. $P_2(n)$) stands for the largest (resp. second largest) prime factor of the integer $n \ge 2$, with $P(1) = P_2(1) = 1$.

Lemma 1. *Let $2 \le k_x \in \mathbb{N}$ be such that $\rho_x := k_x^2/\log\log x \to 0$ as $x \to \infty$. Set $\theta_x := \sqrt{\rho_x}$ and let*

$$\Pi_k^{(0)}(I_{x,y}) = \#\{n \in \mathcal{P}_k \cap I_{x,y} : P_2(n) \ge x^{\theta_x/2k}\},$$
$$N_k^{(0)}(I_{x,y}) = \#\{n \in \mathcal{N}_k \cap I_{x,y} : P_2(n) \ge x^{\theta_x/2k}\}.$$

Then,

$$\max_{2\le k\le k_x} \frac{\Pi_k^{(0)}(I_{x,y})}{\Pi_k(I_{x,y})} \to 0 \quad as \quad x \to \infty. \tag{2.4}$$

Similarly,

$$\max_{2\le k\le k_x} \frac{N_k^{(0)}(I_{x,y})}{N_k(I_{x,y})} \to 0 \quad as \quad x \to \infty. \tag{2.5}$$

Proof. We first prove (2.4). Let $\delta = \theta_x/2k$. Consider the integers $n = = p_1 \cdots p_k \in I_{x,y}$ with $p_1 < \cdots < p_k$ and let p_m be the largest of those prime factors satisfying $p_m < x^\delta$. If n is counted in $\Pi_k^{(0)}(I_{x,y})$, then $m \le k-2$ and $n = p_1 \cdots p_m \nu = a\nu$, say, where $P(\nu) > x^\delta$. We then have $a \le x^{\delta m} < x^{1/2}$. Hence, for fixed $p_1, \dots, p_m$, the number of such ν's is, by Mertens' theorem,

$$< \# \left\{ \nu : \left(\nu, \prod_{\pi \le x^\delta} \pi \right) = 1, \quad \frac{x}{a} \le \nu \le \frac{x}{a} + \frac{y}{a} \right\} \le$$
$$\le \frac{cy}{a} \prod_{\pi < x^\delta} \left(1 - \frac{1}{\pi}\right) \le \frac{c_1 y}{a\delta \log x}.$$

Summing over $m = 0$ (that is, when $a = 1$) and $m = 1, \dots, k-1$ and observing that, for fixed m,

$$\sum_{\substack{a=0\\ \omega(a)=m}}^{k-1} \frac{1}{a} < \frac{1}{(m-1)!}(\log\log x + c)^{m-1},$$

we obtain that

$$\Pi_k^{(0)}(I_{x,y}) \le \frac{c_2 y}{\delta \log x} \sum_{m=0}^{k-2} \frac{(\log\log x + c)^m}{m!} \le$$
$$\le \frac{c_3 y}{\delta \log x} \frac{(\log\log x + c)^{k-2}}{(k-2)!} \le c_4 \Pi_k(I_{x,y}) \frac{k-1}{\log\log x} \cdot \frac{1}{\delta}.$$

Since

$$\frac{k}{\log\log x} \cdot \frac{1}{\delta} \le \frac{2k_x^2}{\log\log x} \cdot \frac{1}{\theta_x} = 2\sqrt{\rho_x} \to 0 \quad as \quad x \to \infty,$$

estimate (2.4) follows immediately.

The proof of (2.5) is similar and will therefore be omitted.

3. Second series of main results

We can prove the following generalizations of Theorems 1 through 5.

