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SOME CLASSES OF UNIFORMLY CONVERGENT INTERPOLATION PROCESSES ON THE ROOTS OF CHEBYSHEV POLYNOMIALS

Zsolt Németh (Budapest, Hungary)

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Abstract. In this paper we construct discrete processes on the roots of four kinds of Chebyshev polynomials supplemented with some endpoints of $[-1,1]$ by using suitable summations generated by a function φ. Our aim is to investigate these methods regarding the interpolation property and uniform convergence in the Banach space $(C[-1,1], \|\cdot\|_\infty)$. With proper conditions on φ we obtain wide classes of interpolation processes which are uniformly convergent for every function $f \in C[-1,1]$.

1. Introduction

Let $C[-1,1]$ denote the linear space of continuous functions defined on $[-1,1]$. Then $(C[-1,1], \|\cdot\|_\infty)$ is a Banach space, where

$$\|f\|_\infty := \max_{x\in[-1,1]} |f(x)| \qquad (f \in C[-1,1])$$

is the maximum norm.

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In order to construct discrete processes we first define some point systems. For this we will use the roots of four kinds of Chebyshev polynomials supplemented with some endpoints of $[-1,1]$. Let $M \in \mathbb{N}_0 := \{0,1,2,\dots\}$ and introduce the following notations and definitions.

Definition 1.1. Let

$$T_M(x) := \cos(M \arccos x) \quad (x \in [-1,1])$$

be the M-th *Chebyshev polynomial of the first kind.* The point system $\mathbf{X}_M^{\mathcal{T}}$ is defined by

$$\mathbf{X}_M^{\mathcal{T}} := \Big\{ x_{k,M} := \cos \frac{2k-1}{2M}\pi \ : \ k = 1,\dots,M \Big\},$$

i.e. $\mathbf{X}_M^{\mathcal{T}}$ are the roots of the Chebyshev polynomials of the first kind.

Definition 1.2. Let

$$U_M(x) := \frac{\sin((M+1)\arccos x)}{\sin(\arccos x)} \qquad (x \in [-1,1])$$

be the M-th *Chebyshev polynomial of the second kind.* The point system $\mathbf{X}_M^{\overline{\mathcal{U}}}$ is defined by

$$\mathbf{X}_M^{\overline{\mathcal{U}}} := \Big\{ x_{k,M+2} := \cos \frac{k-1}{M+1}\pi \ : \ k = 1,\dots,M+2 \Big\},$$

i.e. $\mathbf{X}_M^{\overline{\mathcal{U}}}$ are the roots of the Chebyshev polynomials of the second kind supplemented with the endpoints -1 and 1, so they are the roots of the weighted polynomial

$$\overline{U}_M(x) := \sqrt{1-x^2}\, U_M(x) \qquad (x \in [-1,1]). \tag{1.1}$$

Definition 1.3. Let

$$V_M(x) := \frac{\cos((M+\frac{1}{2})\arccos x)}{\cos(\frac{1}{2}\arccos x)} \qquad (x \in [-1,1])$$

be the M-th *Chebyshev polynomial of the third kind.* The point system $\mathbf{X}_M^{\overline{\mathcal{V}}}$ is defined by

$$\mathbf{X}_M^{\overline{\mathcal{V}}} := \Big\{ x_{k,M+1} := \cos \frac{2k-1}{2M+1}\pi \ : \ k = 1,\dots,M+1 \Big\},$$

i.e. $\mathbf{X}_M^{\overline{\mathcal{V}}}$ are the roots of the Chebyshev polynomials of the third kind supplemented with -1, so they are the roots of the weighted polynomial

$$\overline{V}_M(x) := \sqrt{1+x}\, V_M(x) \qquad (x \in [-1,1]). \tag{1.2}$$

Definition 1.4. Let

$$W_M(x) := \frac{\sin((M+\frac{1}{2})\arccos x)}{\sin(\frac{1}{2}\arccos x)} \qquad (x \in [-1,1])$$

be the M-th *Chebyshev polynomial of the fourth kind.* The point system $\mathbf{X}_M^{\overline{\mathcal{W}}}$ is defined by

$$\mathbf{X}_M^{\overline{\mathcal{W}}} := \Big\{ x_{k,M+1} := \cos\frac{2(k-1)}{2M+1}\pi \ : \ k = 1,\dots,M+1 \Big\},$$

i.e. $\mathbf{X}_M^{\overline{\mathcal{W}}}$ are the roots of the Chebyshev polynomials of the fourth kind supplemented with 1, so they are the roots of the weighted polynomial

$$\overline{W}_M(x) := \sqrt{1-x}\, W_M(x) \qquad (x \in [-1,1]). \tag{1.3}$$

Note that the indices of the point systems may differ from the number of nodes.

In the next step we define four bases consisting the functions defined above. We are going to use these to construct our approximating functions.

Definition 1.5. Let us define the following four bases in $(C[-1,1], \|\cdot\|_\infty)$:

$$\mathcal{T}_n := \{T_0, T_1, \dots, T_n\};$$
$$\overline{\mathcal{U}}_n := \{\overline{U}_{-2}, \overline{U}_{-1}, \dots, \overline{U}_n\};$$
$$\overline{\mathcal{V}}_n := \{\overline{V}_{-1}, \overline{V}_0, \dots, \overline{V}_n\};$$
$$\overline{\mathcal{W}}_n := \{\overline{W}_{-1}, \overline{W}_0, \dots, \overline{W}_n\},$$

where $\overline{U}_{-2} = \overline{V}_{-1} = \overline{W}_{-1} := 1$, and $\overline{U}_{-1}(x) := x$.

