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RATIONAL MODELING OF MULTI-LEAD QRS COMPLEXES IN ECG SIGNALS

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Abstract. The main topic of this paper is the relation between the QRS complexes recorded from different pairs of electrodes of the same ECG signal. The electrode combinations I, II, III, aVR, aVL, aVF will be considered. Our aim is to provide a simple mathematical model for explaining and demonstrating the relation between the records. The model we construct is based on elementary rational functions having a single pole of second order. The records are then represented in a proper three dimensional function space determined by the pole. We show that the same pole turns to be optimal for each of the electrode combinations. For finding the optimal pole we have developed a hyperbolic version of the Nelder–Mead algorithm. We also show that if we extend the function space by adding the elementary rational functions with the same pole of order one, three, and four etc. then a good approximation of QRS complexes can be given for all of the records. We used the Physionet database [3] for testing our model.

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1. Introduction

Heart functioning is accompanied by electric activity of the heart over time. This causes electrical changes on the human skin that is amplified and measured by an ECG device. For this purpose electrodes are placed over the surface of the human body according to a prescribed arrangement. Then the electric voltage between two of them is measured. Each pair is called lead and they detect the same electric heart activity but from a different angle. The 12-lead ECG, in which 12 different electrical signals are recorded, are the most widely used. In this paper the I, II, III, aVR, aVL, aVF leads will be considered. For these leads the corresponding electrodes are coplanar, therefore we may use a two dimensional model for representing the relation between the records. We can view these records as functions that are nearly periodic. A typical period of an ECG record consists of a P wave, a QRS complex, a T wave. They generate a natural segmentation of the signal. The lengths of time intervals determined by the starting and endpoints of the waves contain important medical information along with the local extremal values. The QRS complexes are of special importance in this respect. For details we refer to [2].

2. The mathematical model and the rational functions

The QRS complex is of special diagnostic importance in the analysis of ECG signals. In our model they will be represented by means of a simple function that is analytic on the closed unit disc. We have found rational functions of order two to be adequate for this purpose.

Namely, for an a in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ we take the elementary rational function

$$r_a(z) := \frac{1}{1 - \overline{a}z} \qquad (|z| \leq 1)\,.$$

where the pole is $a/|a|^2$, a is called the inverse pole of r_a. Then we restrict r_a onto the unit circle and decompose it into real and imaginary parts

$$r_a^2(e^{it}) = U_a^1(t) + iU_a^2(t) \qquad (t \in \mathbb{R}, a \in \mathbb{D})\,.$$

Then the QRS complexes will be modeled as proper linear combinations of $U_a^1(t)$, $U_a^2(t)$, and the constant function $U_a^0(t) = 1$. In other words the QRS complexes will be modeled by the elements of the three dimensional subspace

$$\mathcal{L}_a := \operatorname{span}\{U_a^i : i = 0, 1, 2\}\,.$$

The role of the constant function is basically normalization. It has no effect on the shape of the curve. Therefore, it has no morphological importance. This means that our model is essentially two dimensional or in other words a planar one. This modeling of course implies that also the QRS complexes should be understood as real functions defined on an interval of length 2π. The preprocessing of the QRS signals is detailed in the *Tests and Results* section. If the pole a is fixed, i.e. the subspace $\mathcal{L}_a$ is given, then the approximation will be Fourier projection of the QRS generated functions onto the subspace. To this order it is convenient to have an orthonormal basis in $\mathcal{L}_a$, which is quite easy to obtain in this case. Namely, if $a \in \mathbb{D}$ is given in Euler form $a = re^{\alpha i}$ then by

$$\begin{aligned}\frac{1}{(1-\overline{a}z)^2} &= \sum_{n=0}^{\infty}(n+1)(\overline{a}z)^n = \\ &= \sum_{n=0}^{\infty}(n+1)r^n(\cos(n(t-\alpha)) + i\sin(n(t-\alpha))) \qquad (z = e^{it})\end{aligned}$$

we have

$$U_a^1(t) = \sum_{n=0}^{\infty}(n+1)r^n\cos(n(t-\alpha))\,, \quad U_a^2(t) = \sum_{n=0}^{\infty}(n+1)r^n\sin(n(t-\alpha))\,.$$

