ABOUT A CONDITION FOR STARLIKENESS

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Abstract. In this paper a result concerning the starlikeness of the image of the Alexander operator is improved. The techniques of differential subordinations and extreme points are used.

1. Introduction

Let $U(z_0, r)$ be the disc centered at the point $z_0$ and of radius $r$ defined by $U(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| < r \}$. $U$ denotes the open unit disc in $\mathbb{C}$, $U = \{ z \in \mathbb{C} : |z| < 1 \}$. Let $\mathcal{A}$ be the class of analytic functions $f$, which are defined on the unit disc $U$ and have the form: $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$. The subclass of $\mathcal{A}$ consisting of functions for which the range $f(U)$ is starlike with respect to 0, is denoted by $S^*$. An analytic characterization of $S^*$ is given by:

$$S^* = \left\{ f \in \mathcal{A} : \text{Re} \frac{zf'(z)}{f(z)} > 0, \ z \in U \right\}.$$

Another subclass of $\mathcal{A}$ we deal with is the class of close-to-convex functions denoted by $C$. A function $f \in \mathcal{A}$ belongs to the class $C$ if and only if there is

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a starlike function $g \in S^*$, so that $\text{Re} \frac{zf'(z)}{g(z)} > 0$, $z \in U$. We note that $C$ and $S^*$ contain univalent functions. The Alexander integral operator is defined by the equality:

$$A(f)(z) = \int_0^z \frac{f(t)}{t} \, dt.$$ 

The authors of [2] (pp. 310 – 311) proved the following result:

**Theorem 1.1.** Let $A$ be the Alexander operator and let $g \in A$ satisfy

\[(1.1) \quad \text{Re} \frac{zg'(z)}{g(z)} \geq \left| \text{Im} \frac{zg'(z)}{g(z)} \right|, \quad z \in U.\]

If $f \in A$ and

$$\text{Re} \frac{zf'(z)}{g(z)} > 0, \quad z \in U,$$

then $F = A(f) \in S^*$.

This theorem states that a subclass of $C$ is mapped by the Alexander operator to $S^*$. On the other hand we know that $A(C) \not\subset S^*$. In [3] and [4] several improvements of this result are proved, simplifying condition (1.1). Investigating this question, the following theorems have been deduced in [3]:

**Theorem 1.2.** Let $g \in A$ be a function which satisfies the condition:

\[(1.2) \quad \text{Re} \frac{zg'(z)}{g(z)} > 2.273 \left| \text{Im} \frac{zg'(z)}{g(z)} \right|, \quad z \in U.\]

If $f \in A$ satisfies

$$\text{Re} \frac{zf'(z)}{g(z)} > 0, \quad z \in U,$$

then $F = A(f) \in S^*$.

**Theorem 1.3.** If $f, g \in A$ and

\[(1.3) \quad \text{Re} \frac{g(z)}{z} > \frac{100}{83} \left| \text{Im} \frac{g(z)}{z} \right|, \quad z \in U,$

then the condition

$$\text{Re} \frac{zf'(z)}{g(z)} > 0, \quad z \in U$$

implies that $F = A(f) \in S^*$. 
The implications-chain is deduced in [3]: (1.1) ⇒ (1.2) ⇒ (1.3). Thus Theorem 1.2 and Theorem 1.3 are improvements of Theorem 1.1. Consequently, the question to determine the smallest \( c \in [0, \infty) \) for which the following statement holds arises naturally:

**If** \( f, g \in A \) and

\[
\text{Re} \frac{g(z)}{z} > c \left| \text{Im} \frac{g(z)}{z} \right|, \quad z \in U,
\]

then the condition

\[
\text{Re} \left( \frac{zf''(z)}{g(z)} \right) > 0, \quad z \in U
\]

implies that \( F = A(f) \in S^* \).

We are not able to answer this question completely at the moment, but we will prove that the statement holds for \( c = 1 \). This is an improvement of Theorem 1.3. In order to do this, we need the following lemmas.

