ABOUT A CONDITION FOR STARLIKENESS

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Abstract. In this paper a result concerning the starlikeness of the image of the Alexander operator is improved. The techniques of differential subordinations and extreme points are used.

1. Introduction

Let $U(z_0, r)$ be the disc centered at the point z_0 and of radius r defined by $U(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$. U denotes the open unit disc in $\mathbb{C}$, $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}$ be the class of analytic functions f, which are defined on the unit disc U and have the form: $f(z) = z + a_2z^2 + a_3z^3 + \ldots$. The subclass of $\mathcal{A}$ consisting of functions for which the range $f(U)$ is starlike with respect to 0, is denoted by S^*. An analytic characterization of S^* is given by:

$$S^* = \left\{ f \in \mathcal{A} : \mathrm{Re}\,\frac{zf'(z)}{f(z)} > 0,\ z \in U \right\}.$$

Another subclass of $\mathcal{A}$ we deal with is the class of close-to-convex functions denoted by C. A function $f \in \mathcal{A}$ belongs to the class C if and only if there is

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a starlike function $g \in S^*$, so that $\operatorname{Re}\frac{zf'(z)}{g(z)} > 0, \quad z \in U$. We note that C and S^* contain univalent functions. The Alexander integral operator is defined by the equality:

$$A(f)(z) = \int_0^z \frac{f(t)}{t}\, dt.$$

The authors of [2] (pp. $310 - 311$) proved the following result:

Theorem 1.1. *Let A be the Alexander operator and let $g \in \mathcal{A}$ satisfy*

$$\operatorname{Re}\frac{zg'(z)}{g(z)} \geq \left|\operatorname{Im}\frac{z(zg'(z))'}{g(z)}\right|, \ z \in U. \tag{1.1}$$

If $f \in \mathcal{A}$ and

$$\operatorname{Re}\frac{zf'(z)}{g(z)} > 0, \ z \in U,$$

then $F = A(f) \in S^*$.

This theorem states that a subclass of C is mapped by the Alexander operator to S^*. On the other hand we know that $A(C) \not\subset S^*$. In [3] and [4] several improvements of this result are proved, simplifying condition (1.1). Investigating this question, the following theorems have been deduced in [3]:

Theorem 1.2. *Let $g \in \mathcal{A}$ be a function which satisfies the condition:*

$$\operatorname{Re}\frac{zg'(z)}{g(z)} > 2.273\left|\operatorname{Im}\frac{zg'(z)}{g(z)}\right|, \quad z \in U. \tag{1.2}$$

If $f \in \mathcal{A}$ satisfies

$$\operatorname{Re}\frac{zf'(z)}{g(z)} > 0, \ z \in U,$$

then $F = A(f) \in S^*$.

Theorem 1.3. *If $f, g \in \mathcal{A}$ and*

$$\operatorname{Re}\frac{g(z)}{z} > \frac{100}{83}\left|\operatorname{Im}\frac{g(z)}{z}\right|, \ z \in U, \tag{1.3}$$

then the condition

$$\operatorname{Re}\frac{zf'(z)}{g(z)} > 0, \ z \in U$$

implies that $F = A(f) \in S^*$.

The implications-chain is deduced in [3]: (1.1) ⇒ (1.2) ⇒ (1.3). Thus Theorem 1.2 and Theorem 1.3 are improvements of Theorem 1.1. Consequently, the question to determine the smallest $c \in [0,\infty)$ for which the following statement holds arises naturally:

If $f, g \in \mathcal{A}$ and

$$\operatorname{Re}\frac{g(z)}{z} > c\left|\operatorname{Im}\frac{g(z)}{z}\right|,\ z \in U, \tag{1.4}$$

then the condition

$$\operatorname{Re}\frac{zf'(z)}{g(z)} > 0,\ z \in U$$

implies that $F = A(f) \in S^*$.

We are not able to answer this question completely at the moment, but we will prove that the statement holds for $c = 1$. This is an improvement of Theorem 1.3. In order to do this, we need the following lemmas.

