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SOME REMARKS ON BEURLING TYPE INTEGERS GENERATED BY THE SET OF SHIFTED PRIMES

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Dedicated to Dr. Bui Minh Phong
on his sixtieth anniversary

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1. Introduction

The analogues of some theorems for additive and multiplicative functions are proved for Beurling type integers.

1.1. Notation and preliminary results

Let $\mathcal{P}$ be the whole set of the primes, $\omega(n)$ and $\Omega(n)$ be the number of prime factors, and the number of prime power factors of n, respectively. $\omega(n)$ is strongly additive, $\Omega(n)$ is completely additive function. Let

$$N_k(x) := \#\{n \le x \mid \Omega(n) = k\}. \tag{1.1}$$

Let $A(p)$ be a sequence of real numbers such that

$$0 < A(p) < Cp^{1-\Delta}, \tag{1.2}$$

where C and $\Delta < 1$ are arbitrary positive numbers.

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Let

$$\mathcal{P}^* = \{p + A(p) \mid p \in \mathcal{P}\},$$

and $\mathcal{N}_{\mathcal{P}^*}$ be the multiplicative semigroup with unit element 1 generated by $\mathcal{P}^*$. Let $\vartheta(p) = p + A(p)$, and ϑ be a completely multiplicative function over $\mathbb{N}$, i.e. if $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ $(p_1, \dots, p_r \in \mathcal{P})$, then $\vartheta(n) = \vartheta(p_1)^{\alpha_1} \dots \vartheta(p_r)^{\alpha_r}$.

Let

$$N_\vartheta(x) = \#\{\vartheta(n) \le x\}, \tag{1.3}$$

i.e. the number of those elements of $\mathcal{N}_{\mathcal{P}^*}$ which are not greater than x.

Let

$$\kappa(n) = \frac{\vartheta(n)}{n} = \prod_{j=1}^{r} \left(1 + \frac{A(p_j)}{p_j}\right)^{\alpha_j}, \quad \text{if } n = p_1^{\alpha_1} \dots p_r^{\alpha_r}.$$

Then

$$\begin{aligned} F_\vartheta(s) =& \sum \frac{1}{n^s \kappa(n)^s} = \prod_{p \in \mathcal{P}} \frac{1}{1 - \frac{1}{p^s \kappa^s(p)}} = \\ =& \zeta(s) \prod_{p \in \mathcal{P}} \frac{1 - \frac{1}{p^s}}{1 - \frac{1}{p^s \kappa(p)^s}}, \end{aligned}$$

where

$$\zeta(s) = \sum \frac{1}{n^s},$$

i.e.

$$F_\vartheta(s) = H_\vartheta(s)\, \zeta(s), \tag{1.4}$$

where

$$H_\vartheta(s) = \prod_p \frac{1 - \frac{1}{p^s}}{1 - \frac{1}{p^s \kappa(p)^s}}. \tag{1.5}$$

By using the argument of Bateman (see in Tenenbaum [1], II.5, Theorem 4, page 186) we obtain that

$$|N_\vartheta(x) - H_\vartheta(1)\, x| = \mathcal{O}\left(x \exp\left(-c_1 \sqrt{\log x}\right)\right), \tag{1.6}$$

where c_1 is a suitable positive constant.

Let Y be an arbitrary positive number, $\vartheta_Y(n) = n\kappa_Y(n)$, $\kappa_Y(n) = \prod_{\substack{p^\alpha || n \\ p<Y}} \kappa_Y(p^\alpha)$. Then, similarly as above,

$$F_{\vartheta_Y}(s) = \sum \frac{1}{\vartheta_Y(n)^s} = H_{\vartheta_Y}(s)\zeta(s),$$

$$H_{\vartheta_Y}(s) = \prod_{p \le Y} \frac{1-\frac{1}{p^s}}{1-\frac{1}{p^s\kappa(p)^s}}.$$