**2. Preliminaries**

**Lemma 2.1.** ([2]) Let \( p(z) = a + \sum_{k=n}^\infty a_k z^k \) be analytic in \( U \) with \( p(z) \not\equiv a \), \( n \geq 1 \) and let \( q : U(0,1) \to \mathbb{C} \) be a univalent function with \( q(0) = a \). If there are two points \( z_0 \in U(0,1) \) and \( \zeta_0 \in \partial U(0,1) \) so that \( q \) is defined in \( \zeta_0 \), \( p(z_0) = q(\zeta_0) \) and \( p(U(0,r_0)) \subset q(U) \), where \( r_0 = |z_0| \), then there is an \( m \in [n, +\infty) \) so that

(i) \( z_0 p'(z_0) = m \zeta_0 q'(\zeta_0) \) and

(ii) \( \text{Re} \left( 1 + \frac{z_0 p''(z_0)}{p'(z_0)} \right) \geq m \text{Re} \left( 1 + \frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} \right) \).

**Lemma 2.2.** ([2]) Let \( p(z) = a + \sum_{k=n}^\infty a_k z^k, p(z) \not\equiv a \) and \( n \geq 1 \). If \( z_0 \in U \) and

\[
\text{Re} p(z_0) = \min \{ \text{Re} p(z) : |z| \leq |z_0| \},
\]

then

(i) \( z_0 p'(z_0) \leq -\frac{n}{2} \frac{|p(z_0) - a|^2}{\text{Re} (a - p(z_0))} \) and

(ii) \( \text{Re} \left[ \frac{z_0^2 p''(z_0)}{2} \right] + z_0 p'(z_0) \leq 0. \)

Recall that if \( f \) and \( g \) are analytic functions in \( U \) and there is a function \( w \) also analytic, satisfying \( w(0) = 0, |w(z)| \leq |z|, \quad z \in U \) and \( f(z) = g(w(z)) \),
z ∈ U, then the function f is said to be subordinate to g, written \( f \prec g \). If g is univalent then \( f(0) = g(0) \) and \( f(U) \subset g(U) \) implies that \( f \prec g \).

**Lemma 2.3.** ([1]) Let \( F_\alpha(z) = \left( \frac{1+z}{1-z} \right)^\alpha \), \( |c| \leq 1, \ c \neq -1 \). In case of \( \alpha \geq 1 \), the subordination \( f \prec F_\alpha \) holds if and only if there exists a probability measure \( \mu \) on \([0, 2\pi]\) having the property

\[
f(z) = \int_0^{2\pi} \left( \frac{1+z e^{-it}}{1-z e^{-it}} \right)^\alpha \, d\mu(t), \quad z \in U.
\]

The set of extreme points of the class \( \{ f \in A | f \prec F_\alpha \} \) is

\[
\left\{ f_t(z) = \left( \frac{1+z e^{-it}}{1-z e^{-it}} \right)^\alpha, \ t \in [0, 2\pi] \right\}.
\]

Let \( P \) denote the class of analytic functions of the form

\[
p(z) = 1 + c_1 z + c_2 z^2 + \ldots,
\]

and having the property \( \text{Re} \, p(z) > 0, \ z \in U \). We note that this property is equivalent to \( p(z) \prec \frac{1+z}{1-z} \) and Lemma 2.3 implies that there is a probability measure \( \mu \) on the interval \([0, 2\pi]\) such that \( p(z) = \int_0^{2\pi} \frac{1+z e^{-it}}{1-z e^{-it}} \, d\mu(t) \). This equality actually is the Herglotz formula.

**Lemma 2.4.** ([1, Corollary 3.7]) \( p \in P \) if and only if there exist a sequence of functions \( (p_n)_{n \geq 1} \) so that \( p_n \) has the form

\[
q(z) = \sum_{k=1}^m t_k \frac{1+zx_k}{1-zx_k},
\]

where \( |x_k| = 1, \ t_k \geq 0 \) and \( \sum_{k=1}^m t_k = 1 \) and \( p_n \to p \) uniformly on compact subsets of \( U \).

