LAUDATION TO DR. BUI MINH PHONG ON HIS 60TH ANNIVERSARY

by I. Kátai

Bui Minh Phong was born in Vietnam on 11th of February of 1953. He studied in the Teacher's Training College in Eger, where he obtained his diploma in mathematics and physics in 1976. Continuing his studies he went to Hanoi where he obtained a master degree in mathematics at the Pedagogical University in 1978.

In the years 1978-1984 he worked as a first assistant and later as a lecturer at the Pedagogical University of Ho Chi Minh City. In 1985 he was working at the Mathematics Department of Teacher's Training College in Eger. Working from 1986 at the Eötvös Loránd University, he has been appointed to the associate professor in the Computer Algebra Department of the Faculty of Informatics in 2003.

His main scientific interest is number theory. Continuing his scientific carrier he earned the so called candidate degree in mathematics in 1987 by his dissertation "Lehmer sequence and Lehmer pseudoprime numbers". His supervisor was Professor Péter Kiss.

Until now he published 72 papers in different internationally recognized journals. The main topics he was interested are the following:

(1) Recurrence sequences ([1], [2], [3], [11], [14], [15], [16], [23], [24], [29], [33], [47])

(2) Pseudoprime numbers ([4], [5], [6], [7], [8], [9], [10], [11], [13], [17], [33], [35])

(3) Additive functions with regularity properties ([27], [30], [37], [50], [70], [71], [72])

(4) Unimodular multiplicative functions ([20], [28], [30], [34], [41], [51], [55])

(5) Reduced residue systems and a problem for multiplicative functions ([40], [42], [45], [48], [49], [53], [68])

(6) On the pairs of multiplicative functions with a special relation ([42], [49], [65], [68], [69])

(7) Continuous homomorphisms as additive functions with values in compact Abelian groups ([25], [26], [36])

(8) On a problem of Kátai and Subbarao ([61], [64])

(9) Additive uniqueness sets for arithmetical functions ([38, [39], [46], [59], [60], [62], [66])

(10) Arithmetical functions satisfying a congruence property ([12], [18], [19], [21], [22], [31], [32], [43], [44], [52], [54], [56], [57], [63], [67]).

Let $\mathcal{A}$ and $\mathcal{A}^*$ be the set of real additive (completely additive) functions. P. Erdős (1946) observed that if for a function $f \in \mathcal{A}$ either $f(n+1) - f(n) \to \to 0 \quad (n \to \infty)$ or $f(n)$ is monotonic, then $f(n) = c \log n$. Several mathematicians (e.g. J.-L. Mauclaire, P.D.T.A. Elliott, E. Wirsing, A. Hildebrand) were dealing with the generalization of this assertion including myself and Bui Minh Phong. He proved the following interesting theorems:

Theorem I. *Let a, b, c be positive integers and let d be a real constant. Then $f_1 \in \mathcal{A}$ and $f_2 \in \mathcal{A}$ satisfy the condition*

$$\liminf_{x \to \infty} \frac{1}{\log x} \sum_{n \le x} \frac{1}{n} \Big| f_1(an+b) - f_2(cn) - d \Big| = 0$$

if and only if there are a real constant U and functions $F_1 \in \mathcal{A}$, $F_2 \in \mathcal{A}$ such that $f_i(n) = U \log n + F_i(n) \quad (i = 1, 2)$ and

$$F_1(an+b) - F_2(cn) - d + U \log \left(\frac{a}{c} \right) = 0$$

holds for all positive integers n.

Theorem II. *Let a, b and c be positive integers. If $f_1 \in \mathcal{A}$ and $f_2 \in \mathcal{A}$ satisfy the condition*

$$f_1(an+b) - f_2(cn) = O(1) \quad \text{as} \quad n \to \infty,$$

then there are a real constant U and a function $F_1 \in \mathcal{A}$ such that

$$f_1(n) = U \log n + F_1(n), \qquad f_2(n) = U \log n + O(1)$$

and

$$F_1(an+b) = O(1)$$

hold for all positive integers n. In particular, we have

$$F_1(n) = O(1) \quad \textit{for all} \quad (n, a) = 1.$$

I started to investigate those complex valued multiplicative functions g, for which $g(n+1) - g(n) \to 0 \quad (n \to \infty)$, and formulated the conjecture that it holds if and only if either $g(n) \to 0 \quad (n \to \infty)$ or $g(n) = n^s$, where s is a complex number, Re $s < 1$. This has been proved by E. Wirsing.