Hence it follows immediately that the functions U_a^0, $U_a^1 - U_a^0$, U_a^2 are pairwise orthogonal with respect to the usual scalar product

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)g(t)\,dt \qquad (f,\, g \in H)$$

in the real Hilbert space $H = L^2[-\pi, \pi)$. Consequently, this triple forms a basis in $\mathcal{L}_a$. Taking the norm induced by the scalar product we have

$$\|U_a^1 - 1\|^2 = \|U_a^2\|^2 = \frac{1}{2}\sum_{n=1}^{\infty}(n+1)^2r^{2n}\,.$$

We can express it in a closed form by considering

$$\frac{d}{dz}\frac{z}{(1-z)^2} = \frac{1+z}{(1-z)^3} = \sum_{n=1}^{\infty}n^2z^{n-1}\,.$$

Namely, by substituting $z = r^2$ we obtain

$$\sum_{n=1}^{\infty}n^2r^{2(n-1)} = \frac{1+r^2}{(1-r^2)^3}\,.$$

Hence,

$$N^2(r) := \|U_a^1 - 1\|^2 = \|U_a^2\|^2 = \frac{1}{2}\Big(\frac{1+r^2}{(1-r^2)^3} - 1\Big)\,.$$

Then the orthonormal basis we will use in $\mathcal{L}_a$ is

$$u_a^0 := U_a^0 = 1\,, \quad u_a^1 := \frac{U_a^1 - 1}{N(r)}\,, \quad u_a^2 := \frac{U_a^2}{N(r)}\,,$$

which can be written in an explicit form as follows

$$u_a^1(t) = \sqrt{\frac{2}{\frac{1+r^2}{(1-r^2)^3} - 1}}\left(\frac{1 - 2r\cos(t-\alpha) + r^2\cos(2(t-\alpha))}{(1 - 2r\cos(t-\alpha) + r^2)^2} - 1\right)\,,$$

$$u_a^2(t) = \sqrt{\frac{2}{\frac{1+r^2}{(1-r^2)^3} - 1}}\frac{2r\sin(t-\alpha) - r^2\sin(2(t-\alpha))}{(1 - 2r\cos(t-\alpha) + r^2)^2}\,.$$

Then by the Bessel formula we have that the best approximation of a preprocessed QRS complex in the Hilbert subspace $\mathcal{L}_a$ is

$$d_f(a) = \|f\|^2 - \sum_{j=0}^{2} |\langle f, u_a^j\rangle|^2\,.$$

This is the error for a fixed inverse pole a. Since we may choose a arbitrarily in the unit disc the process goes on with minimizing the error function $d_f : \mathbb{D} \to \mathbb{R}$. If the point of minima is $a_m \in \mathbb{D}$ then the QRS complex will be represented by the corresponding Fourier-projection

$$f_{\mathrm{QRS}} := S_{a_m} f = \sum_{j=0}^{2} \langle f, u_{a_m}^j\rangle u_{a_m}^j\,.$$

For this step of the process we have used and developed the hyperbolic version of the Nelder–Mead algorithm.