**Lemma 2.5.** If \( f, g \in A \) and

(2.1) \[ \text{Re} \, \frac{g(z)}{z} > \left| \text{Im} \, \frac{g(z)}{z} \right|, \quad z \in U, \]

and \( F = A(f) \), then the condition

(2.2) \[ \text{Re} \, \frac{zf'(z)}{g(z)} > 0, \quad z \in U \]

implies that there is a probability measure \( \mu \) on \([0, 2\pi]\), such that

\[
\frac{F(z)}{z} = \int_0^{2\pi} \int_0^1 \ln \left( \frac{1+x z e^{-it}}{1-x z e^{-it}} \right) \, dx \, d\mu(t), \quad z \in U.
\]
Proof. Inequality (2.1) is equivalent to

\[ \left| \arg \frac{g(z)}{z} \right| \leq \frac{\pi}{4}, \ z \in U. \]  

Applying Lemma 2.3 in case of \( c = 1, \alpha = 1 \) and \( F_1(z) = \frac{1+z}{1-z} \) it follows that:

\[ f'(z) = \frac{g(z)}{z} 2\pi \int_0^1 \frac{1 + z e^{-it}}{1 - z e^{-it}} d\nu(t), \]

where \( \nu \) is a probability measure on \([0, 2\pi]\). Thus we get:

\[ \left| \arg f'(z) \right| \leq \left| \arg \frac{g(z)}{z} \right| + \left| \arg \int_0^{2\pi} \frac{1 + z e^{-it}}{1 - z e^{-it}} d\nu(t) \right| < \frac{3\pi}{4}, \ z \in U. \]

We introduce the notation \( D = \{ z \in \mathbb{C} : |\arg(z)| \leq \frac{3\pi}{4} \} \). The function

\[ q(z) = \left( \frac{1 + z}{1 - z} \right)^\tau, \ \tau = \frac{3}{2}, \]

is the Riemann mapping from \( U \) to \( D \). (The principal branch of \( \left( \frac{1 + z}{1 - z} \right)^\tau \) is chosen.) The inequality (2.4) implies

\[ f'(z) \prec q(z), \]

and according to Lemma 2.3, this subordination is equivalent to

\[ f'(z) = 2\pi \int_0^{2\pi} \left( \frac{1 + z e^{-it}}{1 - z e^{-it}} \right)^{\frac{3}{2}} d\mu(t), \ z \in U, \]

where \( \mu \) denotes a probability measure on \([0, 2\pi]\). On the other hand, if

\[ q(z) = 1 + \sum_{n=1}^{\infty} a_n z^n, \]

then

\[ f'(z) = 1 + \sum_{n=1}^{\infty} a_n z^n \int_0^{2\pi} e^{-int} d\mu(t), \]

and

\[ \frac{F(z)}{z} = 1 + \sum_{n=1}^{\infty} a_n \frac{z^n}{n} \int_0^{2\pi} e^{-int} d\mu(t). \]
The equalities \( \int_0^1 x^n \ln \frac{1}{x} \, dx = \frac{1}{(n+1)^2}, \quad n \in \mathbb{N} \) imply

\[
F(z) = \frac{1}{z} \int_0^1 \ln \frac{1}{x} \left( 1 + \sum_{n=1}^{\infty} a_n x^n z^n \right) e^{-int} d\mu(t) \, dx.
\]

Lemma 2.4 implies that the second integration can be interchanged with the summation and the first integration and finally we get

\[
F(z) = \frac{1}{z} \int_0^1 \ln \frac{1}{x} \left( 1 + \sum_{n=1}^{\infty} a_n x^n z^n \right) e^{-int} d\mu(t) \, dx = \int_0^1 \int_0^1 \ln \frac{1}{x} \left( 1 + \sum_{n=1}^{\infty} a_n x^n z^n \right) e^{-int} d\mu(t) \, dx,
\]

where \( z \in U \).

Lemma 2.6. The function \( A : [0, \frac{3\pi}{4}] \to \mathbb{R} \),

\[
A(\theta) = (\pi - \theta)(\sin \theta - \cos \theta) \int_0^\infty \left( \frac{e^x - 1}{e^x + 1} \right)^{\frac{3}{2}} \frac{1}{e^x} \, dx - (\sin \theta + \cos \theta) \int_0^\infty \left( \frac{e^x - 1}{e^x + 1} \right)^{\frac{3}{2}} x \frac{1}{e^x} \, dx
\]

is increasing and the function \( B : [\frac{\pi}{6}, \frac{3\pi}{4}] \to \mathbb{R} \) defined by

\[
B(\theta) = \sqrt{2} \int_0^{\frac{\pi}{2}} x \left( \cot \frac{\theta + x}{2} \right)^{\frac{3}{2}} \cos x \, dx
\]

is decreasing.