For the sake of simplicity we will use the notations $T_{-2} = T_{-1} = \overline{V}_{-2} = \overline{W}_{-2} := 0$. We remark that the cases when $\mathcal{T}_n$ is used as a basis were already investigated in [10], so we will omit the proofs for these cases.

2. Constructions of discrete processes

In the construction of discrete processes for approximating a function $f \in C[-1,1]$, we will determine an interpolating function of f (using a minimal number of base functions) for a fixed number of nodes, and later apply summation in that form.

Let $M \geq 1,\ M \in \mathbb{N}$ and let $\overline{\mathcal{B}}_{M-1} \in \{\mathcal{T}_{M-1}, \overline{\mathcal{U}}_{M-1}, \overline{\mathcal{V}}_{M-1}, \overline{\mathcal{W}}_{M-1}\}$ denote one of the bases and $\mathbf{X}_M^{\overline{\mathcal{B}}}$ be the corresponding point system. Since our bases are mixtures of polynomials and weighted polynomials, we will call the linear combination

$$\big(L_M^{\overline{\mathcal{B}}} f\big)(x) = \sum_{j=-2}^{M-1} c_{j,M}^{\overline{\mathcal{B}}}(f) \cdot \overline{B}_j(x), \tag{2.1}$$

the *Lagrange interpolation* of $f \in C[-1,1]$ for the point system $\mathbf{X}_M^{\overline{\mathcal{B}}}$, if it interpolates f at the points of $\mathbf{X}_M^{\overline{\mathcal{B}}}$.

2.1. Lagrange interpolation

In this section we show that the Lagrange interpolation of f on the point system $\mathbf{X}_M^{\overline{\mathcal{B}}}$ is unique and we determine the $c_{j,M}^{\overline{\mathcal{B}}}(f)$ coefficients in the sum (2.1) for all of the four cases.

Lemma 2.1. *If $\overline{\mathcal{B}}_{M-1} = \mathcal{T}_{M-1}$ then $\boldsymbol{X}_M^{\overline{\mathcal{B}}} = \boldsymbol{X}_M^{\mathcal{T}}$ and*

$$c_{0,M}^{\mathcal{T}}(f) = \frac{1}{M} \sum_{k=1}^{M} f(x_{k,M}) T_0(x_{k,M});$$

$$c_{j,M}^{\mathcal{T}}(f) = \frac{2}{M} \sum_{k=1}^{M} f(x_{k,M}) T_j(x_{k,M}),$$

where $j \in \mathbb{N},\ 0 < j < M$.

Lemma 2.2. *If $\overline{\mathcal{B}}_{M-1} = \overline{\mathcal{U}}_{M-1}$ then $\boldsymbol{X}_M^{\overline{\mathcal{B}}} = \boldsymbol{X}_M^{\overline{\mathcal{U}}}$ and*

$$c_{-2,M}^{\overline{\mathcal{U}}}(f) = \frac{f(x_{1,M+2}) + f(x_{M+2,M+2})}{2};$$

$$c_{-1,M}^{\overline{\mathcal{U}}}(f) = \frac{f(x_{1,M+2}) - f(x_{M+2,M+2})}{2};$$

$$c_{j,M}^{\overline{\mathcal{U}}}(f) = \frac{2}{M+1} \sum_{k=1}^{M+2} \Big[f(x_{k,M+2}) - c_{-1,M}^{\overline{\mathcal{U}}}(f) \cdot x_{k,M+2} - c_{-2,M}^{\overline{\mathcal{U}}}(f) \Big] \overline{U}_j(x_{k,M+2}),$$

where $j \in \mathbb{N}_0,\ j < M$.

Note that the $k = 1$ and $k = M+2$ terms are always 0 so it is optional to include them in the formulas.

Proof. Let

$$g(x) := f(x) - \frac{f(1) - f(-1)}{2}x - \frac{f(1) + f(-1)}{2}.$$

Observe that $g(-1) = g(1) = 0$.

The weighted interpolation polynomial of g in the basis

$$\{\sqrt{1-x^2} \cdot U_0, \sqrt{1-x^2} \cdot U_1, .., \sqrt{1-x^2} \cdot U_{M-1}\}$$

(cf. [2, p. 164]) can be written as

$$\begin{aligned}\big(L_M^U g\big)(x) &= \sqrt{1-x^2}\times \\ &\times \sum_{j=0}^{M-1}\left[\sum_{k=2}^{M+1} g(x_{k,M+2}) \cdot U_j(x_{k,M+2}) \cdot 2 \cdot \frac{\sqrt{1-x_{k,M+1}^2}}{M+1}\right] \cdot U_j(x) = \\ &= \sum_{j=0}^{M-1}\Big[\frac{2}{M+1}\sum_{k=2}^{M+1} g(x_{k,M+2}) \cdot \overline{U}_j(x_{k,M+2})\Big]\overline{U}_j(x).\end{aligned}$$

So we know that $\big(L_M^U g\big)(x)$ is 0 at the points $\{-1, 1\}$ and interpolates g at $x_{k,M+2}, (k = 2, .., M+1)$. Using the definition of g we can construct the Lagrange interpolation polynomial of f in the form

$$\big(L_M^{\overline{\mathcal{U}}} f\big)(x) = \frac{f(1) + f(-1)}{2} + \frac{f(1) - f(-1)}{2}x + \big(L_M^U g\big)(x).$$