Let $\mathcal{M}$ and $\mathcal{M}^*$ be the set of complex multiplicative (completely multiplicative) functions, and let $\mathcal{M}(1)$ and $\mathcal{M}^*(1)$ be that subclasses of $\mathcal{M}$ and $\mathcal{M}^*$ for which additionally $\mid g(n) \mid = 1 \quad (n \in \mathbb{N})$ is satisfied.

Bui Minh Phong obtained the following results concerning this topic:

Theorem III. *Let a, b, c be positive integers and let d be a non-zero complex number. Then the functions $g_1 \in \mathcal{M}(1)$ and $g_2 \in \mathcal{M}(1)$ satisfy the condition*

$$\sum_{n \le x} \frac{1}{n} |g_1(an+b) - dg_2(cn)| = o(\log x) \quad \textit{as} \quad x \to \infty$$

if and only if there are functions $g^* \in \mathcal{M}^*(1)$ and $G_1, G_2 \in \mathcal{M}(1)$ such that

$$g_1(n) = g^*(n)G_1(n),\ g_2(n) = g^*(n)G_2(n),\ G_1(an+b) - d\frac{g^*(c)}{g^*(a)}G_2(cn) = 0$$

for all $n \in \mathbb{N}$ and

$$\sum_{n \le x} \frac{1}{n} |g^*(n+1) - g^*(n)| = o(\log x) \quad \textit{as} \quad x \to \infty.$$

Theorem IV. *Let a, b, c be positive integers and let d be a non-zero complex number. Then the functions $g_1 \in \mathcal{M}(1)$ and $g_2 \in \mathcal{M}(1)$ satisfy the condition*

$$\sum_{n \le x} |g_1(an+b) - dg_2(cn)| = o(x) \quad \textit{as} \quad x \to \infty$$

if and only if there are functions $g^* \in \mathcal{M}^*(1)$, $G_1 \in \mathcal{M}(1)$ and $G_2 \in \mathcal{M}(1)$ such that

$$g_1(n) = g^*(n)G_1(n),\ g_2(n) = g^*(n)G_2(n),\ G_1(an+b) - d\frac{g^*(c)}{g^*(a)}G_2(cn) = 0$$

for all $n \in \mathbb{N}$ *and*

$$\sum_{n \le x} |g^*(n+1) - g^*(n)| = o(x) \quad as \quad x \to \infty.$$

Theorem V. *Let* a, b, c *be positive integers and let* d *be a non-zero complex number. Then the functions* $g_1 \in \mathcal{M}(1)$ *and* $g_2 \in \mathcal{M}(1)$ *satisfy the condition*

$$g_1(an+b) - dg_2(cn) = o(1) \quad as \quad n \to \infty$$

if and only if there are a real number τ *and functions* $G_1 \in \mathcal{M}(1)$, $G_2 \in \mathcal{M}(1)$ *such that*

$$g_1(n) = n^{i\tau} G_1(n), \qquad g_2(n) = n^{i\tau} G_2(n)$$

and

$$G_1(an+b) - d\frac{c^{i\tau}}{a^{i\tau}} G_2(cn) = 0$$

hold for all $n \in \mathbb{N}$.

Theorem VI. *Assume that* $A \in \mathbb{N}$, $C \in \mathcal{C} \setminus \{0\}$ *and* $f,\ g \in \mathcal{M}$ *satisfy the condition*

$$g(An+1) - Cf(n) = o(1) \quad as \quad n \to \infty.$$

Then either $f(n) = o(1)$ *and* $g(An+1) = o(1)$ *as* $n \to \infty$ *or there exist a complex number* s *and functions* $F,\ G \in \mathcal{M}$ *such that*

$$f(n) = n^s F(n), \quad g(m) = n^s G(n) \quad (0 \le \text{Re } s < 1),$$

and

$$G(An+1) = \frac{1}{F(2)} F(n)$$

are satisfied for all $n \in \mathbb{N}$.