Proof. Notice that

\[
I_1 = \int_0^\infty \left( \frac{e^x - 1}{e^x + 1} \right)^{\frac{3}{2}} \frac{1}{e^x} \, dx = 0.28..., \quad I_2 = \int_0^\infty \left( \frac{e^x - 1}{e^x + 1} \right)^{\frac{3}{2}} x \frac{1}{e^x} \, dx = 0.51...
\]

and \( I_1 < I_2 < 2I_1 \). Thus it follows that in case \( \theta \in [\frac{\pi}{4}, \frac{3\pi}{4}] \) we have

\[
A'(\theta) = (\pi - \theta)(\sin \theta + \cos \theta) I_1 + (\sin \theta - \cos \theta)(I_2 - I_1) > 0
\]
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and if \( \theta \in [0, \frac{\pi}{4}] \), then

\[ A'(\theta) > [(\pi - \theta)(\sin \theta + \cos \theta) + \sin \theta - \cos \theta] I_1 > 0. \]

Consequently the first part of the assertion is proved.

In the following we will prove that: \( B'(\theta) \leq 0, \theta \in [\frac{\pi}{6}, \frac{3\pi}{4}] \). We have:

\[
B'(\theta) = -\frac{3\sqrt{2}}{4} \int_0^{\pi/4 - \theta} x \left( \cot \left( \frac{\theta + x}{2} \right) \right)^{1/2} \left( \sin \left( \frac{\theta + x}{2} \right) \right)^{-2} \cos x dx, \theta \in \left[ \frac{\pi}{6}, \frac{3\pi}{4} \right].
\]

The claimed inequality holds evidently in case \( \theta \in [\frac{\pi}{2}, \frac{3\pi}{4}] \).

We will use the following equality to prove \( B'(\theta) \leq 0 \) in case \( \theta \in \left[ \frac{\pi}{6}, \frac{\pi}{2} \right] \):

\[
B'(\theta) = -\frac{3\sqrt{2}}{4} \int_0^{\pi/6 - \theta} x \left( \cot \left( \frac{\pi}{4} + \theta + x \right) \right)^{1/2} \left( \sin \left( \frac{\pi}{4} + \theta + x \right) \right)^{-2} \sin x dx - \frac{3\sqrt{2}}{4} \int_0^{\pi/6 - \theta} \left( \frac{\theta + x}{2} \right)^{1/2} \left( \sin \frac{\theta + x}{2} \right)^{-2} \cos x dx.
\]

Some elementary calculations lead to the following inequalities:

\[
\left( \cot \left( \frac{\theta + x}{2} \right) \right)^{1/2} \geq (1 + \sqrt{2}) \left( \cot \left( \frac{\pi}{4} + \theta + x \right) \right)^{1/2}, \quad x \in [0, \frac{\pi}{2} - \theta]
\]

\[
\left( \sin \left( \frac{\theta + x}{2} \right) \right)^{-2} \geq 2 \left( \sin \left( \frac{\pi}{4} + \theta + x \right) \right)^{-2}, \quad x \in [0, \frac{\pi}{2} - \theta]
\]

\[
x \cos x \geq \frac{\pi}{6} \tan \left( \frac{\pi}{2} \right) \sin x, \quad x \in [0, \frac{\pi}{2} - \theta].
\]

These inequalities imply that in case \( x \in [0, \frac{\pi}{2} - \theta] \) we have:

\[
x \left( \cot \left( \frac{\theta + x}{2} \right) \right)^{1/2} \left( \sin \left( \frac{\theta + x}{2} \right) \right)^{-2} \cos x \geq \frac{4(1 + \sqrt{2})}{5\sqrt{3}} \left( \frac{\pi}{4} + \theta + x \right)^{1/2} \left( \sin \left( \frac{\pi}{4} + \theta + x \right) \right)^{-2} \sin x \geq \left( \frac{\pi}{4} + \theta + x \right)^{1/2} \left( \sin \left( \frac{\pi}{4} + \theta + x \right) \right)^{-2} \sin x,
\]

and finally we get:

\[
\int_0^{\pi/6 - \theta} x \left( \cot \left( \frac{\pi}{4} + \theta + x \right) \right)^{1/2} \left( \sin \left( \frac{\pi}{4} + \theta + x \right) \right)^{-2} \cos x dx \geq \int_0^{\pi/6 - \theta} \left( \frac{\theta + x}{2} \right)^{1/2} \left( \sin \frac{\theta + x}{2} \right)^{-2} \sin x dx.
\]
The inequality $B' (\theta) \leq 0, \ \theta \in \left[ \frac{\pi}{6}, \frac{\pi}{2} \right]$ follows from (2.5) and (2.6).