Then the proof can be finished by collecting the coefficients. ■

Lemma 2.3. *If* $\overline{\mathcal{B}}_{M-1} = \overline{\mathcal{V}}_{M-1}$ *then* $\boldsymbol{X}_M^{\overline{\mathcal{B}}} = \boldsymbol{X}_M^{\overline{\mathcal{V}}}$ *and*

$$c_{-1,M}^{\overline{\mathcal{V}}}(f) = f(x_{M+1,M+1});$$

$$c_{j,M}^{\overline{\mathcal{V}}}(f) = \frac{2}{2M+1}\sum_{k=1}^{M+1}\Big[f(x_{k,M+1}) - c_{-1,M}^{\overline{\mathcal{V}}}\Big]\overline{V}_j(x_{k,M+1}),$$

where $j \in \mathbb{N}_0,\ j < M$.

Proof. Let

$$g(x) := f(x) - f(-1).$$

Then $g(-1) = 0$, and the statement can be proved analogously to Lemma 2.2 by using the interpolation formulae from [2, p. 166]. ■

Lemma 2.4. *If* $\overline{\mathcal{B}}_n = \overline{\mathcal{W}}_n$ *then* $\boldsymbol{X}_M^{\overline{\mathcal{B}}} = \boldsymbol{X}_M^{\overline{\mathcal{W}}}$ *and*

$$c_{-1,M}^{\overline{\mathcal{W}}}(f) = f(x_{1,M+1});$$

$$c_{j,M}^{\overline{\mathcal{W}}}(f) = \frac{2}{2M+1} \sum_{k=1}^{M+1} \left[f(x_{k,M+1}) - c_{-1,M}^{\overline{\mathcal{W}}} \right] \overline{W}_j(x_{k,M+1}),$$

where $j \in \mathbb{N}_0,\ j < M$.

Proof. Let

$$g(x) := f(x) - f(1).$$

Then $g(1) = 0$, and the statement can be proved analogously to Lemma 2.2 by using the interpolation formulae from [2, p. 166]. ■

Notice that in the above cases we received the $c_{j,M}^{\overline{\mathcal{B}}}(f)$ coefficients for $j < M$ only. From now on, let us define $c_{j,M}^{\overline{\mathcal{B}}}(f)$ for all $j \in \mathbb{N}_0$ by the formulas in the previous four lemmas.

2.2. Summation functions and discrete processes

We will investigate summation processes generated by a function φ. First denote by Φ the set of *summation functions* $\varphi : [-1, +\infty) \to \mathbb{R}$ satisfying the following requirements:

i) supp $\varphi \subset [-1, 1]$;

ii) if $t \in [-1, 0)$ then $\varphi(t) := 1$;

iii) $\lim\limits_{t \to 0+} \varphi(t) = \varphi(0) := 1$ and $\lim\limits_{t \to 1-} \varphi(t) = \varphi(1) := 0$;

iv) the limits

$$\varphi(t_0 \pm 0) := \lim_{t \to t_0 \pm 0} \varphi(t)$$

exists and are finite in every $t_0 \in (0, +\infty)$;

v) for all $t \in \mathbb{R}$ the function value $\varphi(t)$ lies in the closed interval determined by $\varphi(t-0)$ and $\varphi(t+0)$.

Condition iii) ensures that every $\varphi \in \Phi$ is Riemann integrable on [0,1] (cf. [9, p. 161]). Therefore φ is continuous except at most countable many points of $[0, 1]$.

Now let $f \in C[-1, 1]$ and let us fix the natural numbers $M, n \geq 2$. Let $\overline{\mathcal{B}}_n \in$ $\in \left\{ \mathcal{T}_n, \overline{\mathcal{U}}_n, \overline{\mathcal{V}}_n, \overline{\mathcal{W}}_n \right\}$ be a basis and $\mathbf{X}_M^{\overline{\mathcal{B}}}$ be the corresponding point system.

Consider the least-degree interpolation (2.1). For a function $\varphi \in \Phi$, let us define

$$(2.2)\qquad \big(S^{\varphi,\overline{\mathcal{B}}}_{n,M} f\big)(x) := \sum_{j=-2}^{n} \varphi\Big(\frac{j}{n}\Big) \cdot c^{\overline{\mathcal{B}}}_{j,M}(f) \cdot \overline{B}_j(x),$$

where the coefficients $c^{\overline{\mathcal{B}}}_{j,M}(f)$ are defined at the end of the previous section. Note that we also weakened the degree constraint of the Lagrange interpolation.

2.3. Properties of the coefficients

We will investigate the interpolatory properties and uniform convergence of summation processes defined in (2.2). In order to do so, first we need to take a closer look at the coefficients.

Theorem 2.1. *The coefficients $c^{\overline{\mathcal{B}}}_{j,M}(f)$ have the following properties:*

i) *Symmetry property:*

$$c^{\overline{\mathcal{B}}}_{M+j,M}(f) = (-1) \cdot c^{\overline{\mathcal{B}}}_{M-j,M}(f)$$

for any $\overline{\mathcal{B}} \in \{\mathcal{T}, \overline{\mathcal{U}}, \overline{\mathcal{V}}, \overline{\mathcal{W}}\}$, $0 \le j \le M,\ j \in \mathbb{N}$.