2. Preliminaries

Lemma 2.1. ([2]) *Let $p(z) = a + \sum\limits_{k=n}^{\infty} a_k z^k$ be analytic in U with $p(z) \not\equiv a$, $n \geq 1$ and let $q : U(0,1) \to \mathbb{C}$ be a univalent function with $q(0) = a$. If there are two points $z_0 \in U(0,1)$ and $\zeta_0 \in \partial U(0,1)$ so that q is defined in ζ_0, $p(z_0) = q(\zeta_0)$ and $p(U(0,r_0)) \subset q(U)$, where $r_0 = |z_0|$, then there is an $m \in [n,+\infty)$ so that*

(i) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$ *and*

(ii) $\operatorname{Re}\left(1 + \dfrac{z_0 p''(z_0)}{p'(z_0)}\right) \geq m\operatorname{Re}\left(1 + \dfrac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}\right)$.

Lemma 2.2. ([2]) *Let $p(z) = a + \sum\limits_{k=n}^{\infty} a_k z^k$, $p(z) \not\equiv a$ and $n \geq 1$.*
If $z_0 \in U$ and

$$\operatorname{Re} p(z_0) = \min\{\operatorname{Re} p(z) : |z| \leq |z_0|\},$$

then

(i) $z_0 p'(z_0) \leq -\dfrac{n}{2}\dfrac{|p(z_0) - a|^2}{\operatorname{Re}(a - p(z_0))}$ *and*

(ii) $\operatorname{Re}[z_0^2 p''(z_0)] + z_0 p'(z_0) \leq 0$.

Recall that if f and g are analytic functions in U and there is a function w also analytic, satisfying $w(0) = 0$, $|w(z)| \leq |z|$, $z \in U$ and $f(z) = g(w(z))$,

$z \in U$, then the function f is said to be subordinate to g, written $f \prec g$. If g is univalent then $f(0) = g(0)$ and $f(U) \subset g(U)$ implies that $f \prec g$.

Lemma 2.3. ([1]) *Let* $F_\alpha(z) = \left(\frac{1+cz}{1-z}\right)^\alpha,\ |c| \le 1,\ c \ne -1$. *In case of* $\alpha \ge 1$, *the subordination* $f \prec F_\alpha$ *holds if and only if there exists a probability measure* μ *on* $[0, 2\pi]$ *having the property*

$$f(z) = \int_0^{2\pi} \left(\frac{1+ze^{-it}}{1-ze^{-it}}\right)^\alpha d\mu(t),\ z \in U.$$

The set of extreme points of the class $\{f \in \mathcal{A} \big| f \prec F_\alpha\}$ *is*

$$\left\{ f_t(z) = \left(\frac{1+ze^{-it}}{1-ze^{-it}}\right)^\alpha,\ t \in [0, 2\pi] \right\}.$$

Let $\mathcal{P}$ denote the class of analytic functions of the form

$$p(z) = 1 + c_1 z + c_2 z^2 + \dots,$$

and having the property $\operatorname{Re} p(z) > 0,\ z \in U$. We note that this property is equivalent to $p(z) \prec \frac{1+z}{1-z}$ and Lemma 2.3 implies that there is a probability measure μ on the interval $[0, 2\pi]$ such that $p(z) = \int_0^{2\pi} \frac{1+ze^{-it}}{1-ze^{-it}} d\mu(t)$. This equality actually is the Herglotz formula.

Lemma 2.4. ([1, Corollary 3.7]) $p \in \mathcal{P}$ *if and only if there exist a sequence of functions* $(p_n)_{n \ge 1}$ *so that* p_n *has the form*

$$q(z) = \sum_{k=1}^{m} t_k \frac{1+zx_k}{1-zx_k},$$

where $|x_k| = 1,\ t_k \ge 0$ *and* $\sum_{k=1}^m t_k = 1$ *and* $p_n \to p$ *uniformly on compact subsets of* U.

Lemma 2.5. *If* $f, g \in \mathcal{A}$ *and*

$$\operatorname{Re} \frac{g(z)}{z} > \left| \operatorname{Im} \frac{g(z)}{z} \right|,\ z \in U, \tag{2.1}$$

and $F = A(f)$, *then the condition*

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0,\ z \in U \tag{2.2}$$

implies that there is a probability measure μ *on* $[0, 2\pi]$, *such that*

$$\frac{F(z)}{z} = \int_0^{2\pi} \int_0^1 \ln \frac{1}{x} \left(\frac{1+xze^{-it}}{1-xze^{-it}}\right)^{\frac{3}{2}} dx d\mu(t),\ z \in U.$$