**Lemma 2.7.** If

$$F(z) = \int_0^1 \left( \frac{1 + xz}{1 - xz} \right)^{\frac{3}{2}} \ln \frac{1}{x} \, dx,$$

then

$$\text{Re} F(e^{i\theta}) \geq \text{Im} F(e^{i\theta}), \ \theta \in [0, \pi].$$

**Proof.** We begin with the observation that the change of variable $x = e^{-t}$

leads to

$$F(e^{i\theta}) = \lim_{R \to \infty} \int_{\gamma_1} f(z) \, dz = - \lim_{R \to \infty} \left[ \int_{\gamma_2} f(z) \, dz + \int_{\gamma_3} f(z) \, dz + \int_{\gamma_4} f(z) \, dz \right].$$

Now consider the function:

$$f(z) = \left( \frac{e^z + e^{i\theta}}{e^z - e^{i\theta}} \right)^{\frac{3}{2}} z.$$  

We integrate it on $\Gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, where $\gamma_1(t) = t, \ t \in [0, R], \ \gamma_2(t) = R - it, \ t \in [0, \pi - \theta], \ \gamma_3(t) = R - t + i(\theta - \pi), \ t \in [0, R]$ and $\gamma_4(t) = i(\theta - \pi + t), \ t \in [0, \pi - \theta]$. Because $f$ is analytic in the interior of $\Gamma$ we have,

$$\int_{\Gamma} f(z) \, dz = 0$$

which leads to

$$F(e^{i\theta}) = \lim_{R \to \infty} \int_{\gamma_1} f(z) \, dz = - \lim_{R \to \infty} \left[ \int_{\gamma_2} f(z) \, dz + \int_{\gamma_3} f(z) \, dz + \int_{\gamma_4} f(z) \, dz \right].$$

The change of variable $\theta - \pi + t = -x$ in the second integral implies the equality

$$F(e^{i\theta}) = \int_0^\infty \left( \frac{e^x - 1}{e^x + 1} \right)^{\frac{3}{2}} (x + i(\theta - \pi))(- \cos \theta + i \sin \theta) \, dx -$$

$$- \int_0^{\pi - \theta} x \left( \cot \frac{\theta + x}{2} \right)^{\frac{3}{2}} e^{i(x + \frac{\pi}{2})} \, dx.$$
Thus it follows that

\[ \text{Re } F(e^{i\theta}) - \text{Im } F(e^{i\theta}) = (\pi - \theta)(\sin \theta - \cos \theta) \int_0^\infty \left( \frac{e^x - 1}{e^x + 1} \right)^2 \frac{1}{e^x} \, dx - \\
- (\sin \theta + \cos \theta) \int_0^\infty \left( \frac{e^x - 1}{e^x + 1} \right)^2 \frac{x}{e^x} \, dx + \sqrt{2} \int_0^{\pi - \theta} x \left( \cot \frac{\theta + x}{2} \right)^2 \cos x \, dx = \\
\int_0^{\pi} \left( \frac{e^x - 1}{e^x + 1} \right)^2 \frac{1}{e^x} \, dx - \\
- (\sin \theta + \cos \theta) \int_0^\infty \left( \frac{e^x - 1}{e^x + 1} \right)^2 \frac{x}{e^x} \, dx + \sqrt{2} \int_0^{\pi - \theta} x \left( \cot \frac{\theta + x}{2} \right)^2 \cos x \, dx = \]

(2.7) \[ \text{ } A(\theta) + B(\theta). \]

According to the monotonicity of \( A \) and \( B \), the inequalities hold

\[ B(\theta) + A(\theta) \geq B(\theta_k) + A(\theta_{k-1}), \quad \theta \in [\theta_{k-1}, \theta_k], \quad k = 21, 90. \]

Now, if we check that

(2.8) \[ B(\theta_k) + A(\theta_{k-1}) > 0, \quad \theta_k = \frac{k\pi}{120}, \quad k = 21, 90 \]

we obtain

\[ B(\theta) + A(\theta) > 0, \quad \theta \in [\theta_{k-1}, \theta_k], \quad k = 21, 90 \]

and the proof is done in case of \( \theta \in \left[ \frac{\pi}{6}, \frac{3\pi}{4} \right] \). Inequalities (2.8) can be checked easily by using a computer program. The inequality \( \text{Re } F(e^{i\theta}) \geq \text{Im } F(e^{i\theta}) \), \( \theta \in \left[ \frac{3\pi}{4}, \pi \right] \) follows from (2.7). It remains to prove the assertion in case \( \theta \in \left[ 0, \frac{\pi}{6} \right] \).