ii) *Periodicity:*

$$\begin{gathered} c^{\mathcal{T}}_{j,M}(f) = (-1) \cdot c^{\mathcal{T}}_{j+2M,M}(f); \\ c^{\overline{\mathcal{U}}}_{j,M}(f) = c^{\overline{\mathcal{U}}}_{j+2M+2,M}(f); \\ c^{\overline{\mathcal{V}}}_{j,M}(f) = (-1) \cdot c^{\overline{\mathcal{V}}}_{j+2M+1,M}(f); \\ c^{\overline{\mathcal{W}}}_{j,M}(f) = c^{\overline{\mathcal{W}}}_{j+2M+1,M}(f) \end{gathered}$$

for any $j \in \mathbb{N}_0$.

iii) *Zero coefficients:*

$$c^{\overline{\mathcal{B}}}_{M,M}(f) = 0$$

for any $\overline{\mathcal{B}} \in \{\mathcal{T}, \overline{\mathcal{U}}, \overline{\mathcal{V}}, \overline{\mathcal{W}}\}$. Also, we have

$$c^{\overline{\mathcal{U}}}_{2M+1,M}(f) = 0.$$

Proof. i) For $\overline{\mathcal{B}} = \overline{\mathcal{U}}$, observe that

$$U_{M+j}(x_{k,M+2}) = \frac{\sin((M+j+1)\frac{k-1}{M+1}\pi)}{\sin(\frac{k-1}{M+1}\pi)},$$

where

$$\sin\left[(M+j+1)\frac{k-1}{M+1}\pi\right] = \sin\left[\Big((2M+2)-(M-j+1)\Big)\frac{k-1}{M+1}\pi\right] =$$
$$= \sin\left[(2M+2)\frac{k-1}{M+1}\pi\right]\cdot\cos\left[(M-j+1)\frac{k-1}{M+1}\pi\right] -$$
$$-\cos\left[(2M+2)\frac{k-1}{M+1}\pi\right]\cdot\sin\left[(M-j+1)\frac{k-1}{M+1}\pi\right] =$$
$$= -\sin\left[(M-j+1)\frac{k-1}{M+1}\pi\right].$$

Using this, we have

$$U_{M+j}(x_{k,M+2}) = (-1)\cdot U_{M-j}(x_{k,M+2}), \tag{2.3}$$

and considering Lemma 2.2 one can now easily see that

$$c^{\overline{\mathcal{U}}}_{M+j,M}(f) = (-1)\cdot c^{\overline{\mathcal{U}}}_{M-j,M}(f).$$

The same method can be used to prove i) for $\overline{\mathcal{B}} \in \{\mathcal{T}, \overline{\mathcal{V}}, \overline{\mathcal{W}}\}$.

ii) Now observe that for the point system $\mathbf{X}^{\overline{\mathcal{U}}}_M$ we have

$$\sin\left[(j+1+2M+2)\frac{k-1}{M+1}\pi\right] =$$
$$= \sin\left[(j+1)\frac{k-1}{M+1}\pi\right]\cdot\cos\left[(2M+2)\frac{k-1}{M+1}\pi\right] +$$
$$+\cos\left[(j+1)\frac{k-1}{M+1}\pi\right]\cdot\sin\left[(2M+2)\frac{k-1}{M+1}\pi\right] =$$
$$= \sin\left[(j+1)\frac{k-1}{M+1}\pi\right],$$

which means that

$$U_j(x_{k,M+2}) = U_{j+2M+2}(x_{k,M+2}).$$

Considering Lemma 2.2 we can see that

$$c^{\overline{\mathcal{U}}}_{j,M}(f) = c^{\overline{\mathcal{U}}}_{j+2M+2,M}(f).$$

The remaining three periodic properties can be proved the same way.

iii) It is obvious that

$$U_M(x_{k,M+2}) = 0$$

when $x_{k,M+2} \in \mathbf{X}^{\overline{\mathcal{U}}}_M$.

Then

$$c^{\overline{\mathcal{U}}}_{M,M}(f)=0$$

immediately follows, and the same holds for the systems $\mathbf{X}^{\mathcal{T}}_M, \mathbf{X}^{\overline{\mathcal{V}}}_M, \mathbf{X}^{\overline{\mathcal{W}}}_M$, so

$$c^{\overline{\mathcal{B}}}_{M,M}(f)=0.$$

Finally, from

$$\sin\left[(2M+2)\frac{k-1}{M+1}\pi\right]=0,$$

we have $U_{2M+1}(x_{k,M+2})=0$, which means $c^{\overline{\mathcal{U}}}_{2M+1,M}(f)=0$. ■

3. Interpolatory properties

In this section we give a necessary and sufficient condition on the summation function regarding the interpolatory property of the processes defined before. Our result is analogous to [8, Lemma 3].

For the basis $\overline{\mathcal{B}}_n$ and the corresponding point system $\mathbf{X}^{\overline{\mathcal{B}}}_M$ we have already defined the function $S^{\varphi,\overline{\mathcal{B}}}_{n,M}f$ for a given $\varphi\in\Phi$ (see (2.2)). Some of these interpolate f at the points $\mathbf{X}^{\overline{\mathcal{B}}}_M$. We give a necessary and sufficient condition for φ satisfying this requirement.

Theorem 3.1. *Let $M\ge 2$ be an integer, $M\le n\le 2M$, $\overline{\mathcal{B}}_n\in\{\mathcal{T}_n,\overline{\mathcal{U}}_n,\overline{\mathcal{V}}_n,\overline{\mathcal{W}}_n\}$, $\boldsymbol{X}^{\overline{\mathcal{B}}}_M$ be the corresponding point system and $\varphi\in\Phi$ be a summation function. Then $S^{\varphi,\overline{\mathcal{B}}}_{n,M}f$ interpolates $f:[-1,1]\to\mathbb{R}$ at the points of $\boldsymbol{X}^{\overline{\mathcal{B}}}_M$ if and only if*

$$\varphi\Big(\frac{j}{n}\Big)+\varphi\Big(1-\frac{j}{n}\Big)=1\qquad (j=0,1,\dots,n;\ j\ne M).$$

(Note that if $M=n$ then $\varphi(M/n)=0$, otherwise it is arbitrary.)