Proof. Inequality (2.1) is equivalent to

$$\left|\arg \frac{g(z)}{z}\right| \le \frac{\pi}{4},\ z \in U. \tag{2.3}$$

Applying Lemma 2.3 in case of $c = 1$, $\alpha = 1$ and $F_1(z) = \frac{1+z}{1-z}$ it follows that:

$$f'(z) = \frac{g(z)}{z} \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\nu(t),$$

where ν is a probability measure on $[0, 2\pi]$. Thus we get:

$$|\arg f'(z)| \le \left|\arg \frac{g(z)}{z}\right| + \left|\arg \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\nu(t)\right| < \frac{3\pi}{4},\ z \in U. \tag{2.4}$$

We introduce the notation $\mathcal{D} = \left\{z \in \mathbb{C} : |\arg(z)| \le \frac{3\pi}{4}\right\}$. The function

$$q(z) = \left(\frac{1+z}{1-z}\right)^{\tau}, \quad \tau = \frac{3}{2},$$

is the Riemann mapping from U to $\mathcal{D}$. (The principal branch of $\left(\frac{1+z}{1-z}\right)^{\tau}$ is chosen.) The inequality (2.4) implies

$$f'(z) \prec q(z),$$

and according to Lemma 2.3, this subordination is equivalent to

$$f'(z) = \int_0^{2\pi} \left(\frac{1 + ze^{-it}}{1 - ze^{-it}}\right)^{\frac{3}{2}} d\mu(t),\ z \in U,$$

where μ denotes a probability measure on $[0, 2\pi]$. On the other hand, if

$$q(z) = 1 + \sum_{n=1}^{\infty} a_n z^n,$$

then

$$f'(z) = 1 + \sum_{n=1}^{\infty} a_n z^n \int_0^{2\pi} e^{-int} d\mu(t),$$

and

$$\frac{F(z)}{z} = 1 + \sum_{n=1}^{\infty} a_n \frac{z^n}{(n+1)^2} \int_0^{2\pi} e^{-int} d\mu(t).$$

The equalities $\int_0^1 x^n \ln\frac{1}{x}dx = \frac{1}{(n+1)^2},\quad n\in\mathbb{N}$ imply

$$\frac{F(z)}{z} = \int_0^1 \ln\frac{1}{x}\left(1+\sum_{n=1}^{\infty} a_n x^n z^n \int_0^{2\pi} e^{-int}d\mu(t)\right)dx.$$

Lemma 2.4 implies that the second integration can be interchanged with the summation and the first integration and finally we get

$$\frac{F(z)}{z} = \int_0^1 \ln\frac{1}{x}\int_0^{2\pi}\left(\frac{1+xze^{-it}}{1-xze^{-it}}\right)^{\frac{3}{2}} d\mu(t)dx =$$

$$= \int_0^{2\pi}\int_0^1 \ln\frac{1}{x}\left(\frac{1+xze^{-it}}{1-xze^{-it}}\right)^{\frac{3}{2}} dxd\mu(t),\ z\in U. \qquad \blacksquare$$

Lemma 2.6. *The function* $A:[0,\frac{3\pi}{4}]\to\mathbb{R}$,

$$A(\theta) = (\pi-\theta)(\sin\theta-\cos\theta)\int_0^{\infty}\left(\frac{e^x-1}{e^x+1}\right)^{\frac{3}{2}}\frac{1}{e^x}dx-$$

$$-(\sin\theta+\cos\theta)\int_0^{\infty}\left(\frac{e^x-1}{e^x+1}\right)^{\frac{3}{2}}\frac{x}{e^x}dx$$

is increasing and the function $B:[\frac{\pi}{6},\frac{3\pi}{4}]\to\mathbb{R}$ *defined by*

$$B(\theta) = \sqrt{2}\int_0^{\pi-\theta} x\left(\cot\frac{\theta+x}{2}\right)^{\frac{3}{2}}\cos x dx$$

is decreasing.

Proof. Notice that

$$I_1 = \int_0^{\infty}\left(\frac{e^x-1}{e^x+1}\right)^{\frac{3}{2}}\frac{1}{e^x}dx = 0.28...,\quad I_2 = \int_0^{\infty}\left(\frac{e^x-1}{e^x+1}\right)^{\frac{3}{2}}\frac{x}{e^x}dx = 0.51...$$

and $I_1 < I_2 < 2I_1$. Thus it follows that in case $\theta\in[\frac{\pi}{4},\frac{3\pi}{4}]$ we have

$$A'(\theta) = (\pi-\theta)(\sin\theta+\cos\theta)I_1 + (\sin\theta-\cos\theta)(I_2-I_1) > 0$$

and if $\theta \in [0, \frac{\pi}{4}]$, then

$$A'(\theta) > [(\pi - \theta)(\sin\theta + \cos\theta) + \sin\theta - \cos\theta] I_1 > 0.$$

Consequently the first part of the assertion is proved.