3. Hyperbolic Nelder–Mead algorithm

The Nelder–Mead simplex algorithm [8] is a fast and widely used direct search method for multidimensional unconstrained minimization. Despite of its age it is still a very popular method for practitioners. It is simple and does not contain differentiation. It is based on concepts and transformations in the

usual Eucledian geometry. Concerning the use of Nelder–Mead algorithm for finding the poles of rational functions we refer to [7]. In this section we take the Poincaré model on $\mathbb{D}$ of the hyperbolic geometry. The reason behind taking the hyperbolic model is that we need to keep the inverse poles within the unit circle. We note that the hyperbolic model turned to be useful in system and control theories as well (see e.g. [1]). In the Poincaré model the arcs intersecting the unit circle perpendicularly and the diameters play the role of straight lines. These can be described by means of the Blaschke functions

$$B_a(z) := \frac{z-a}{1-\overline{a}z} \qquad (|z| \le 1,\, |a| < 1\,, a,\, z \in \mathbb{C})\,.$$

It is known that $B_a : \mathbb{D} \to \mathbb{D}$, and $B_a : \mathbb{T} \to \mathbb{T}$ are bijections for any $a \in \mathbb{D}$, and every hyperbolic line can be given in a parametric form

$$(-1,1) \ni t \to B_{\mathfrak{a}}(t) := \epsilon B_a(t)\,, \qquad \text{where } a \in \mathbb{D}, \quad \text{and } \epsilon \in \mathbb{T}\,.$$

It can be shown that for any pairs $w_1, w_2 \in \mathbb{D}$, $w_1 \neq w_2$ there exist a unique parameter $\mathfrak{a} := (a, \epsilon) \in \mathbb{D} \times \mathbb{T}$, and a number $p \in (0,1)$ such that $B_{\mathfrak{a}}(0) = w_1$, $B_{\mathfrak{a}}(p) = w_2$. Moreover the parametrization of the hyperbolic line connecting w_1, and w_2 is $B_{\mathfrak{a}}$, and $B_{\mathfrak{a}}$ maps the interval $[0,p]$ onto the hyperbolic segment connecting w_1, and w_2. These parameters can be calculated as follows

$$p = |B_{w_1}(w_2)|, \quad \epsilon = \frac{B_{w_1}(w_2)}{|B_{w_1}(w_2)|}, \quad a = -\epsilon w_1\,.$$

The so-called pseudo-hyperbolic metric on $\mathbb{D}$ is defined by

$$\rho(z_1, z_1) := \frac{|z_1 - z_2|}{|1 - \overline{z}_1 z_2|} = |B_{z_1}(z_2)| \qquad (z_1,\, z_2 \in \mathbb{D})\,.$$

Then $(\mathbb{D}, \rho)$ is a complete metric space which is invariant with respect to the Blaschke transforms, i.e.

$$\rho(B_{\mathfrak{a}}(z_1), B_{\mathfrak{a}}(z_2)) = \rho(z_1, z_2) \qquad (z_1,\, z_2 \in \mathbb{D},\, \mathfrak{a} \in \mathbb{D} \times \mathbb{T})\,.$$

One can prove that the group of hyperbolic congruences can be identified with the collection of the transforms $\{B_{\mathfrak{a}} : \mathfrak{a} \in \mathbb{D} \times \mathbb{T}\}$. In particular, the geometric operations in the Nelder–Mead algorithm can all be expressed in the hyperbolic plain by means of the $B_{\mathfrak{a}}$ functions. We note that the interval $(-1,1)$ itself is a hyperbolic line in which the distance between the points $-1 < q < p < 1$ is

$$\rho(p,q) = \frac{p-q}{1-pq}.$$

We take the $\overline{0p}$ hyperbolic line segment as a special case. Let the hyperbolic middle point of it be denoted by p_F and let p_T be the reflection of 0 with

respect to the point p. Then the following equations of second degree hold for them

$$p_F = \frac{p - p_F}{1 - pp_F}\,, \quad p = \frac{p_T - p}{1 - pp_T}\,.$$

It is easy to see that they both have a unique solution in $(0, 1)$. For any hyperbolic line segment $\overline{w_1 w_2}$ the middle point w_F, and the reflection w_T of w_1 with respect to w_2 can be be given in the form