Proof. We show the statement for $\overline{\mathcal{B}}=\overline{\mathcal{U}}$ only, the other cases are similar. Since

$$\varphi\Big(\frac{-2}{n}\Big)=\varphi\Big(\frac{-1}{n}\Big)=1,$$

we have by Lemma 2.2 that $S^{\varphi,\overline{\mathcal{U}}}_{n,M}f$ interpolates f at the points $\{-1,1\}$ for every $\varphi\in\Phi$.

Let

$$g(x):=f(x)-\frac{f(1)-f(-1)}{2}x-\frac{f(1)+f(-1)}{2},$$

and observe that $g(1) = g(-1) = 0$ so

$$c^{\overline{\mathcal{U}}}_{-2,M}(g) = c^{\overline{\mathcal{U}}}_{-1,M}(g) = 0.$$

$S^{\varphi,\overline{\mathcal{U}}}_{n,M} f$ interpolates f at the points of $\mathbf{X}^{\overline{\mathcal{U}}}_M$ if and only if $S^{\varphi,\overline{\mathcal{U}}}_{n,M} g$ interpolates g at the same points. Therefore it is enough to prove the theorem for the latter one.

Let us define the summation function $\nu \in \Phi$ as follows

$$\nu(x) := \begin{cases} \varphi\left(\frac{2M}{n}x\right), & x \le \frac{n}{2M}; \\ 0, & x > \frac{n}{2M}. \end{cases}$$

Now considering the equality

$$\nu\left(\frac{j}{2M}\right) = \varphi\left(\frac{j}{n}\right) \qquad (j = 0, 1, \ldots, n)$$

we get $S^{\varphi,\overline{\mathcal{U}}}_{n,M} g = S^{\nu,\overline{\mathcal{U}}}_{2M,M} g$, so it is enough to prove the statement for $n := 2M$.

We have

$$\begin{aligned} \left(S^{\varphi,\overline{\mathcal{U}}}_{2M,M} g\right)(x) &= \sum_{j=0}^{2M} \varphi\Big(\frac{j}{2M}\Big) \cdot c^{\overline{\mathcal{U}}}_{j,M}(g) \cdot \overline{U}_j(x) = \\ = \sum_{j=0}^{2M} \varphi\Big(\frac{j}{2M}\Big) &\cdot \Big[\frac{2}{M+1} \sum_{k=1}^{M+2} g(x_{k,M+2}) \cdot \overline{U}_j(x_{k,M+2})\Big] \cdot \overline{U}_j(x) = \\ = \sum_{k=1}^{M+2} g(x_{k,M+2}) &\cdot \Big\{\frac{2}{M+1} \sum_{j=0}^{2M} \varphi\Big(\frac{j}{2M}\Big) \cdot \overline{U}_j(x_{k,M+2}) \cdot \overline{U}_j(x)\Big\}. \end{aligned}$$

Let us introduce the notation

$$\ell^{\varphi,\overline{\mathcal{U}}}_{k,M+2}(x) := \frac{2}{M+1} \sum_{j=0}^{2M} \varphi\Big(\frac{j}{2M}\Big) \cdot \overline{U}_j(x_{k,M+2}) \cdot \overline{U}_j(x).$$

One can prove that the polynomial $S^{\varphi,\overline{\mathcal{U}}}_{2M,M} g$ interpolates g at the points of $\mathbf{X}^{\overline{\mathcal{U}}}_M$ if and only if

$$\ell^{\varphi,\overline{\mathcal{U}}}_{k,M+2}(x_{l,M+2}) = \delta_{k,l} \qquad (k, l = 1, 2, \ldots, M+2). \tag{3.1}$$

Similarly, we can write the Lagrange interpolation polynomial of g from Lemma 2.2 in the form

$$L^{\overline{\mathcal{U}}}_M(x) = \sum_{k=1}^{M+2} g(x_{k,M+2}) \cdot \Big\{\frac{2}{M+1} \sum_{j=0}^{M-1} \overline{U}_j(x_{k,M+2}) \cdot \overline{U}_j(x)\Big\}.$$

Set

$$\ell^{\overline{\mathcal{U}}}_{k,M+2}(x) := \frac{2}{M+1}\sum_{j=0}^{M-1}\overline{U}_j(x_{k,M+2})\cdot\overline{U}_j(x).$$

Since $L^{\overline{\mathcal{U}}}_M(x)$ interpolates g at the points of $\mathbf{X}^{\overline{\mathcal{U}}}_M$, we have

$$\ell^{\overline{\mathcal{U}}}_{k,M+2}(x_{l,M+2}) = \delta_{k,l} \qquad (k,l = 1,2,\dots,M+2). \tag{3.2}$$