We put in the integral \( \int_0^{\pi - \theta} x \left( \cot \frac{\theta + x}{2} \right)^2 \cos x \, dx \) the change of variable \( x + \theta = u \) and we obtain

\[ \text{Re } F(e^{i\theta}) - \text{Im } F(e^{i\theta}) = (\pi - \theta)(\sin \theta - \cos \theta) \int_0^\infty \left( \frac{e^x - 1}{e^x + 1} \right)^2 \frac{1}{e^x} \, dx - \\
- (\sin \theta + \cos \theta) \int_0^\infty \left( \frac{e^x - 1}{e^x + 1} \right)^2 \frac{x}{e^x} \, dx + \sqrt{2} \int_0^{\pi} \left( \cot \frac{u}{2} \right)^2 \cos (u - \theta) \, du. \]

This can be rewritten as follows

\[ \text{Re } F(e^{i\theta}) - \text{Im } F(e^{i\theta}) = \\
= \sin \theta \left( (\pi - \theta)I_1 - I_2 + \sqrt{2} \int_0^{\pi} (u - \theta) \left( \cot \frac{u}{2} \right)^2 \sin u \, du \right) + \\
+ \cos \theta \left( -(\pi - \theta)I_1 - I_2 + \sqrt{2} \int_0^{\pi} (u - \theta) \left( \cot \frac{u}{2} \right)^2 \cos u \, du \right). \]
(1 and 2 are defined in the proof of the previous lemma.) We observe that the mapping \( C : [0, \pi/6] \) defined by

\[
C(\theta) = (\pi - \theta)I_1 - I_2 + \sqrt{2} \int_0^\pi (u - \theta)(\cot u/2)^2 \sin u \, du
\]

is strictly decreasing. This implies the inequality: \( C(\theta) \geq C(\pi/6) \geq 6.8... \) Thus it follows that

\[
\text{Re } F(e^{i\theta}) - \text{Im } F(e^{i\theta}) \geq \\
\geq \cos \left( 6.8 \tan \theta - (\pi - \theta)I_1 - I_2 + \sqrt{2} \int_0^\pi (u - \theta)(\cot u/2)^2 \cos u \, du \right).
\]

Let the functions \( D \) and \( E \) be defined by the equalities

\[
D(\theta) = 6.8 \tan \theta - (\pi - \theta)I_1 - I_2
\]

and

\[
E(\theta) = \sqrt{2} \int_0^\pi (u - \theta)(\cot u/2)^2 \cos u \, du.
\]

It is simple to show that \( D \) is strictly increasing and \( E \) is strictly decreasing. The monotonicity of these functions imply

\[
D(\theta) + E(\theta) > D(\theta_{k-1}) + E(\theta_k), \quad \theta_k = k\pi/120, \quad k = 1, 20.
\]

If we prove that \( D(\theta_{k-1}) + E(\theta_k) > 0, \quad \theta_k = k\pi/120, \quad k = 1, 20 \), then it follows that \( \text{Re } F(e^{i\theta}) \geq \text{Im } F(e^{i\theta}) \), \( \theta \in [0, \pi/6] \) and the proof is done. The inequalities \( D(\theta_{k-1}) + E(\theta_k) > 0, \quad k = 1, 20 \) can be checked easily by using a computer program.

3. The main result

**Theorem 3.1.** If \( f, g \in \mathcal{A} \) and

\[
\text{Re } \frac{g(z)}{z} > \left| \text{Im } \frac{g(z)}{z} \right|, \quad z \in U,
\]
then the condition
\[ \text{Re} \frac{zf'(z)}{g(z)} > 0, \ z \in U \]
implies that
\[ (3.1) \quad \text{Re} \frac{F(z)}{z} > \left| \text{Im} \frac{F(z)}{z} \right|, \ z \in U, \]
where \( F = A(f) \).