$$B_{\mathfrak{a}}(p_F) = w_F\,, \quad B_{\mathfrak{a}}(p_T) = w_T.$$

The hyperbolic version of the Nelder–Mead algorithm is designed for minimization of functions of type $F : \mathbb{D} \to \mathbb{R}$. In our problems related to the QRS complexes we will need the following variant of it. In order to define the main step of the algorithm let z_1, z_2, z_3 belong to $\mathbb{D}$ and be indexed according to the relation

$$F(z_3) \le F(z_2) \le F(z_1)\,.$$

Furthermore, let the middle point of the hyperbolic line segment $\overline{z_2 z_3}$ be denoted by z_0, and the hyperbolic reflection of z_1 with respect to z_0 by z_e. Then a new point z' that depends on the value $F(z_e)$ will be defined. Finally, we replace z_1 by z' (*Case I.*), or modify the original triple (*Case II.*) in order to get the new triple of points. We note that in *Case I.* we define z' so that the condition

$$\max\{F(z_2), F(z_3), F(z')\} < \max\{F(z_1), F(z_2), F(z_3)\}$$

holds for it.

Case I.:

a) If $F(z_3) \le F(z_e) < F(z_2)$ then let $z' = z_e$.

b) If $F(z_e) < F(z_3)$ then let z_4 be the hyperbolic reflection of z_0 with respect to z_e. This is illustrated in Figure 1. Then by comparing the values $F(z_4)$ and $F(z_e)$ the point z' is defined as follows:
If $F(z_4) < F(z_e)$ then let $z' = z_4$, otherwise let $z' = z_e$.

c) If $F(z_2) \le F(z_e) < F(z_1)$ then let us take the middle point of the hyperbolic line segment $\overline{z_0 z_e}$ and denote it by z_4. In case $F(z_4) \le F(z_e)$ let $z' = z_4$, otherwise turn to *Case II.* below.

d) If $F(z_1) \le F(z_e)$ then let z_4 be the middle point of the hyperbolic line segment $\overline{z_0 z_1}$. In case $F(z_4) < F(z_1)$ let $z' = z_4$, otherwise turn to *Case II.* below.

Case II.: Let z_1' be the middle point of $\overline{z_1 z_3}$. Similarly, let z_2' be the middle point of $\overline{z_2 z_3}$. Then the triangle $z_1 z_2 z_3$ will be replaced by $z_1' z_2' z_3$. This is illustrated in Figure 2.

It is easy to check that by this construction the condition made above for z' is fulfilled. By repeating the steps the triangle shrinks around the best vertex and the process can be stopped when the desired accuracy is reached.

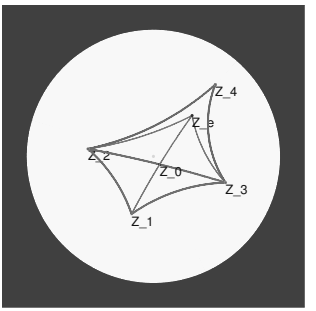

Figure 1.

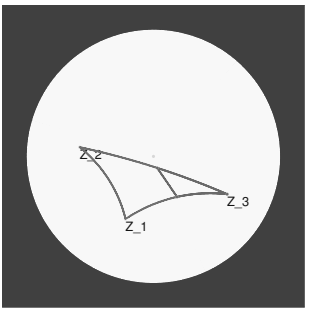

Figure 2.

We note that generally the limit of the process may depend on the starting triangle. So it should be chosen according to the nature of the problem. For instance, we experienced that if it is applied for the record of an entire heartbeat rather than for the QRS complex only than there are two local minima, and the process may converge to any of them. In that case we can ensure the proper convergence by taking the initial values close to the expected limit. On the other hand the same experiences showed that in case of QRS complexes and the linear subspace $\mathcal{L}_a$ defined above the minimum is unique and the process converges to the point of minimum.