In the following we will prove that: $B'(\theta) \le 0,\ \theta \in [\frac{\pi}{6}, \frac{3\pi}{4}]$. We have:

$$B'(\theta) = -\frac{3\sqrt{2}}{4} \int\limits_0^{\pi-\theta} x\left(\cot\frac{\theta+x}{2}\right)^{\frac{1}{2}}\left(\sin\frac{\theta+x}{2}\right)^{-2}\cos x dx,\ \theta \in [\frac{\pi}{6}, \frac{3\pi}{4}].$$

The claimed inequality holds evidently in case $\theta \in [\frac{\pi}{2}, \frac{3\pi}{4}]$.

We will use the following equality to prove $B'(\theta) \le 0$ in case $\theta \in [\frac{\pi}{6}, \frac{\pi}{2}]$:

$$B'(\theta) = \frac{3\sqrt{2}}{4} \int\limits_0^{\frac{\pi}{2}-\theta} (x + \frac{\pi}{2})\left(\cot(\frac{\pi}{4} + \frac{\theta+x}{2})\right)^{\frac{1}{2}}\left(\sin(\frac{\pi}{4} + \frac{\theta+x}{2})\right)^{-2}\sin x dx -$$

$$-\frac{3\sqrt{2}}{4}\int_0^{\frac{\pi}{2}} x\left(\cot\frac{\theta+x}{2}\right)^{\frac{1}{2}}\left(\sin\frac{\theta+x}{2}\right)^{-2}\cos x dx. \tag{2.5}$$

Some elementary calculations lead to the following inequalities:

$$\left(\cot\frac{\theta+x}{2}\right)^{\frac{1}{2}} \ge (1+\sqrt{2})\left(\cot(\frac{\pi}{4} + \frac{\theta+x}{2})\right)^{\frac{1}{2}}, \quad x \in [0, \frac{\pi}{2} - \theta]$$

$$\left(\sin\frac{\theta+x}{2}\right)^{-2} \ge 2\left(\sin(\frac{\pi}{4} + \frac{\theta+x}{2})\right)^{-2}, \quad x \in [0, \frac{\pi}{2} - \theta]$$

$$x\cos x \ge \frac{\frac{\pi}{3}}{\frac{5\pi}{6}\tan\left(\frac{\pi}{3}\right)}(\frac{\pi}{2} + x)\sin x, \quad x \in [0, \frac{\pi}{2} - \theta].$$

These inequalities imply that in case $x \in [0, \frac{\pi}{2} - \theta]$ we have:

$$x\left(\cot\frac{\theta+x}{2}\right)^{\frac{1}{2}}\left(\sin\frac{\theta+x}{2}\right)^{-2}\cos x \ge$$

$$\ge \frac{4(1+\sqrt{2})}{5\sqrt{3}}(x+\frac{\pi}{2})\left(\cot(\frac{\pi}{4} + \frac{\theta+x}{2})\right)^{\frac{1}{2}}\left(\sin(\frac{\pi}{4} + \frac{\theta+x}{2})\right)^{-2}\sin x \ge$$

$$\ge (x+\frac{\pi}{2})\left(\cot(\frac{\pi}{4} + \frac{\theta+x}{2})\right)^{\frac{1}{2}}\left(\sin(\frac{\pi}{4} + \frac{\theta+x}{2})\right)^{-2}\sin x,$$

and finally we get:

$$\int_0^{\frac{\pi}{2}-\theta} x\left(\cot\frac{\theta+x}{2}\right)^{\frac{1}{2}}\left(\sin\frac{\theta+x}{2}\right)^{-2}\cos x dx \ge \tag{2.6}$$

$$\ge \int_0^{\frac{\pi}{2}-\theta} (x+\frac{\pi}{2})\left(\cot(\frac{\pi}{4} + \frac{\theta+x}{2})\right)^{\frac{1}{2}}\left(\sin(\frac{\pi}{4} + \frac{\theta+x}{2})\right)^{-2}\sin x dx.$$

The inequality $B'(\theta) \leq 0,\ \theta \in [\frac{\pi}{6}, \frac{\pi}{2}]$ follows from (2.5) and (2.6). ■