Z. Daróczy and I started to investigate the regular behaviour of additive functions the values of which are elements of an additively written compact Abelian topological group. Bui Minh Phong continued the investigation and found interesting theorems.

Let G be an additively written, metrically compact Abelian topological group, $\mathbb{N}$ be the set of all positive integers. A function $f : \mathbb{N} \to G$ will be called additive, if $f(nm) = f(n) + f(m)$ holds for all coprime pairs $n,\ m \in \mathbb{N}$, while if the above relation holds for each couple of positive integers $n,\ m \in \mathbb{N}$, then we say it is completely additive. Let $\mathcal{A}_G$ and $\mathcal{A}_G^*$ denote the class of all additive, and the class of all completely additive functions, respectively.

Let $A > 0$ and $B \neq 0$ be fixed integers. We shall say that an infinite sequence $\{x_\nu\}_{\nu=1}^\infty$ in G is of property $D[A,B]$ if for any convergent subsequence $\{x_{\nu_n}\}_{n=1}^\infty$ the sequence $\{x_{A\nu_n+B}\}_{n=1}^\infty$ has a limit, too. We say that it is of property $E[A,B]$ if for any convergent subsequence $\{x_{A\nu_n+B}\}_{n=1}^\infty$ the sequence $\{x_{\nu_n}\}_{n=1}^\infty$ is convergent. We shall say that an infinite sequence $\{x_\nu\}_{\nu=1}^\infty$ in G is of property $\Delta[A,B]$ if the sequence $\{x_{A\nu+B} - x_\nu\}_{\nu=1}^\infty$ has a limit.

Let $\mathcal{A}_G(D[A,B])$, $\mathcal{A}_G(E[A,B])$ and $\mathcal{A}_G(\Delta[A,B])$ be the classes of those $f \in \mathcal{A}_G$ for which $\{x_\nu = f(\nu)\}_{\nu=1}^\infty$ is of property $D[A,B]$, $E[A,B]$ and $\Delta[A,B]$, respectively. The classes $\mathcal{A}_G^*(D[A,B])$, $\mathcal{A}_G^*(E[A,B])$ and $\mathcal{A}_G^*(\Delta[A,B])$ are defined as follows:

$$\mathcal{A}_G^*(D[A,B]) = \mathcal{A}_G(D[A,B]) \cap \mathcal{A}_G^*, \quad \mathcal{A}_G^*(E[A,B]) = \mathcal{A}_G(E[a,B]) \cap \mathcal{A}_G^*$$

and

$$\mathcal{A}_G^*(\Delta[A,B]) = \mathcal{A}_G(\Delta[a,B]) \cap \mathcal{A}_G^*.$$

It is obvious that

$$\mathcal{A}_G(\Delta[A,B]) \subseteq \mathcal{A}_G(D[A,B]), \quad \mathcal{A}_G(\Delta[A,B]) \subseteq \mathcal{A}_G(E[A,B])$$

and

$$\mathcal{A}_G^*(\Delta[A,B]) \subseteq \mathcal{A}_G^*(D[A,B]), \quad \mathcal{A}_G^*(\Delta[A,B]) \subseteq \mathcal{A}_G^*(E[A,B]).$$

Bui Minh Phong proved:

Theorem VII. *For any fixed integers $A > 0$ and $B \neq 0$, we have*

$$\mathcal{A}_G^*(E[A,B]) = \mathcal{A}_G^*(\Delta[A,B]).$$

If

$$f \in \mathcal{A}_G^*(E[A,B]) = \mathcal{A}_G^*(\Delta[A,B]),$$

then there exists a continuous homomorphism $\Phi : \mathbb{R}_* \to G$, $\mathbb{R}_*$ denotes the multiplicative group of the positive reals, such that f is a restriction of Φ on the set $\mathbb{N}$, i.e.

$$f(n) = \Phi(n)$$

for all $n \in \mathbb{N}$.