4. Tests and results

We have used signals from the Physionet ([3], `http://www.physionet.org`) ECG database to test our model. More precisely, 77 records of the 52 healthy subjects there, the first 10 heartbeats for each of them, altogether 770 heartbeats were taken. They can be found in the PTB Diagnostic ECG Database (`http://www.physionet.org/physiobank/database/ptbdb/`) of Physionet.

In order to use our model we had to transform the QRS complexes to 2π periodic functions. This preprocessing of the QRS complex went as follows.

First we used the segmentation program built in Physionet, called ecgpuwave (`http://www.physionet.org/physiotools/ecgpuwave`). For more information about the program we refer to
`http://www.physionet.org/physiotools/wag/ecgpuw-1.htm♯sect8`.
It is available as part of PhysioToolkit.

After the segmentation the result was a function restricted onto an interval. The values at the two endpoints were usually not equal. Therefore we extended the function linearly by taking the slopes at the two endpoints and also used vertical shift. Then a Tukey window was applied to generate a 2π periodic signal. The parameters of the window were set so that the constant one part of the window corresponded to the original segmentation interval, and the transition part was about 10 percent on each sides. Outside of it the function was defined to be zero. We performed it for the six leads: I, II, III, aVR, aVL, aVF. Then we used the hyperbolic Nelder–Mead algorithm for finding the best pole for the rational approximation detailed above. During the algorithm the distance between the rational Fourier projection and the QRS function was calculated only for the interval on which the QRS complex was originally supported. This included about a hundred sample points.

In our first test we were interested in the dependence of the best poles for the different leads. For this reason we compared the 6 poles received from the records of 6 leads of the same heartbeat. We found that the best poles for the different leads are very close to each other even for real records that contain noise.

In Figure 3. we show the positions of the inverse poles in the unit circle for the signal marked as "s0479" in the ptbdb database. The average of the six inverse poles is marked by $*$. In Figure 4. we demonstrate the result of the test performed for all the 770 heartbeats. We calculated the maximum of the distances of the 6 inverse poles from their average. The test shows that the inverse poles are within a 0.1 radius circle for about 65% of the records and within a 0.2 radius circle for more than 95% of them.

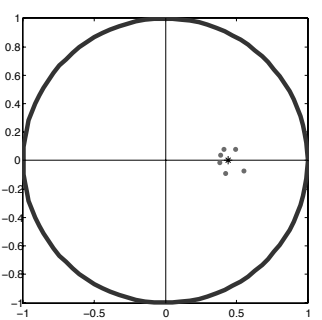

Figure 3.

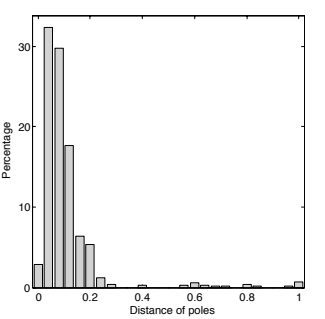

Figure 4.

On the basis of the test we may consider the pole to be invariant with respect to the leads, and therefore we may conclude that the pole is characteristic for the heartbeat itself rather than for the different leads.

After having found the poles the Fourier projection of the QRS complexes onto the corresponding $\mathcal{L}_a$ can be calculated. Then a simple approximation for the QRS complexes in the 6 leads will be received. We can not of course expect a high accuracy from this approximation. All we want to demonstrate is the surprising result, that even these simple rational functions show the basic characters of the records. In Figure 5. we took the same records as in Figure 3.