**Proof.** Let \( \Lambda \) be the set of probability measures on \([0, 2\pi]\). We introduce the notation
\[ B = \left\{ \int_0^{2\pi} \int_0^1 \ln \left( \frac{1 + xze^{-it}}{1 - xze^{-it}} \right)^{1/2} dx d\mu(t) \mid \mu \in \Lambda \right\}. \]

According to Lemma 2.5 we have \( F \in B \). Let \( z_0 \in U \) be an arbitrarily fixed point, and let \( p_{z_0} \) be the functional defined by
\[ p_{z_0} : B \to \mathbb{R}, \quad p_{z_0}(F) = \text{Re} F(z_0) - \left| \text{Im} F(z_0) \right|. \]

If we prove that \( p_{z_0}(F) \geq 0 \) for every \( F \in B \) in case of an arbitrarily fixed point \( z_0 \in U \), then inequality (3.1) follows. Since the functional \( p_{z_0} \) is concave, according to Lemma 2.5, we have to verify \( p_{z_0}(F) \geq 0 \) only for the extreme points of the class \( B \). It follows from Lemma 2.5 that the extreme points of this class are
\[ F_t(z) = \int_0^1 \ln \left( \frac{1 + xze^{-it}}{1 - xze^{-it}} \right)^{1/2} dx, \ t \in [0, 2\pi]. \]

For \( z_0 = r_0e^{i\theta_0} \), the inequality \( p_{z_0}(F_t) \geq 0 \) is equivalent to
\[ \int_0^1 \ln \left( \frac{1 + x^2r_0^2 + 2xr_0 \cos(\theta_0 - t)}{1 + x^2r_0^2 - 2xr_0 \cos(\theta_0 - t)} \right)^{1/2} \cos \left( \frac{3}{2} \arctan \frac{2xr_0 \sin(\theta_0 - t)}{1 - x^2r_0^2} \right) dx \geq \left| \int_0^1 \ln \left( \frac{1 + x^2r_0^2 + 2xr_0 \cos(\theta_0 - t)}{1 + x^2r_0^2 - 2xr_0 \cos(\theta_0 - t)} \right)^{1/2} \sin \left( \frac{3}{2} \arctan \frac{2xr_0 \sin(\theta_0 - t)}{1 - x^2r_0^2} \right) dx \right|. \]

Denoting \( \theta_0 - t \) by \( \beta \), we obtain
\[ \int_0^1 \ln \left( \frac{1 + x^2r_0^2 + 2xr_0 \cos \beta}{1 + x^2r_0^2 - 2xr_0 \cos \beta} \right)^{1/2} \cos \left( \frac{3}{2} \arctan \frac{2xr_0 \sin \beta}{1 - x^2r_0^2} \right) dx \geq \left| \int_0^1 \ln \left( \frac{1 + x^2r_0^2 + 2xr_0 \cos \beta}{1 + x^2r_0^2 - 2xr_0 \cos \beta} \right)^{1/2} \sin \left( \frac{3}{2} \arctan \frac{2xr_0 \sin \beta}{1 - x^2r_0^2} \right) dx \right|. \]
and we have to prove this inequality in case of \( r \in [0, 1], \beta \in [0, 2\pi] \). Replacing \( \beta \) by \( 2\pi - \beta \), we get the same inequality. This shows that we have to prove (3.2) only in the case \( \beta \in [0, \pi] \) and \( r_0 \in [0, 1) \). Since

\[
\int_0^1 \ln \frac{1}{x} \left( \frac{1 + x^2 r_0^2 + 2xr_0 \cos \beta}{1 + x^2 r_0^2 - 2xr_0 \cos \beta} \right)^{\frac{3}{2}} \sin \left( \frac{3}{2} \arctan \frac{2xr_0 \sin \beta}{1 - x^2 r_0^2} \right) dx \geq 0, \quad \beta \in [0, \pi],
\]

inequality (3.2) is equivalent to

\[
\int_0^1 \ln \frac{1}{x} \left( \frac{1 + x^2 r_0^2 + 2xr_0 \cos \beta}{1 + x^2 r_0^2 - 2xr_0 \cos \beta} \right)^{\frac{3}{2}} \cos \left( \frac{3}{2} \arctan \frac{2xr_0 \sin \beta}{1 - x^2 r_0^2} \right) dx \geq \int_0^1 \ln \frac{1}{x} \left( \frac{1 + x^2 r_0^2 + 2xr_0 \cos \beta}{1 + x^2 r_0^2 - 2xr_0 \cos \beta} \right)^{\frac{3}{2}} \sin \left( \frac{3}{2} \arctan \frac{2xr_0 \sin \beta}{1 - x^2 r_0^2} \right) dx,
\]

(3.3)

\( \beta \in [0, \pi], \ r_0 \in [0, 1) \).