Then, for $N_{\vartheta_Y}(x) = \#\{\vartheta_Y(n) \le x\}$ we have

$$|N_{\vartheta_Y}(x) - H_{\vartheta_Y}(1)x| = \mathcal{O}\left(x \exp\left(-c_1\sqrt{\log x}\right)\right). \tag{1.7}$$

1.2. Main theorems

Let $\alpha = \prod_{i=1}^{r} \vartheta(p_i)^{a_i}$, $\beta = \prod_{j=1}^{h} \vartheta(q_j)^{b_j}$. We say that $U : \mathcal{N}_{\mathcal{P}^*} \to \mathbb{C}$ is completely multiplicative, if $U(\alpha\beta) = U(\alpha)\cdot U(\beta)$ holds for every $\alpha, \beta \in \mathcal{N}_{\mathcal{P}^*}$, and $U(1) = 1$. We say that $V : \mathcal{N}_{\mathcal{P}^*} \to \mathbb{R}$ is completely additive, if $V(\alpha\beta) = V(\alpha) + V(\beta)$ holds for every $\alpha, \beta \in \mathcal{N}_{\mathcal{P}^*}$, and $V(1) = 0$.

Assume that U is completely multiplicative in $\mathcal{N}(\mathcal{P}^*)$. Let us define $u(n) := U(\vartheta(n))$. Then $u(n)$ is completely multiplicative in $\mathbb{N}$. Similarly, if V is completely additive in $\mathcal{N}(\mathcal{P}^*)$, then $v(n) := V(\vartheta(n))$ is completely additive in $\mathbb{N}$.

The following analogue of the theorem of Halász holds.

Theorem 1. *Let $\mathcal{P}^*$, $\mathcal{N}_{\mathcal{P}^*}$ be defined as in 1.1. Let $G : \mathcal{N}_{\mathcal{P}^*} \to \mathbb{C}$ be completely multiplicative, $|G(\vartheta(n))| = 1$ $(\forall \vartheta(n) \in \mathcal{N}_{\mathcal{P}^*})$.*

Let

$$S(x) := \sum_{\vartheta(n) \le x} G(\vartheta(n)). \tag{1.8}$$

Then there exist a complex constant C_1, a real number τ, a slowly oscillating function $L_0(u)$, such that $|L_0(u)| = 1$, $\frac{L_0(u_1)}{L_0(u)} \to 1$ uniformly as $u \to \infty$, $u \le u_1 \le 2u$, such that

$$S(x) = C_1 x^{1+i\tau} L_0(\log x) + o(x). \tag{1.9}$$

Moreover, $\frac{S(x)}{N_\vartheta(x)} \to 0$ *as* $x \to \infty$ *if and only if*

$$\sum_{\vartheta(p)\in\mathcal{P}^*} \frac{1-\mathrm{Re}\left(G(\vartheta(p))\cdot\vartheta(p)^{-i\tau}\right)}{\vartheta(p)} \tag{1.10}$$

diverges for every real τ.

Assume that (1.10) is convergent for some τ. *Then*

$$\frac{S(x)}{N_\vartheta(x)} = C_2 x^{i\tau} L_0(\log x) + o_x(1) \qquad (x\to\infty). \tag{1.11}$$

The condition

$$\lim_{x\to\infty} \frac{S(x)}{N_\vartheta(x)} = M \neq 0 \tag{1.12}$$

holds if and only if

$$\sum_{\vartheta(p)\in\mathcal{P}^*} \frac{1-F(\vartheta(p))}{\vartheta(p)} \tag{1.13}$$

is convergent.

Remark 1. Theorem 1 remains true without almost any restriction for multiplicative, not only for completely multiplicative functions. The proof becomes somewhat more complicated.