Note that in the sum $\ell^{\varphi,\overline{\mathcal{U}}}_{k,M+2}(x)$ the $j = M$ member is 0. Consider the following transformation:

$$\begin{aligned}\ell^{\varphi,\overline{\mathcal{U}}}_{k,M+2}(x) &= \frac{2}{M+1}\sum_{j=0}^{M-1}\overline{U}_j(x_{k,M+2})\cdot\overline{U}_j(x)+\\ &+\frac{2}{M+1}\sum_{j=0}^{M-1}\Big[\varphi\Big(\frac{j}{2M}\Big)-1+\varphi\Big(1-\frac{j}{2M}\Big)\Big]\cdot\overline{U}_j(x_{k,M+2})\cdot\overline{U}_j(x)-\\ &-\frac{2}{M+1}\sum_{j=0}^{M-1}\varphi\Big(1-\frac{j}{2M}\Big)\cdot\overline{U}_j(x_{k,M+2})\cdot\overline{U}_j(x)+\\ &+\frac{2}{M+1}\sum_{j=M+1}^{2M}\varphi\Big(\frac{j}{2M}\Big)\cdot\overline{U}_j(x_{k,M+2})\cdot\overline{U}_j(x).\end{aligned}$$

Observe that the first sum is $\ell^{\overline{\mathcal{U}}}_{k,M+2}(x)$ from the Lagrange interpolation polynomial of g. Also note that the last two sums have common values of φ, so they add up to

$$\frac{2}{M+1}\sum_{j=M+1}^{2M}\varphi\Big(\frac{j}{2M}\Big)\cdot C(x);$$

$$C(x) = \overline{U}_j(x_{k,M+2})\cdot\overline{U}_j(x) - \overline{U}_{2M-j}(x_{k,M+2})\cdot\overline{U}_{2M-j}(x).$$

Now we consider $\ell^{\varphi,\overline{\mathcal{U}}}_{k,M+2}(x_{l,M+2})$. From (2.3) we can immediately prove that

$$C(x_{k,M+2}) = 0.$$

Hence

$$\ell^{\varphi,\overline{\mathcal{U}}}_{k,M+2}(x_{l,M+2}) = \ell^{\overline{\mathcal{U}}}_{k,M+2}(x_{l,M+2}) + A(x_{l,M+2});$$

$$A(x) = \frac{2}{M+1}\sum_{j=0}^{M-1}\Big[\varphi\Big(\frac{j}{2M}\Big)-1+\varphi\Big(1-\frac{j}{2M}\Big)\Big]\cdot\overline{U}_j(x_{k,M+2})\cdot\overline{U}_j(x).$$

From (3.2) it follows immediately that $S_{2M,M}^{\varphi,\overline{\mathcal{U}}} g$ interpolates g at the points of $\mathbf{X}_M^{\overline{\mathcal{U}}}$ if and only if

$$A(x_{l,M+2}) = 0 \qquad (l = 1, 2, \ldots, M+2).$$

Then $A(x)$ has M distinct roots on the interval $(-1,1)$, and it can be written in the form $A(x) = \sqrt{1-x^2} \cdot P(x)$, where $P(x)$ is a polynomial of degree at most $M-1$. Since $\sqrt{1-x^2} \neq 0$ for $x \in (-1,1)$ we have that $P(x)$ is the zero polynomial. One can easily see that it is true if and only if

$$\varphi\Big(\frac{j}{2M}\Big) - 1 + \varphi\Big(1 - \frac{j}{2M}\Big) = 0 \qquad (j = 0, 1, \ldots, M-1),$$

which proves the statement. ■

4. Convergence

In this section we show that if the Fourier transform of the summation function φ is Lebesgue integrable on $\mathbb{R}_0^+ := [0, \infty)$ then for a wide range of sequences of (2.2) we have uniform convergence on $[-1,1]$ for all $f \in C[-1,1]$. This can be considered as the analogoue of the well-known theorem of G. M. Natanson and V. V. Zuk (see [3]).

Denote by $L^1(\mathbb{R}_0^+)$ the linear space of measurable functions $g : \mathbb{R}_0^+ \to \mathbb{R}$ for which the Lebesgue integral

$$\int_{\mathbb{R}_0^+} |g|$$

is finite.

The functional

$$\|g\|_{L^1(\mathbb{R}_0^+)} := \int_0^{+\infty} |g(x)| dx \qquad (g \in L^1(\mathbb{R}_0^+))$$

is a norm on $L^1(\mathbb{R}_0^+)$ and $\Big(L^1(\mathbb{R}_0^+), \|\cdot\|_{L^1(\mathbb{R}_0^+)}\Big)$ is a Banach space.

The Fourier transform of $g \in L^1(\mathbb{R}_0^+)$ is defined by

$$\hat{g}(x) := \frac{1}{2\pi} \int_0^{+\infty} g(t) \cos(tx) dt \qquad (x \in \mathbb{R}_0^+).$$

With these definitions at hand, we prove the following convergence theorem:

Theorem 4.1. *Let $\overline{\mathcal{B}}_n \in \{\mathcal{T}_n, \overline{\mathcal{U}}_n, \overline{\mathcal{V}}_n, \overline{\mathcal{W}}_n\}$ be a basis and $\boldsymbol{X}_M^{\overline{\mathcal{B}}}$ be the corresponding point system. Suppose that*

$$n_k \to +\infty \quad (k \to +\infty) \qquad \textit{and} \qquad n_k \le 2M_k \quad (k \in \mathbb{N}).$$

Moreover let $\varphi \in \Phi$ be a summation function.

If $\hat{\varphi} \in L^1(\mathbb{R}_0^+)$ then the sequence $S_{n_k,M_k}^{\varphi,\overline{\mathcal{B}}} f$ uniformly converges on $[-1,1]$ to f for all $f \in C[-1,1]$.