Lemma 2.7. *If*

$$F(z) = \int_0^1 \left(\frac{1+xz}{1-xz}\right)^{\frac{3}{2}} \ln\frac{1}{x} dx,$$

then

$$\mathrm{Re}F(e^{i\theta}) \geq \mathrm{Im}F(e^{i\theta}), \quad \theta \in [0, \pi].$$

Proof. We begin with the observation that the change of variable $x = e^{-t}$ leads to

$$F(e^{i\theta}) = \int_0^\infty \left(\frac{e^t + e^{i\theta}}{e^t - e^{i\theta}}\right)^{\frac{3}{2}} \frac{t}{e^t} dt.$$

Now consider the function:

$$f(z) = \left(\frac{e^z + e^{i\theta}}{e^z - e^{i\theta}}\right)^{\frac{3}{2}} \frac{z}{e^z}.$$

We integrate it on $\Gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, where $\gamma_1(t) = t,\ t \in [0, R],\ \gamma_2(t) = R - it,\ t \in [0, \pi - \theta],\ \gamma_3(t) = R - t + i(\theta - \pi),\ t \in [0, R]$ and $\gamma_4(t) = i(\theta - \pi + t),\ t \in [0, \pi - \theta]$. Because f is analytic in the interior of Γ we have, $\int_\Gamma f(z)dz = 0$ which leads to

$$F(e^{i\theta}) = \lim_{R\to\infty} \int_{\gamma_1} f(z)dz = -\lim_{R\to\infty} \Big[\int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz +$$

$$+ \int_{\gamma_4} f(z)dz\Big] = \int_0^\infty \left(\frac{e^x - 1}{e^x + 1}\right)^{\frac{3}{2}} \frac{(x + i(\theta - \pi))(-\cos\theta + i\sin\theta)}{e^x} dx +$$

$$+ \int_0^{\pi-\theta} \big(\tan\frac{t}{2}\big)^{\frac{3}{2}} e^{i\frac{3\pi}{4}} \frac{\theta - \pi + t}{e^{i(\theta - \pi + t)}} dt.$$

The change of variable $\theta - \pi + t = -x$ in the second integral implies the equality

$$F(e^{i\theta}) = \int_0^\infty \left(\frac{e^x - 1}{e^x + 1}\right)^{\frac{3}{2}} \frac{(x + i(\theta - \pi))(-\cos\theta + i\sin\theta)}{e^x} dx -$$

$$- \int_0^{\pi-\theta} x\left(\cot\frac{\theta + x}{2}\right)^{\frac{3}{2}} e^{i(x + \frac{3\pi}{4})} dx.$$

Thus it follows that

$$\operatorname{Re} F(e^{i\theta}) - \operatorname{Im} F(e^{i\theta}) = (\pi-\theta)(\sin\theta - \cos\theta)\int\limits_0^\infty \left(\frac{e^x-1}{e^x+1}\right)^{\frac{3}{2}} \frac{1}{e^x}dx -$$

$$-(\sin\theta+\cos\theta)\int\limits_0^\infty \left(\frac{e^x-1}{e^x+1}\right)^{\frac{3}{2}} \frac{x}{e^x}dx + \sqrt{2}\int\limits_0^{\pi-\theta} x\left(\cot\frac{\theta+x}{2}\right)^{\frac{3}{2}} \cos x dx =$$

$$= A(\theta)+B(\theta). \tag{2.7}$$

According to the monotonicity of A and B, the inequalities hold

$$B(\theta)+A(\theta) \geq B(\theta_k)+A(\theta_{k-1}), \quad \theta\in[\theta_{k-1},\theta_k], \quad k=\overline{21,90}.$$

Now, if we check that

$$B(\theta_k)+A(\theta_{k-1}) > 0, \qquad \theta_k = \frac{k\pi}{120}, \qquad k=\overline{21,90} \tag{2.8}$$

we obtain

$$B(\theta)+A(\theta) > 0, \quad \theta\in[\theta_{k-1},\theta_k], \quad k=\overline{21,90}$$