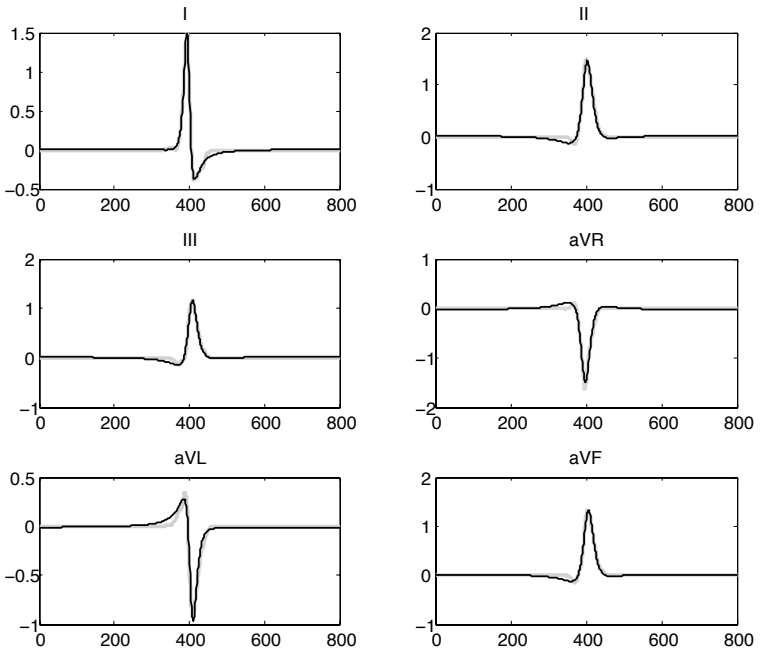

Figure 5.

The green curves correspond to the QRS complexes, and the black curves are the rational approximates. Besides the geometric similarity we note that the PRD's, the percent root mean square differences, calculated on the support of the QRS complexes fall between 8.7 and 13 for all of the leads.

Finally we were interested in improving the accuracy of the approximation. For this reason we kept the optimal pole calculated but instead of just using the corresponding elementary rational function of degree 2 we added those of degree 1, 3, 4 and higher. In other words we took the linear space spanned by the real and imaginary parts of the functions $r_a^n(e^{it})$ $(t \in \mathbb{R},\, n = 0, \dots N,\, N \in \mathbb{N})$.

Taking an orthonormal basis in this $2N+1$ dimensional real subspace the approximation is the Fourier projection of the functions corresponding to the QRS complexes onto the subspace. The degree of approximation of the Fourier projection improves by increasing N. At the same time the compression ratio decreases since more and more coefficients should be stored. This relation is demonstrated in Figure 6. by taking 5 records, (s0306lre, s0301lre, s0324lre, s0479_re, s0552_re) from the database. We performed the computation for $N = 12$ for the 770 heartbeats, and calculation the error in terms of PRD. The result is presented in Figure 7. It shows that for more than 80% of the records the PRD is not greater than 5%.

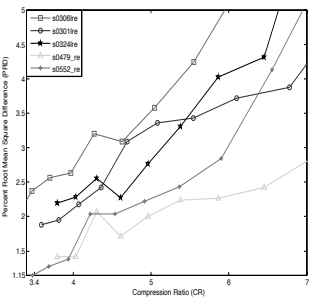

Figure 6.

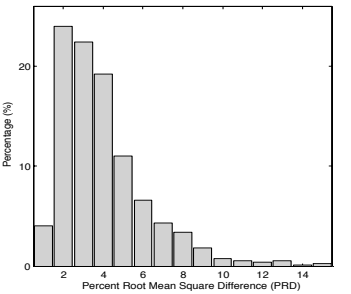

Figure 7.

We note that this method is related to the theory of discrete Laguerre systems, which are special Malmquist–Takenaka systems (see e.g. [5]). Concerning our latest results in this respect we refer to [4], and [9].

5. Future work

Finally we would like to remark that the model and the results presented above naturally induce several questions that may initiate further investigations. Here we only mention two as examples.

a) Our model in this paper is a planar one and therefore can be applied for 6 leads only. We plan to develop a spatial model in order to demonstrate the relation between all the 12 leads.

b) As it was shown in the paper the best poles for the different leads are characteristic for the heartbeat itself rather than for the leads. So far we have only used a database of healthy subjects. We will continue this test for patients showing various symptoms in order to find out whether the position of the pole could be an indicator for certain malfunctions of the heart.

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