Remark 2. A reformulation of Theorem 1 is the following

Theorem 1'. *Assume that the conditions of Theorem 1 hold. Let* $g(n) :=$ $:= G(\vartheta(n))$. *Then* g *is completely multiplicative in* $\mathbb{N}$, $|g(n)| = 1 \quad (n\in\mathbb{N})$,

$$S(x) = \sum_{\vartheta(n)\le x} g(n). \tag{1.14}$$

The sum

$$\sum_{\vartheta(p)\in\mathcal{P}^*} \left\{ \frac{\mathrm{Re}\left(G(\vartheta(p))\cdot\vartheta(p)^{-i\tau}\right)}{\vartheta(p)} - \frac{\mathrm{Re}\left(g(p)\cdot p^{-i\tau}\right)}{p} \right\} \tag{1.15}$$

is absolutely convergent, consequently (1.10) is divergent for some τ, if and only if

$$\sum_{p\in\mathcal{P}^*}\frac{1-\operatorname{Re}\left(g(p)\cdot p^{-i\tau}\right)}{p} \tag{1.16}$$

is divergent. The condition (1.13) is equivalent to the convergence of (1.17), where

$$\sum_{p\in\mathcal{P}^*}\frac{1-g(p)}{p}. \tag{1.17}$$

Moreover, $\frac{S(x)}{N_\vartheta(x)}\to 0$ as $x\to\infty$, if (1.15) is divergent for every real τ.

Assume that (1.16) is convergent for some τ. Then

$$\frac{S(x)}{N_\vartheta(x)} = C_2 x^{i\tau} L_0(\log x) + o_x(1) \qquad (x\to\infty). \tag{1.18}$$

The condition

$$\lim_{x\to\infty}\frac{S(x)}{N_\vartheta(x)} = M \neq 0 \tag{1.19}$$

holds if only if (1.13) is convergent.

2. Proof of Theorem 1'

Let Y be fixed, $Q=\prod_{\substack{p\le Y\\ p\in\mathcal{P}}} p$,

$$E(x) := \sum_{n\le x} g(n), \tag{2.1}$$

$$S_Y(x) = \sum_{\vartheta_Y(n)\le x} g(n). \tag{2.2}$$

Let

$$E(x \mid Q) = \sum_{\substack{n \le x \\ (n,Q)=1}} g(n). \tag{2.3}$$

It is clear that

$$\begin{aligned} E(x \mid Q) &= \sum_{n \le x} g(n) \sum_{\delta \mid (Q,n)} \mu(\delta) = \\ &= \sum_{\delta \mid Q} \mu(\delta)\, g(\delta)\, E\left(\frac{x}{\delta}\right) \end{aligned} \tag{2.4}$$

and furthermore

$$S_Y(x) = \sum_D g(D) \sum_{\substack{mD\kappa(D) \le x \\ (m,Q)=1}} g(m), \tag{2.5}$$

where D runs over the integers, the largest prime factor of which is at most Y.

Thus

$$S_Y(x) = \sum_D g(D)\, E\left(\frac{x}{D\kappa(D)} \;\Big|\; Q\right). \tag{2.6}$$

Let us assume first that $\frac{E(x)}{x} \to 0$ as $x \to \infty$. Then $\lim_{x\to\infty} \frac{E(x|Q)}{x} = 0$, furthermore

$$\limsup \frac{|S_Y(x)|}{x} \le \sum_{D > Y^Y} \frac{1}{D\kappa(D)} \le \frac{1}{Y^2}, \tag{2.7}$$

say. Since $\kappa(n) \ge \kappa_Y(n)$, therefore $\vartheta_Y(n) \le \vartheta(n)$, consequently

$$|S(x) - S_Y(x)| \le |H_{\vartheta_Y}(1) - H_\vartheta(1)|\, x + \mathcal{O}\left(x \exp\left(-c_1\sqrt{\log x}\right)\right) \tag{2.8}$$

(see (1.6), (1.7)).