and the proof is done in case of $\theta\in[\frac{\pi}{6},\frac{3\pi}{4}]$. Inequalities (2.8) can be checked easily by using a computer program. The inequality $\operatorname{Re} F(e^{i\theta}) \geq \operatorname{Im} F(e^{i\theta})$, $\theta\in[\frac{3\pi}{4},\pi]$ follows from (2.7). It remains to prove the assertion in case $\theta\in[0,\frac{\pi}{6}]$. We put in the integral $\int_0^{\pi-\theta} x\big(\cot\frac{\theta+x}{2}\big)^{\frac{3}{2}}\cos x dx$ the change of variable $x+\theta=u$ and we obtain

$$\operatorname{Re} F(e^{i\theta}) - \operatorname{Im} F(e^{i\theta}) = (\pi-\theta)(\sin\theta - \cos\theta)\int\limits_0^\infty \left(\frac{e^x-1}{e^x+1}\right)^{\frac{3}{2}} \frac{1}{e^x}dx -$$

$$-(\sin\theta+\cos\theta)\int\limits_0^\infty \left(\frac{e^x-1}{e^x+1}\right)^{\frac{3}{2}} \frac{x}{e^x}dx + \sqrt{2}\int\limits_\theta^{\pi} (u-\theta)\left(\cot\frac{u}{2}\right)^{\frac{3}{2}} \cos(u-\theta) dx.$$

This can be rewritten as follows

$$\operatorname{Re} F(e^{i\theta}) - \operatorname{Im} F(e^{i\theta}) =$$

$$= \sin\theta\left((\pi-\theta)I_1 - I_2 + \sqrt{2}\int\limits_\theta^\pi (u-\theta)(\cot\frac{u}{2})^{\frac{3}{2}}\sin u du\right)+$$

$$+\cos\theta\left(-(\pi-\theta)I_1 - I_2 + \sqrt{2}\int\limits_\theta^\pi (u-\theta)(\cot\frac{u}{2})^{\frac{3}{2}}\cos u du\right).$$

(I_1 and I_2 are defined in the proof of the previous lemma.) We observe that the mapping $C : [0, \frac{\pi}{6}]$ defined by

$$C(\theta) = (\pi - \theta)I_1 - I_2 + \sqrt{2}\int_{\theta}^{\pi}(u-\theta)(\cot\frac{u}{2})^{\frac{3}{2}}\sin u du$$

is strictly decreasing. This implies the inequality: $C(\theta) \geq C(\frac{\pi}{6}) \geq 6.8...$ Thus it follows that

$$\operatorname{Re} F(e^{i\theta}) - \operatorname{Im} F(e^{i\theta}) \geq$$

$$\geq \cos\theta\left(6.8\tan\theta - (\pi-\theta)I_1 - I_2 + \sqrt{2}\int_{\theta}^{\pi}(u-\theta)(\cot\frac{u}{2})^{\frac{3}{2}}\cos u du\right).$$

Let the functions D and E be defined by the equalities

$$D(\theta) = 6.8\tan\theta - (\pi-\theta)I_1 - I_2$$

and

$$E(\theta) = \sqrt{2}\int_{\theta}^{\pi}(u-\theta)(\cot\frac{u}{2})^{\frac{3}{2}}\cos u du.$$

It is simple to show that D is strictly increasing and E is strictly decreasing. The monotonicity of these functions imply

$$D(\theta) + E(\theta) > D(\theta_{k-1}) + E(\theta_k),\ \theta_k = \frac{k\pi}{120},\ k = \overline{1, 20}.$$

If we prove that $D(\theta_{k-1}) + E(\theta_k) > 0,\ \theta_k = \frac{k\pi}{120},\ k = \overline{1, 20}$, then it follows that $\operatorname{Re}F(e^{i\theta}) \geq \operatorname{Im}F(e^{i\theta}),\ \ \theta \in [0, \frac{\pi}{6}]$ and the proof is done. The inequalities $D(\theta_{k-1}) + E(\theta_k) > 0,\ \ k = \overline{1, 20}$ can be checked easily by using a computer program. ∎

3. The main result

Theorem 3.1. *If $f, g \in \mathcal{A}$ and*

$$\operatorname{Re}\frac{g(z)}{z} > \left|\operatorname{Im}\frac{g(z)}{z}\right|,\ z \in U,$$

then the condition

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0, \ z \in U$$

implies that

$$\operatorname{Re} \frac{F(z)}{z} > \left| \operatorname{Im} \frac{F(z)}{z} \right|, \ z \in U, \tag{3.1}$$

where $F = A(f)$.