Theorem 6. *Let g be as in the statement of Theorem 1 and let*

$$M_a(g) := \prod_{(p,a)=1} \left(1 + \frac{g(p)-1}{p-1}\right) \quad for \quad 1 \le a \le x^{\varepsilon}.$$

Then,

$$\max_{1 \le a \le x^{\varepsilon/2}} \max_{x^{7/12+\varepsilon} \le y \le x} \left| \frac{1}{\pi(I_{x,y})} \sum_{p \in I_{x,y}} g(ap+1) - M_a(g) \right| \to 0 \quad as \quad x \to \infty.$$

Theorem 7. *Let f, φ and F be as in Theorem 2 and let $A(x)$ be as in Theorem 2. Moreover, let*

$$F_{I_{x,y}}^{(k)}(u) := \frac{1}{\Pi_k(I_{x,y})} \#\{m \in I_{x,y} \cap \wp_k : f(m+1) - A(x) < u\}.$$

Then,

$$\lim_{x\to\infty}\sup_{k\le k_x}\max_{x^{7/12+\varepsilon}\le y\le x}\max_{u\in\mathbb{R}}\left|F^{(k)}_{I_{x,y}}(u)-F(u)\right|=0.$$

Theorem 8. *Let f, $A(x)$ and $B(x)$ be as in Theorem 3, with $B(x)\to\infty$ as $x\to\infty$. Moreover, let*

$$G^{(k)}_{I_{x,y}}(u):=\frac{1}{\Pi_k(I_{x,y})}\#\left\{m\in I_{x,y}\cap\wp_k:\frac{f(m+1)-A(x)}{B(x)}<u\right\}.$$

Then,

$$\lim_{x\to\infty}\sup_{k\le k_x}\max_{x^{7/12+\varepsilon}\le y\le x}\sup_{u\in\mathbb{R}}\left|G^{(k)}_{I_{x,y}}(u)-\Phi(u)\right|=0.$$

Theorem 9. *Let f, h, η, $A(x)$ and $B(x)$ be as in Theorem 4, with $B(x)\to\to\infty$ as $x\to\infty$. Moreover, let*

$$H^{(k)}_{I_{x,y}}(u):=\frac{1}{\Pi_k(I_{x,y})}\#\left\{m\in I_{x,y}\cap\wp_k:\frac{f(|h(m)|)-A(x)}{B(x)}<u\right\}.$$

Then,

$$\lim_{x\to\infty}\sup_{k\le k_x}\max_{x^{7/12+\varepsilon}\le y\le x}\sup_{u\in\mathbb{R}}\left|H^{(k)}_{I_{x,y}}(u)-\Phi(u)\right|=0.$$

Theorem 10. *Let f, h and η be as in Theorem 5. Moreover, assume that the two series*

$$\sum_p\eta(p)\frac{f(p)}{p-1}\qquad\text{and}\qquad\sum_p\eta(p)\frac{f^2(p)}{p-1}$$

are convergent, and let

$$F(z):=\lim_{x\to\infty}\frac{1}{\pi(x)}\#\{p\le x:f(|h(p)|)<z\}.$$

Then,

$$\lim_{x\to\infty}\max_{x^{7/12+\varepsilon}\le y\le x}\max_{z\in\mathbb{R}}\left|\frac{1}{\pi_k(I_{x,y})}\#\left\{m\in I_{x,y}\cap\wp_k:f(|h(m)|)<z\right\}-F(z)\right|=0.$$

Remark 3. *Using the result of Germán [7], one can prove the analogue of Theorems 6 and 7 for the non short interval case.*

4. Proof of Theorem 6

Since the proof of Theorem 6 is essentially a model for the proofs of Theorems 7-10, we will only prove Theorem 6.

Let $K_1(x) < K_2(x)$ be two numbers such that $\lim_{x\to\infty} K_1(x) = \infty$ and set $K_2(x) = x^\delta$, where $\delta > 0$ is a small number. For each number $H > 0$, set

$$g_H(p) = \begin{cases} g(p) & \text{if } p \le H, \\ 1 & \text{if } p > H. \end{cases}$$

Define implicitly the strongly additive function f by $g(p) = \exp\{if(p)\}$ for $f(p) \in [-\pi, \pi)$. Further define

$$f_H(p) = \begin{cases} f(p) & \text{if } p \le H, \\ 0 & \text{if } p > H. \end{cases}$$

In light of the condition $\lim_{p\to\infty} g(p) = 1$, we obtain that

$$\max_{n\in I_{x,y}} |g(n) - g_{K_2(x)}(n)| \to 0 \qquad \text{as} \quad x \to \infty.$$