Let us observe furthermore that $H_{\vartheta_Y}(1) \to H_\vartheta(1)$ as $Y \to \infty$. We have

$$\begin{aligned} \left|\frac{S(x)}{N_\vartheta(x)}\right| &\le \frac{|S_Y(x)|}{N_\vartheta(x)} + \frac{|S(x) - S_Y(x)|}{N_\vartheta(x)} \le \\ &\le c_1 \frac{|S_Y(x)|}{x} + c_2 \frac{|S(x) - S_Y(x)|}{x}, \end{aligned}$$

with constants c_1, c_2 which may depend only on ϑ.

From (2.7), (2.8) we obtain that

$$\limsup_{x\to\infty} \frac{|S(x)|}{N_\vartheta(x)} \le \frac{c_1}{Y^2} + c_2 |H_\vartheta(1) - H_{\vartheta_Y}(1)|. \tag{2.9}$$

Since the inequality (2.9) remains true for $Y \to \infty$, it follows that $\frac{S(x)}{N_\vartheta(x)} \to$ $\to 0 \quad (x \to \infty)$.

Assume that (1.16) is divergent for τ. Then $E(x) = Cx^{1+i\tau} L_0(\log x) +$ $+ o(x) \quad (x \to \infty)$, according to the theorem of G. Halász.

From (2.4) we obtain that

$$E(x \mid Q) = Cx^{1+i\tau} L_0(\log x) \prod_{p<Y} \left(1 - \frac{g(p)}{p^{1+i\tau}}\right), \tag{2.10}$$

$$S_Y(x) = Cx^{1+i\tau} \sum_{D<Y^Y} \frac{g(D)}{(D\kappa(D))^{1+i\tau}} L_0(\log x) \prod_{p<Y} \left(1 - \frac{g(p)}{p^{1+i\tau}}\right) + \\ + \mathcal{O}\left(\frac{x}{Y^2}\right).$$

Thus

$$\frac{S_Y(x)}{x} = Cx^{i\tau} L_0(\log x) \prod_{p<Y} \left(1 - \frac{g(p)}{p^{1+i\tau}}\right) \cdot \prod_{p<Y} \left(\frac{1}{1 - \frac{g(p)}{p^{1+i\tau}\kappa(p)^{1+i\tau}}}\right) + \\ + \mathcal{O}\left(\frac{1}{Y^2}\right).$$

Let

$$\eta(Y) = \prod_{p<Y} \frac{1 - \frac{g(p)}{p^{1+i\tau}}}{1 - \frac{g(p)}{p^{1+i\tau}\kappa(p)^{1+i\tau}}}.$$

Since $\kappa(p) = 1 + \mathcal{O}\left(p^{\Delta-1}\right)$, it follows that $\lim_{Y\to\infty} \eta(Y) = \eta$ exists and $\eta \ne 0$. Continuing as in the proof of the first assertion, we obtain that

$$\frac{S(x)}{N_\vartheta(x)} = C_1 \eta x^{i\tau} L_0(\log x) + o_x(1).$$

Here $C_1 = C \lim_{x\to\infty} \frac{N_\vartheta(x)}{x}$.

In the case (1.17) we have

$$\lim_{x\to\infty}\frac{E(x)}{x}=M_0\neq 0.$$

Arguing as above we obtain (1.19).

3. Analogues of the Erdős-Wintner and Erdős-Kac theorems

From Theorem 1' one can deduce the analogues of the Erdős-Wintner and the Erdős-Kac theorems.