Conversely, let $\Phi : \mathbb{R}_* \to G$ *be an arbitrary continuous homomorphism. Then the function*

$$f(n) := \Phi(n) \quad (\textit{for all} \quad n \in \mathbb{N})$$

belongs to $\mathcal{A}_G^*(E[A,B]) = \mathcal{A}_G^*(\Delta[A,B])$.

Theorem VIII. *Let* $A > 0$ *and* $B \neq 0$ *be fixed integers for which* $(A,B) = 1$. *If* $f \in \mathcal{A}_G^*(D[A,B])$, *then there are* $U \in \mathcal{A}_G^*$ *and a continuous homomorphism* $\Phi : \mathbb{R}_* \to G$, $\mathbb{R}_*$ *denotes the multiplicative group of the positive reals, such that*

- $f(n) = \Phi(n) + U(n)$ *for all* $n \in \mathbb{N}$,
- $U(n + A) = U(n)$ *for all* $n \in \mathbb{N}, (n, A) = 1$.
- *If* X_1, Γ *denote the set of all limit points of* $\{\Phi(n)|n \in \mathbb{N}\}$ *and* $\{U(n)|n \in \mathbb{N}\}$, *respectively, then* $X_1 \cap \Gamma = \{0\}$ *and* Γ *is the smallest closed group generated by*

$$\{U(m) \mid 1 \le m \le A, (m, A) = 1\} \quad \textit{and} \quad \{U(p)| \quad p \quad \textit{is prime}, \quad p|A\}.$$

Conversely, let $\Phi : \mathbb{R}_* \to G$ *be an arbitrary continuous homomorphism,* X_1 *be the smallest compact subgroup generated by* $\{\Phi(n)|n \in \mathbb{N}\}$. *Let* $U \in \in \mathcal{A}_G^*$ *be so chosen that* $U(n + A) = U(n)$ *for all* $n \in \mathbb{N}$, $(n, A) = 1$ *and the smallest closed group* Γ *generated by* $U(\mathbb{N})$ *has the property* $X_1 \cap \Gamma = \{0\}$. *Then the function*

$$f(n) := \Phi(n) + U(n)$$

belongs to $\mathcal{A}_G^*(D[A,B])$.

In the following let := $\{\mathtt{z} \ : \ |\mathtt{z}| = 1\}$ and for each $k \in \mathbb{N}$ let $\mathcal{U}$ be the set of all k-th roots of unity, i.e. $\mathcal{U} = \{\omega \in \mathbb{C} \mid \omega^k = 1\ \}$.

In a joint paper with M.V. Subbarao we formulated the following conjecture:

Let $f \in \mathcal{M}^*(\mathbb{T})$ *and* $\mathcal{A}_f = \{\alpha_1, \ldots, \alpha_k\}$ *be the set of limit points of the sequence* $\{f(n+1)\overline{f}(n) \mid n \in \mathbb{N}\}$. *Then* $\mathcal{A}_f = \mathcal{U}$ *and* $f(n) = n^{i\tau}F(n)$ *with some* $\tau \in \mathbb{R}$, *where* $F(n)^k = 1$ *for all* $n \in \mathbb{N}$.

A somewhat weaker assertion has been proved by E. Wirsing.

Bui Minh Phong proved the following nice results.

Theorem IX. *If a function* $F \in \mathcal{M}^*$ *and a positive integer* $\ell \le 5$ *satisfy the condition* $F(\mathbb{N}) = \mathcal{U}_\ell$, *then* $\{F(n+1)\overline{F}(n) \mid n \in \mathbb{N}\} = \mathcal{U}_\ell$.

Theorem X. *Assume that $\mathbb{G}$ is any Abelian group and $F_1,\ F_2 \in \mathcal{M}^*(\mathbb{G})$. If*

$$\{F_1(An+B)(F_2(An))^{-1} \mid n \in \mathbb{N}\}$$

is a finite set, then there are finite subgroups $\mathbb{G}_1$ and $\mathbb{G}_2$ of $\mathbb{G}$ such that $\mathbb{G}_2 \subseteq \mathbb{G}_1$ and $F_i(\mathbb{N})$ is a subgroup of $\mathbb{G}_i$ (i=1,2).

Bui Minh Phong is very active in the scientific life. He visited several conferences, universities, worked together with other mathematicians.