Let x be a large number with corresponding numbers $K_1 < K_2$. Finally, set

$$u(n) = \sum_{\substack{p|n \\ K_1 \le p < K_2}} f(p).$$

Using (1.2), we can obtain a Turán-Kubilius type inequality. Indeed, letting

$$A_a := \sum_{\substack{p\in[K_1,K_2] \\ p\nmid a}} \frac{f(p)}{p-1}, \qquad B_a^2 = \sum_{\substack{p\in[K_1,K_2] \\ p\nmid a}} \frac{f^2(p)}{p-1},$$

we have

$$\sum_{p\in I_{x,y}} (u(ap+1) - A_a)^2 \le c\pi(I_{x,y}) B_a^2.$$

Now

$$A_a = A_1 - \sum_{\substack{p\in(K_1,K_2] \\ p|a}} \frac{f(p)}{p-1} = A_1 - D_a,$$

say. From the conditions stated in the theorem, we obtain that $B_a \to 0$ and $A_a \to 0$ uniformly for $a \le x^\varepsilon$ as $x \to \infty$. We have thus obtained that, uniformly for $a \in [1, x^\varepsilon]$,

$$\frac{1}{\pi(I_{x,y})} \sum_{p \in I_{x,y}} \left| g(ap+1) - g_{K_1}(ap+1)e^{-iD_a} \right| \to 0 \quad \text{as} \quad x \to \infty.$$

Now, let $g_{K_1}(n) = \sum_{d|n} h_{K_1}(d)$. Since g is strongly multiplicative, it follows that $h(p^\alpha) = 0$ if $\alpha \ge 2$ and also that $h_{K_1}(p) = 0$ if $p > K_1$.

Let us now choose $K_1 = \delta \log x$, where δ is a small positive number. Then, if $h_{K_1}(d) \ne 0$, we then have that

$$d \,\Big|\, \prod_{\pi \le K_1} \pi \le e^{2K_1} = x^{2\delta}, \qquad \text{so that} \;\; d \le x^{2\delta}.$$

On the other hand,

$$\sum_{p \in I_{x,y}} g_{K_1}(ap+1) = \sum_{(d,a)=1} h_{K_1}(d) \pi(I_{x,y} | d, \ell_d), \tag{4.1}$$

where ℓ_d is the solution of $a\ell_d + 1 \equiv 0 \pmod d$. Since $g(p) \to 1$ as $p \to \infty$, it follows that $h(p) \to 0$ as $p \to \infty$. Thus, $h_{K_1}(d)$ is bounded. Hence, from (1.2) and (4.1), we have

$$\sum_{p \in I_{x,y}} g_{K_1}(ap+1) = \sum_{(d,a)=1} \frac{h_{K_1}(d)}{\phi(d)} \pi(I_{x,y}) + o(\pi(I_{x,y})) = E_a \pi(I_{x,y}) + o(\pi(I_{x,y})), \tag{4.2}$$

say. But it is clear that

$$E_a = \prod_{\substack{p \le K_1 \\ p \nmid a}} \left(1 + \frac{g(p) - 1}{p - 1} \right).$$

Using this last estimate in (4.2) and recalling the definition of D_a, we obtain that

$$\frac{1}{\pi(I_{x,y})} \sum_{p \in I_{x,y}} g(ap+1) = e^{-iD_a} \prod_{\substack{p \le K_1 \\ p \nmid a}} \left(1 + \frac{g(p) - 1}{p - 1} \right) + o(1) \qquad (x \to \infty).$$

Now, observe that

$$\prod_{\substack{p>K_1\\p|a}}\left(1+\frac{g(p)-1}{p-1}\right)=\prod_{\substack{p>K_1\\p|a}}\left(1+\frac{e^{if(p)}-1}{p-1}\right)=$$
$$=\prod_{\substack{p>K_1\\p|a}}\left(1+\frac{if(p)}{p-1}+O\left(\frac{f^2(p)}{(p-1)^2}\right)\right)=$$
$$=e^{-iD_a}(1+o(1)).$$

Since

$$\prod_{p>K_1}\left(1+\frac{g(p)-1}{p-1}\right)\to 1 \quad \text{as} \quad x\to\infty,$$

the proof of Theorem 6 is complete.

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