Theorem 2 (Erdős-Wintner). *Let F be a completely additive function in $\mathcal{N}_{\mathcal{P}^*}$. Let*

$$H_x(y):=\frac{1}{N_\vartheta(x)}\#\{\vartheta(n)\le x \mid F(\vartheta(n))<y\}. \tag{3.1}$$

Then

$$\lim_{x\to\infty}H_x(y)=:H(y) \tag{3.2}$$

exists for almost all $y\in\mathbb{R}$, and $H(y)$ is a distribution function if and only if the next three series are convergent:

$$\sum_{|F(\vartheta(p))|\le 1}\frac{F(\vartheta(p))}{\vartheta(p)}, \tag{3.3}$$

$$\sum_{|F(\vartheta(p))|\le 1}\frac{F^2(\vartheta(p))}{\vartheta(p)}, \tag{3.4}$$

$$\sum_{|F(\vartheta(p))|>1}\frac{1}{\vartheta(p)}. \tag{3.5}$$

Let D_x be a sequence of real numbers, such that

$$T_x(y):=\frac{1}{N_\vartheta(x)}\#\{\vartheta(n)\le x \mid F(\vartheta(n))-D_x<y\} \tag{3.6}$$

tends to a distribution function $T(y)$ for almost all $y \in \mathbb{R}$. Then $D_x = Y_x + c + o_x(1)$, where c is an arbitrary real constant,

$$Y_x := \sum_{\substack{|\vartheta(p)|\le x \\ |F(\vartheta(p))|\le 1}} \frac{F(\vartheta(p))}{\vartheta(p)}, \tag{3.7}$$

furthermore (3.4), (3.5) are convergent.

In the opposite direction, if (3.4), (3.5) are convergent, then (3.6) has a limit for almost all y.

Theorem 3 (Erdős-Kac). *Let F be a completely additive function in $\mathcal{N}_{\mathcal{P}^*}$, $F(\vartheta(p)) = \mathcal{O}(1) \quad (\vartheta(p) \in \mathcal{P}^*)$. Let*

$$M_x := \sum_{\vartheta(p)\le x} \frac{F(\vartheta(p))}{\vartheta(p)}, \quad \sigma_x^2 = \sum_{\vartheta(p)\le x} \frac{F^2(\vartheta(p))}{\vartheta(p)}.$$

Then

$$\lim_{x\to\infty} \frac{1}{N_\vartheta(x)} \# \left\{ \vartheta(n) \le x \;\Big|\; \frac{F(\vartheta(p)) - M_x}{\sigma_x} < y \right\} = \phi(y)$$

holds for every $y \in \mathbb{R}$. Here ϕ is the Gaussian law.

Theorems 2 and 3 can be proved by reformulating these theorems for additive functions in $\mathbb{N}$, defining $f(n) := F(\vartheta(n))$, and applying Theorem 1' for the characteristic function $g_\tau(n) = e^{i\tau f(n)}$. We omit the details.

4. Counting $\vartheta(n)$ when n has a fixed number of prime factors

Let

$$N_{\vartheta,k}(x) := \#\{\vartheta(n) \le x \mid \Omega(n) = k\}. \tag{4.1}$$

Assume in this section that $A_p \ge 0$.

We shall write $\xi_k = \xi_k(x) = \frac{1}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!}$. Assume that for some positive constants ϱ, $\varrho \le \frac{k}{\log\log x} \le 2 - \varrho$. Let $\eta = \frac{k}{\log\log x}$.

As we know

$$N_k(x) = (1 + o_x(1))\, x\xi_k(x) \tag{4.2}$$

uniformly in the interval $\eta \in [\varrho, 2-\varrho]$. (See [1], Theorem 5, page 205.)

We shall prove that

$$N_{\vartheta,k}(x) = (1+o_x(1))\,\psi N_k(x) \tag{4.3}$$

uniformly in $\eta \in [\varrho, 2-\varrho]$. Here

$$\psi = \prod_p \frac{1-\frac{\eta}{p}}{1-\frac{\eta}{p\kappa(p)}}. \tag{4.4}$$

It is easy to show that in the interval $\eta \in [\varrho, 2-\varrho]$,

$$N_{k-l}(x) = (1+o_x(1))\,N_k(x)\,\eta^l \tag{4.5}$$

for every fixed l, furthermore

$$N_k(ax) = aN_k(x)(1+o_x(1)) \tag{4.6}$$

for every fixed $a>0$.