Proof. We show the statement for $\overline{\mathcal{B}} = \overline{\mathcal{U}}$ only, the other cases are similar. As before, for any $f \in C[-1,1]$ let

$$g(x) := f(x) - \frac{f(1) - f(-1)}{2} x - \frac{f(1) + f(-1)}{2}.$$

We shall use the Banach-Steinhaus theorem. The polynomials

$$\{\overline{\mathcal{U}}_j : j \ge -2, j \in \mathbb{Z}\}$$

form a closed system in the space $(C[-1,1], \|\cdot\|_\infty)$, therefore we have to show that two conditions hold:

i) For every fixed $j \ge -2$, $j \in \mathbb{Z}$

$$\left\| S_{n_k,M_k}^{\varphi,\overline{\mathcal{U}}} \overline{\mathcal{U}}_j - \overline{\mathcal{U}}_j \right\|_\infty \to 0 \qquad (k \to +\infty). \tag{4.1}$$

ii) The sequence of norms of the operators is uniformly bounded, i.e. there exists a constant $c > 0$ independent of k such that

$$\left\| S_{n_k,M_k}^{\varphi,\overline{\mathcal{U}}} \right\| \le c \qquad (k \in \mathbb{N}). \tag{4.2}$$

First we prove (4.1). The ceses $j = -2$ and -1 are easy to see. Let $j \ge 0$ and k large enough for

$$\min\{n_k, M_k\} > j.$$

Then we have

$$\begin{gathered}
\big(S_{n_k,M_k}^{\varphi,\overline{\mathcal{U}}} \overline{U}_j\big)(x) = \\
= \sum_{i=0}^{n_k} \Big[\frac{2}{M_k+1} \cdot \varphi\Big(\frac{i}{n_k}\Big) \cdot \sum_{l=2}^{M_k+1} \overline{U}_i(x_{l,M_k+2}) \cdot \overline{U}_j(x_{l,M_k+2})\Big] \overline{U}_i(x) = \\
= \varphi\Big(\frac{j}{n_k}\Big) \cdot \overline{U}_j(x) + \\
+\varphi\Big(\frac{2M_k - j}{n_k}\Big) \frac{2}{M_k+1} \Big[\sum_{l=2}^{M_k+1} \overline{U}_j(x_{l,M_k+2}) \overline{U}_{2M_k-j}(x_{l,M_k+2})\Big] \overline{U}_{2M_k-j}(x),
\end{gathered}$$

because of

$$\sum_{l=2}^{M_k+1} \overline{U}_i(x_{l,M_k+2}) \cdot \overline{U}_j(x_{l,M_k+2}) = \begin{cases} 0, & i \neq j \leq M_k+1; \\ \frac{M_k+1}{2}, & i = j, \end{cases}$$

(see [2, (6.25)]), and the symmetry property of the coefficients.

Observe that if $k \to +\infty$ then

$$\varphi\Big(\frac{j}{n_k}\Big) \to \varphi(0) = 1$$

and

$$\varphi\Big(\frac{2M_k - j}{n_k}\Big) = \varphi\Big(1 - \frac{j}{n_k} + \frac{2M_k - n_k}{n_k}\Big) \to 0,$$

so

$$\big(S^{\varphi,\overline{\mathcal{U}}}_{n_k,M_k} \overline{U}_j\big)(x) \to \overline{U}_j(x),$$

and (4.1) follows from the above relations.

Now we prove (4.2).

$$\begin{aligned} \Big\| S^{\varphi,\overline{\mathcal{U}}}_{n_k,M_k} \Big\| &= \sup_{\substack{f \in C[-1,1] \\ \|f\|_\infty = 1}} \| S^{\varphi,\overline{\mathcal{U}}}_{n_k,M_k} f \|_\infty = \\ &= \sup_{\substack{f \in C[-1,1] \\ \|f\|_\infty = 1}} \| S^{\varphi,\overline{\mathcal{U}}}_{n_k,M_k} g + c^{\overline{\mathcal{U}}}_{-1,M}(f) \cdot x + c^{\overline{\mathcal{U}}}_{-2,M}(f) \|_\infty \leq \\ &\leq \sup_{\substack{g \in C[-1,1] \\ \|g\|_\infty \leq 3}} \Big\{ \| S^{\varphi,\overline{\mathcal{U}}}_{n_k,M_k} g \|_\infty : g(-1) = g(1) = 0 \Big\} + 2, \end{aligned}$$

since $x \in [-1,1]$ and $\|f\|_\infty = 1$ implies

$$\| c^{\overline{\mathcal{U}}}_{-1,M}(f) \cdot x + c^{\overline{\mathcal{U}}}_{-2,M}(f) \|_\infty \leq 1 + 1 = 2$$

and $\|g\|_\infty \leq 3$.