Proof. Let Λ be the set of probability measures on $[0, 2\pi]$. We introduce the notation

$$\mathcal{B} = \left\{ \int_0^{2\pi} \int_0^1 \ln \frac{1}{x} \left(\frac{1 + xze^{-it}}{1 - xze^{-it}} \right)^{\frac{3}{2}} dx d\mu(t) \ \middle| \ \mu \in \Lambda \right\}.$$

According to Lemma 2.5 we have $F \in \mathcal{B}$. Let $z_0 \in U$ be an arbitrarily fixed point, and let p_{z_0} be the functional defined by

$$p_{z_0} : \mathcal{B} \to \mathbb{R}, \ p_{z_0}(F) = \operatorname{Re} F(z_0) - \left| \operatorname{Im} F(z_0) \right|.$$

If we prove that $p_{z_0}(F) \geq 0$ for every $F \in \mathcal{B}$ in case of an arbitrarily fixed point $z_0 \in U$, then inequality (3.1) follows. Since the functional p_{z_0} is concave, according to Lemma 2.5, we have to verify $p_{z_0}(F) \geq 0$ only for the extreme points of the class $\mathcal{B}$. It follows from Lemma 2.5 that the extreme points of this class are

$$F_t(z) = \int_0^1 \ln \frac{1}{x} \left(\frac{1 + xze^{-it}}{1 - xze^{-it}} \right)^{\frac{3}{2}} dx, \quad t \in [0, 2\pi].$$

For $z_0 = r_0 e^{i\theta_0}$, the inequality $p_{z_0}(F_t) \geq 0$ is equivalent to

$$\int_0^1 \ln \frac{1}{x} \left(\frac{1 + x^2 r_0^2 + 2xr_0 \cos(\theta_0 - t)}{1 + x^2 r_0^2 - 2xr_0 \cos(\theta_0 - t)} \right)^{\frac{3}{4}} \cos\left(\frac{3}{2} \arctan \frac{2xr_0 \sin(\theta_0 - t)}{1 - x^2 r_0^2} \right) dx \geq$$

$$\geq \left| \int_0^1 \ln \frac{1}{x} \left(\frac{1 + x^2 r_0^2 + 2xr_0 \cos(\theta_0 - t)}{1 + x^2 r_0^2 - 2xr_0 \cos(\theta_0 - t)} \right)^{\frac{3}{4}} \sin\left(\frac{3}{2} \arctan \frac{2xr_0 \sin(\theta_0 - t)}{1 - x^2 r_0^2} \right) dx \right|.$$

Denoting $\theta_0 - t$ by β, we obtain

$$\begin{aligned} &\int_0^1 \ln \frac{1}{x} \left(\frac{1 + x^2 r_0^2 + 2xr_0 \cos\beta}{1 + x^2 r_0^2 - 2xr_0 \cos\beta} \right)^{\frac{3}{4}} \cos\left(\frac{3}{2} \arctan \frac{2xr_0 \sin\beta}{1 - x^2 r_0^2} \right) dx \geq \\ &\geq \left| \int_0^1 \ln \frac{1}{x} \left(\frac{1 + x^2 r_0^2 + 2xr_0 \cos\beta}{1 + x^2 r_0^2 - 2xr_0 \cos\beta} \right)^{\frac{3}{4}} \sin\left(\frac{3}{2} \arctan \frac{2xr_0 \sin\beta}{1 - x^2 r_0^2} \right) dx \right|, \end{aligned} \tag{3.2}$$

and we have to prove this inequality in case of $r \in [0,1]$, $\beta \in [0,2\pi]$. Replacing β by $2\pi - \beta$, we get the same inequality. This shows that we have to prove (3.2) only in the case $\beta \in [0,\pi]$ and $r_0 \in [0,1)$. Since

$$\int_0^1 \ln\frac{1}{x}\left(\frac{1+x^2r_0^2+2xr_0\cos\beta}{1+x^2r_0^2-2xr_0\cos\beta}\right)^{\frac{3}{4}} \sin\left(\frac{3}{2}\arctan\frac{2xr_0\sin\beta}{1-x^2r_0^2}\right)dx \geq 0,\ \ \beta\in[0,\pi],$$

inequality (3.2) is equivalent to

$$\begin{gathered}\int_0^1 \ln\frac{1}{x}\left(\frac{1+x^2r_0^2+2xr_0\cos\beta}{1+x^2r_0^2-2xr_0\cos\beta}\right)^{\frac{3}{4}} \cos\left(\frac{3}{2}\arctan\frac{2xr_0\sin\beta}{1-x^2r_0^2}\right)dx \geq \\ \geq \int_0^1 \ln\frac{1}{x}\left(\frac{1+x^2r_0^2+2xr_0\cos\beta}{1+x^2r_0^2-2xr_0\cos\beta}\right)^{\frac{3}{4}} \sin\left(\frac{3}{2}\arctan\frac{2xr_0\sin\beta}{1-x^2r_0^2}\right)dx, \\ \beta\in[0,\pi],\ \ r_0\in[0,1).\end{gathered} \tag{3.3}$$