Let Y be a large constant, $Q = \prod\limits_{p\le Y} p$. Let

$$N_k(x \mid Q) = \#\{m \le x \mid (m,Q)=1\}. \tag{4.7}$$

Since

$$\begin{aligned} N_k(x \mid Q) &= \sum_{\substack{n\le x\\ \Omega(n)=k}} \sum_{\delta \mid (n,Q)} \mu(\delta) = \\ &= \sum_{\delta \mid Q} \mu(\delta)\, N_{k-\omega(\delta)}\left(\frac{x}{\delta}\right), \end{aligned} \tag{4.8}$$

from (4.5), (4.6) we have

$$N_k(x \mid Q) = (1+o_x(1))\,N_k(x) \sum_{\delta \mid Q} \frac{\mu(\delta)\,\eta^{\omega(\delta)}}{\delta}. \tag{4.9}$$

Let

$$\kappa_Y(n) = \prod_{\substack{p^\alpha \| n\\ p\le Y}} \left(1+\frac{A_p}{p}\right)^\alpha; \qquad g_2(n) = \prod_{\substack{p^\alpha \| n\\ p>Y}} \left(1+\frac{A_p}{p}\right)^\alpha, \ \vartheta_y(n) = n\kappa_Y(n),$$

$$N_{\vartheta_Y,k}(x) = \#\{\vartheta_Y(n) \le x,\ \Omega(n) = k\}. \tag{4.10}$$

Counting the elements in (4.10), write $n = Dm$, where the largest prime factor of D is less than Y, $(m, Q) = 1$. We have

$$N_{\vartheta_Y,k}(x) = \sum_{\substack{\kappa_Y(D)Dm \le x \\ \Omega(m)=k-\Omega(D)}} 1 = \sum_D N_{k-\Omega(D)}\left(\frac{x}{\kappa_Y(D)D} \;\Big|\; Q\right) = \\ = \Sigma_1 + \Sigma_2,$$

where in Σ_1 we sum over $D \le Y^Y$, and in Σ_2 over $D > Y^Y$.

From (4.9) we obtain that

$$\Sigma_1 = (1 + o_x(1)) \prod_{p|Q}\left(1 - \frac{\eta}{p}\right) \sum_{D<Y^Y} \frac{1}{\kappa_Y(D)D} N_k(x) = \\ = (1 + o_x(1))(1 + o_Y(1)) \prod_{p|Q} \frac{1 - \frac{\eta}{p}}{1 - \frac{1}{\kappa_Y(p)p}} N_k(x).$$

To estimate Σ_2, we subdivide Σ_2 as $\Sigma_{2,1} + \Sigma_{2,2}$, where in $\Sigma_{2,1}$ we sum over $Y^Y \le D < \sqrt{x}$, and in $\Sigma_{2,2}$ over $D > \sqrt{x}$. $\Sigma_{2,2}$ is clearly less than $\mathcal{O}\left(x^{\frac{3}{4}}\right)$, say. Using the Hardy-Ramanujan inequality according to which

$$N_k(x) \le c_1 x \xi_k(x),$$

uniformly as $k \le (2 - \varrho)\log\log x$, $c_1 = c_1(\varrho)$, we obtain that

$$\Sigma_{2,1} \le c_1 \sum_{D>Y^Y} \frac{\eta^{\Omega(D)}}{\kappa_Y(D)D} N_k(x).$$

Since

$$\sum_{D>Y^Y} \frac{\eta^{\Omega(D)}}{\kappa_Y(D)D} \le \frac{1}{Y^{\frac{Y}{2}}} \sum \frac{\eta^{\Omega(D)}}{\sqrt{D}} \le \frac{1}{Y^{\frac{Y}{2}}} \prod_{p<Y} \frac{1}{1 - \frac{\eta}{\sqrt{p}}} \le \\ \le \frac{1}{Y^{\frac{Y}{2}}} \exp\left(-\eta Y^{\frac{1}{3}}\right) \to 0 \quad \text{as} \quad Y \to \infty,$$

we have $\Sigma_{2,1} = o_Y(1) N_k(x)$.