So we get the estimation

$$\Big\| S^{\varphi,\overline{\mathcal{U}}}_{n_k,M_k} \Big\| \leq 3 \cdot \sup_{\substack{g \in C[-1,1] \\ \|g\|_\infty = 1}} \Big\{ \| S^{\varphi,\overline{\mathcal{U}}}_{n_k,M_k} g \|_\infty : g(-1) = g(1) = 0 \Big\} + 6. \tag{4.3}$$

This means that we have to show that

$$\sup_{\substack{g\in C[-1,1]\\ \|g\|_\infty=1}} \left\{ \|S^{\varphi,\overline{\mathcal{U}}}_{n_k,M_k} g\|_\infty : g(-1)=g(1)=0 \right\} =$$

$$\sup_{\substack{g\in C[-1,1]\\ \|g\|_\infty=1}} \left\| \sum_{j=0}^{n_k} \Big[\frac{2}{M_k+1} \sum_{l=2}^{M_k+1} \varphi\Big(\frac{j}{n_k}\Big) \cdot g(x_{l,M_k+2}) \cdot \overline{U}_j(x_{l,M_k+2}) \Big] \overline{U}_j(x) \right\|_\infty =$$

$$= \sup_{x\in[-1,1]} \sum_{l=2}^{M_k+1} \left| \frac{2}{M_k+1} \sum_{j=0}^{n_k} \varphi\Big(\frac{j}{n_k}\Big) \cdot \overline{U}_j(x_{l,M_k+2}) \cdot \overline{U}_j(x) \right|$$

is uniformly bounded. Note that we excluded the $l=1$ and $l=M_k+2$ terms of the sum, since $g(-1)=g(1)=0$ implies that they are 0.

Let $x =: \cos\vartheta$ ($\vartheta\in[0,\pi]$) and $x_{l,M+2} =: \cos\vartheta_{l,M+2}$ ($l=2,..,M+1$). For every $n,M\in\mathbb{N}$ we have

$$\left| \sum_{j=0}^{n} \varphi\Big(\frac{j}{n}\Big) \cdot \overline{U}_j(x_{l,M+2}) \cdot \overline{U}_j(x) \right| =$$

$$= \left| \sum_{j=0}^{n} \varphi\Big(\frac{j}{n}\Big) \cdot \sin\vartheta_{l,M+2} \cdot \frac{\sin(j+1)\vartheta_{l,M+2}}{\sin\vartheta_{l,M+2}} \cdot \sin\vartheta \cdot \frac{\sin(j+1)\vartheta}{\sin\vartheta} \right| =$$

$$= \left| \sum_{j=0}^{n} \varphi\Big(\frac{j}{n}\Big) \cdot \sin(j+1)\vartheta_{l,M+2} \cdot \sin(j+1)\vartheta \right| =$$

$$= \frac{1}{2} \left| \frac{1}{2} + \sum_{j=1}^{n+1} \varphi\Big(\frac{j-1}{n}\Big) \cdot \cos j(\vartheta+\vartheta_{l,M+2}) - \frac{1}{2} - \right.$$

$$\left. - \sum_{j=1}^{n+1} \varphi\Big(\frac{j-1}{n}\Big) \cdot \cos j(\vartheta-\vartheta_{l,M+2}) \right| =:$$

$$=: \left| D^{\varphi}_{n+1}(\vartheta+\vartheta_{l,M+2}) - D^{\varphi}_{n+1}(\vartheta-\vartheta_{l,M+2}) \right| \le$$

$$\le \left| D^{\varphi}_{n+1}(\vartheta+\vartheta_{l,M+2}) \right| + \left| D^{\varphi}_{n+1}(\vartheta-\vartheta_{l,M+2}) \right|.$$

Therefore we have

$$\sup_{x\in[-1,1]} \sum_{l=2}^{M+1} \left| \frac{2}{M+1} \sum_{j=0}^{n} \varphi\Big(\frac{j}{n}\Big) \cdot \overline{U}_j(x_{l,M+2}) \cdot \overline{U}_j(x) \right| \le$$

$$\max_{\vartheta\in[0,\pi]} \frac{2}{M+1} \sum_{l=2}^{M+1} \left| D^{\varphi}_{n+1}(\vartheta+\vartheta_{l,M+2}) \right| + \max_{\vartheta\in[0,\pi]} \frac{2}{M+1} \sum_{l=2}^{M+1} \left| D^{\varphi}_{n+1}(\vartheta-\vartheta_{l,M+2}) \right|.$$

From [9, (26) and (27)] we have

$$\max_{\vartheta\in[0,\pi]} \frac{2}{M+1} \sum_{l=2}^{M+1} \left|D_{n+1}^{\varphi}(\vartheta \pm \vartheta_{l,M+2})\right| \le C \left(1 + 2\frac{n+1}{M+2}\pi\right) \|D_{n+1}^{\varphi}\|_1,$$

where

$$\|D_{n+1}^{\varphi}\|_1 := \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_{n+1}^{\varphi}(t)|dt.$$

Also

$$2\sup_{k\in\mathbb{N}} \|D_{n_k+1}^{\varphi}\|_1 = \|\hat{\varphi}\|_{L^1(\mathbb{R}_0^+)},$$

thus $\hat{\varphi} \in L^1(\mathbb{R}_0^+)$ ensures (4.2). ■

In general, the Fourier transform of a function from $L^1(\mathbb{R}_0^+)$ does not belong to the space $L^1(\mathbb{R}_0^+)$. Verifying $\hat{g} \in L^1(\mathbb{R}_0^+)$ is not always easy but the following sufficient condition is known (cf. [3, p. 176]):

Theorem 4.2. *If $g : \mathbb{R}_0^+ \to \mathbb{R}$ is a continuous function supported in $[0,1]$ and $g \in$Lipβ $(\beta > 1/2)$ on $[0,1]$ then $\hat{g} \in L^1(\mathbb{R}_0^+)$.*

Using the last two theorems one can easily construct discrete processes uniformly convergent on the whole interval $[-1,1]$.

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Zs. Németh
Department of Numerical Analysis
Faculty of Informatics
Eötvös Loránd University
H-1117 Budapest, Pázmány P. sétány 1/C
Hungary
birka0@gmail.com