Let $t = 0$ and

$$F_0(z) = \int_0^1 \left(\frac{1+xz}{1-xz}\right)^{\frac{3}{2}} \ln\frac{1}{x}dx.$$

The function Φ defined by the equality

$$\Phi(r,\beta) = \operatorname{Re} F_0(re^{i\beta}) - \operatorname{Im} F_0(re^{i\beta})$$

is harmonic on $D = \{z \in \mathbb{C} : |z| < 1,\ \operatorname{Im} z > 0\}$. Inequality (3.3) is equivalent to

$$\Phi(r,\beta) = \operatorname{Re} F_0(z) - \operatorname{Im} F_0(z) > 0,\ \ z = re^{i\beta} \in D.$$

Thus, according to the maximum principle for harmonic functions we have to check the inequality $\Phi(r,\beta) > 0$ only on the frontier of D, namely in case of $z = e^{i\beta}$, $\beta \in [0,\pi]$, and in case of $z = u \in (-1,1)$. Lemma 2.7 implies that the inequality

$$\Phi(1,\beta) > 0,\ \ \beta\in[0,\pi]$$

holds. In case of $z = u \in (-1,1)$ we have

$$\Phi(r,\beta) = \int_0^1 \left(\frac{1+xu}{1-xu}\right)^{\frac{3}{2}} \ln\frac{1}{x}dx > 0$$

and the proof is completed. ■

The following theorem is an improvement of Theorem 1.3 and brings us closer to the best possible result.

Theorem 3.2. *Suppose $f, g \in \mathcal{A}$ and*

$$\operatorname{Re} \frac{g(z)}{z} > \left| \operatorname{Im} \frac{g(z)}{z} \right|, \; z \in U, \tag{3.4}$$

then the condition

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0, \; z \in U \tag{3.5}$$

implies that

$$F \in S^* \tag{3.6}$$

where $F = A(f)$.

Proof. Differentiating the equality $F = A(f)$ twice, we obtain

$$F'(z) + zF''(z) = f'(z).$$

The notations $p(z) = \frac{zF'(z)}{F(z)}$, $P(z) = \frac{F(z)}{g(z)}$ lead to

$$P(z)\big(zp'(z) + p^2(z)\big) = \frac{zf'(z)}{g(z)}, \quad z \in U.$$

The conditions of the theorem imply that

$$\operatorname{Re} P(z)\big(zp'(z) + p^2(z)\big) > 0, \quad z \in U. \tag{3.7}$$

First, we prove the inequality $\operatorname{Re} P(z) > 0$, $z \in U$. According to Theorem 3.1, inequalities (3.4) and (3.5) imply that

$$\operatorname{Re} \frac{F(z)}{z} > \left| \operatorname{Im} \frac{F(z)}{z} \right|, \; z \in U.$$

This inequality and (3.4), imply that $\operatorname{Re} P(z) = \frac{F(z)}{g(z)} > 0$, $z \in U$.

We are now in the position of proving $\operatorname{Re} p(z) > 0$, $z \in U$.

If $\operatorname{Re} p(z) > 0$, $z \in U$ is not true, then, according to Lemma 2.2, there are two real numbers $s, t \in \mathbb{R}$ and a point $z_0 \in U$, such that $p(z_0) = is$ and $z_0 p'(z_0) = t \leq -\frac{1}{2}(s^2 + 1)$. Thus

$$P(z_0)\big(z_0 p'(z_0) + p^2(z_0)\big) = P(z_0)(t - s^2)$$

and $\operatorname{Re} P(z_0) > 0$ implies that

$$\operatorname{Re} \left[P(z_0)\big(z_0 p'(z_0) + p^2(z_0)\big)\right] < 0.$$

This inequality contradicts (3.7), so we have $\operatorname{Re} p(z) = \operatorname{Re} \frac{zF'(z)}{F(z)} > 0$, $z \in U$. ■

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