Hence it follows that

$$N_{\vartheta_Y,k}(x) = (1 + o_x(1))(1 + o_Y(1)) \prod_{p|Q} \frac{1 - \frac{\eta}{p}}{1 - \frac{1}{\kappa_Y(p)}p} N_k(x), \tag{4.11}$$

uniformly as $\eta \in [\varrho, 2-\varrho]$.

We shall overestimate

$$\Sigma^{(0)} := \sum_{\substack{n \le x \\ \Omega(n)=k}} \log g_2(n).$$

Since $\log g_2(n) \le 2 \sum_{p^a \| n} \frac{A_p a}{p}$, we have

$$\begin{aligned} \Sigma^{(0)} &\le 2 \sum_{Y<p<x} \frac{A_p a}{p} N_{k-a}\left(\frac{x}{p^a}\right) \le \\ &\le 2 \sum_{\substack{p^a<\sqrt{x} \\ p>Y}} \frac{aA_p}{p^{a+1}} N_k(x) + \sqrt{x} \sum_{p>\sqrt{x}} \frac{A_p}{p^2} \le \\ &\le \frac{cN_k(x)}{Y^{\Delta}}. \end{aligned} \tag{4.12}$$

Thus

$$\#\left\{ n \le x,\ \Omega(n)=k \;\middle|\; \log g_2(n) \ge \frac{1}{Y^{\frac{\Delta}{2}}} \right\} \le c \frac{N_k(x)}{Y^{\frac{\Delta}{2}}}. \tag{4.13}$$

To estimate $N_{\vartheta,k}(x)$ we observe that $\vartheta(n) \ge \vartheta_Y(n)$, and so $N_{\vartheta_Y}(x) \ge N_\vartheta(x)$.

$$(0 \le)\ N_{\vartheta_Y}(x) - N_\vartheta(x) = \#\{n \mid \Omega(n)=k,\ \vartheta_Y(n) \le x \le \vartheta(n)\}. \tag{4.14}$$

If n is counted in the right hand side of (4.14), then either

(a) $$\log g_2(n) \ge \frac{1}{Y^{\frac{\Delta}{2}}}, \quad \text{i.e.} \quad g_2(n) \ge e^{\frac{1}{Y^{\frac{\Delta}{2}}}} \ge 1 + \frac{1}{Y^{\frac{\Delta}{2}}}$$

or

(b) $$\frac{x}{g_2(n)} \le \vartheta_Y(n) < x, \quad g_2(n) - 1 < \frac{1}{Y^{\frac{\Delta}{2}}}.$$

The size of the integers in (a) is less than $\frac{cN_k(x)}{Y^{\frac{\Delta}{2}}}$. From (b) we obtain that $\vartheta_Y(n) \in \left[x - \frac{2x}{Y^{\frac{\Delta}{2}}}, x\right]$, and so the size of the integers in (b) is less than $o_Y(1) N_k(x)$. This follows from (4.11). Let us observe that

$$\prod_{p \mid Q} \frac{1-\frac{\eta}{p}}{1-\frac{1}{p\kappa_Y(p)}} \to \psi \qquad \text{as } Y \to \infty.$$

Collecting our results we obtain that

$$\limsup_{x\to\infty}\left|\frac{N_{\vartheta,k}(x)}{N_k(x)}-\psi\right|\le o_Y(1)$$

uniformly as $\eta\in(\varrho,2-\varrho)$. Since Y is arbitrary large, therefore (4.3) is true.

By using the same method we are able to prove the following assertions.