Let \( t = 0 \) and

\[
F_0(z) = \int_0^1 \left( \frac{1 + xz}{1 - xz} \right)^{\frac{3}{2}} \ln \frac{1}{x} dx.
\]

The function \( \Phi \) defined by the equality

\[
\Phi(r, \beta) = \text{Re} F_0(re^{i\beta}) - \text{Im} F_0(re^{i\beta})
\]

is harmonic on \( D = \{ z \in \mathbb{C} : |z| < 1, \ \text{Im}z > 0 \} \). Inequality (3.3) is equivalent to

\[
\Phi(r, \beta) = \text{Re} F_0(z) - \text{Im} F_0(z) > 0, \ z = re^{i\beta} \in D.
\]

Thus, according to the maximum principle for harmonic functions we have to check the inequality \( \Phi(r, \beta) > 0 \) only on the frontier of \( D \), namely in case of \( z = e^{i\beta}, \ \beta \in [0, \pi] \), and in case of \( z = u \in (-1, 1) \). Lemma 2.7 implies that the inequality

\[
\Phi(1, \beta) > 0, \ \beta \in [0, \pi]
\]

holds. In case of \( z = u \in (-1, 1) \) we have

\[
\Phi(r, \beta) = \int_0^1 \left( \frac{1 + xu}{1 - xu} \right)^{\frac{3}{2}} \ln \frac{1}{x} dx > 0
\]

and the proof is completed. \( \blacksquare \)

The following theorem is an improvement of Theorem 1.3 and brings us closer to the best possible result.
Theorem 3.2. Suppose \( f, g \in \mathcal{A} \) and
\[
\text{Re} \frac{g(z)}{z} > \left| \text{Im} \frac{g(z)}{z} \right|, \quad z \in U,
\]
then the condition
\[
\text{Re} \frac{zf'(z)}{g(z)} > 0, \quad z \in U
\]
implies that
\[
F \in S^*
\]
where \( F = A(f) \).

Proof. Differentiating the equality \( F = A(f) \) twice, we obtain
\[
F'(z) + zF''(z) = f'(z).
\]
The notations \( p(z) = \frac{RF(z)}{g(z)} \), \( P(z) = \frac{F(z)}{9(z)} \) lead to
\[
P(z)(zp'(z) + p^2(z)) = \frac{zf'(z)}{g(z)}, \quad z \in U.
\]
The conditions of the theorem imply that
\[
\text{Re} P(z)(zp'(z) + p^2(z)) > 0, \quad z \in U.
\]
First, we prove the inequality \( \text{Re} P(z) > 0, z \in U \). According to Theorem 3.1, inequalities (3.4) and (3.5) imply that
\[
\text{Re} \frac{F(z)}{z} > \left| \text{Im} \frac{F(z)}{z} \right|, \quad z \in U.
\]
This inequality and (3.4), imply that \( \text{Re} P(z) = \frac{F(z)}{g(z)} > 0, z \in U \).

We are now in the position of proving \( \text{Re} p(z) > 0, z \in U \).

If \( \text{Re} p(z) > 0, z \in U \) is not true, then, according to Lemma 2.2, there are two real numbers \( s, t \in \mathbb{R} \) and a point \( z_0 \in U \), such that \( p(z_0) = is \) and \( z_0p'(z_0) = t \leq -s^2/2 \). Thus
\[
P(z_0)(z_0p'(z_0) + p^2(z_0)) = P(z_0)(t - s^2)
\]
and \( \text{Re} P(z_0) > 0 \) implies that
\[
\text{Re} \left[ P(z_0)(z_0p'(z_0) + p^2(z_0)) \right] < 0.
\]
This inequality contradicts (3.7), so we have \( \text{Re} p(z) = \text{Re} \frac{zF'(z)}{g(z)} > 0, z \in U \).
References


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