Theorem 4. *Let g be a multiplicative function, $|g(n)|=1\quad(n\in\mathbb{N})$, assume that*

$$\sum_p\frac{1-g(p)}{p}$$

is convergent. Let

$$M_\eta(g)=\prod_p e_p(\eta),\quad e_p(\eta)=\left(1-\frac{\eta}{p}\right)\left(1+\frac{g(p)\eta}{p}+\frac{g(p^2)\eta^2}{p^2}+\cdots\right).$$

We have

$$\lim_{x\to\infty}\sup_{\eta=\frac{k}{\log\log x}\in[\varrho,2-\varrho]}\left|\frac{1}{N_{k,\vartheta}(x)}\sum_{\substack{\vartheta(n)\le x\\ \Omega(n)=k}}g(n)-M_\eta(g)\right|=0.$$

Theorem 5. *Let f be an additive function, assume that the "three series", i.e.*

$$\sum_{|f(p)|<1}\frac{f(p)}{p},\qquad\sum_{|f(p)|<1}\frac{f^2(p)}{p},\qquad\sum_{|f(p)|\ge1}\frac{1}{p}$$

are convergent.

For some $\eta\in(0,2)$ let $\xi_p=\xi_p(\eta)$ be the random variable distributed by $P(\xi_p=f(p^\alpha))=\left(1-\frac{\eta}{p}\right)\left(\frac{\eta}{p}\right)^\alpha\quad(\alpha=0,1,2,\dots)$. Assume that $\xi_p\quad(p\in\mathcal{P})$ are completely independent, $\theta(\eta):=\sum_p\xi_p(\eta)$. Let $F_\eta(y):=P(\theta(\eta)<y)$. Let furthermore

$$F_{k,x,\vartheta}(y):=\frac{1}{N_{k,\vartheta}(x)}\#\{\vartheta(n)\le x,\ \Omega(n)=k,\ f(n)<y\}.$$

Let $0<\varrho<\frac12$. Then

$$\lim_{x\to\infty}\max_{\eta=\frac{k}{\log\log x}\in[\varrho,2-\varrho]}\sup_{y\in\mathbb{R}}|F_{k,x,\vartheta}(y)-F_\eta(y)|=0.$$

Theorem 6. *Let f be an additive function bounded on the set of prime powers p^α. Let $A_x = \sum_{p\le x} \frac{f(p)}{p}$. Let $f(p)$ be additive defined on prime powers p^α by $f^*(p^\alpha) = f(p^\alpha) - \frac{\alpha A_x}{\log\log x}$. Let $B_x^2 = \sum_{p\le x} \frac{(f^*(p)(p))^2}{p}$. Let $B_x \to \infty$, $\eta = \frac{k}{\log\log x}$. Then*

$$\lim_{x\to\infty} \max_{\eta\in[\varrho,2-\varrho]} \left| \frac{1}{N_{k,\vartheta}(x)} \# \left\{ \vartheta(n) \le x,\ \Omega(n) = k,\ \frac{f^*(n)}{B_x\sqrt{\eta}} < y \right\} - \phi(y) \right| = 0.$$

Here $(0<)\,\varrho\,(<1)$ is an arbitrary constant, ϕ is the standard Gaussian law.

5. Further remarks

Let $A > 1$, $B > 0$ be fixed numbers, $\tilde{\mathcal{P}} = \{\tilde{\vartheta}(p) = Ap + B\}$, $\mathcal{N}_{\tilde{\mathcal{P}}}$ be the semigroup with unit elements 1 generated by the elements of $\tilde{\mathcal{P}}$.

We can obtain analogue theorems of Theorem 3, 4, 5 in this case. It is enough to observe that

$$\tilde{\vartheta}(n) < x,\ \Omega(n) = k$$

holds if and only if $\vartheta(n) < \frac{x}{A^k}$, $\Omega(n) = k$, where $\vartheta(p) = p + \frac{B}{A}$.

References

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