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THE COMMUTER'S PARADOX: WHY IT TAKES LONGER TO GET HOME THAN TO GET TO WORK

Péter Burcsi (Budapest, Hungary)

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Abstract. We examine a simple model of public transport where under certain conditions, there is an asymmetry in the traveling time between two nodes in the two different directions. This asymmetry is surprising at first sight, as it may also occur when every individual line is symmetric with respect to direction – although only when line changes are made during travel. The time-irreversibility of the journey is caused by the asymmetry of waiting times when there are several possible ways to reach the destination. We analyze the phenomenon under different assumptions made on the waiting times and parallelism in the transport graph.

1. Introduction

In this short note we will discuss an interesting asymmetry in public transport travel times the analysis of which has been motivated by personal experience. After moving to an outer district of Budapest a few years ago and having used the public transport for a longer period, I observed that travel times tend to be longer in the homebound direction than in the opposite direction. A few measurements revealed that the difference (which turned out to be smaller in

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reality than in my mind) was mainly due to the differences in waiting times between changes.

Below we examine models of public transport where travel time is not symmetric in the two directions. For simplicity, in most cases in the discussion we will talk about means of transport as buses (or in a few cases metro lines to emphasize larger line frequency), and assume that bus frequency at each stop is the same in both directions. This is not always realistic, since for example buses in the morning towards the city center may have slightly larger frequency than backwards, but as we are mainly interested in comparing morning traffic in one direction with evening traffic in the other, the symmetry assumption is reasonable. We will also not try to analyze all of a city's transport, so we only concentrate on small parts of the transport graph. We make plausible simplifying assumptions about expected waiting times.

Our model consists of the following: a graph $G = (V, E)$ where V is the set of bus stops and E is the set of undirected edges between nodes connected by at least one bus line. The set of bus lines is denoted by L, and to every bus line $l \in L$ corresponds a walk (typically a path) in G, denoted by route_l. For each line l and edge e on route_l we suppose that the travel time of line l along edge e is described by a random variable $X_{l,e}$ in both directions. At the two termini of each line, a stochastic process describes the starting times. We suppose that these processes are independent (so we do not take into account that if a bus arrives late at a terminus, it will probably depart late in the opposite direction), and also that the $X_{l,e}$ are pairwise independent and also independent from the dispatch times at the termini.

At any specific time instant at any stop $v \in V$ the waiting time for line l in a specific direction is described by a random variable. If we know the starting times at the termini and $X_{l,e}$, then this random variable for the waiting time can be calculated. We assume that the waiting time distribution averaged over long time intervals converges to a limiting distribution. Thus it will make sense to say that when we arrive at a stop at a "random time instant", the waiting time is described by a random variable. In what follows, we will assume that this variable is already known, and is the same in both directions. We denote it by $W_{l,v}$. We only consider situations when the passenger arrives at the stop at a random time instant (they do not know the schedule).

A bus journey is defined as a series of bus lines $l_1, l_2, \ldots, l_n$ together with subpaths p_i of route_{l_i} (or possibly of route_{l_i} reversed) for $i = 1, \ldots, n$ such that the endpoint of p_i is the starting point of p_{i+1} for $i = 1, \ldots, n-1$. The time a bus journey takes (a random variable itself) is calculated as the sum of the waiting times at the starting points of $p_1, \ldots, p_{n-1}$ plus the sum of the times needed for l_i to travel along p_i.

We will assume that the aim of the passenger is to minimize the expected travel time by choosing the best available journey. We do not require that they plan the entire journey before starting, so decisions based on observations during the journey ("on-line" decisions) are allowed. The only type of on-line decision we consider in the present paper is that the passenger can decide in a stop whether to take bus 1 or continue waiting for bus 2 if bus 1 arrives first.

The paper is built up as follows: in section 2 we examine two simple scenarios to explain the importance of choices in the case of parallel lines. In sections 3 and 4 we analyze these situations by making the assumptions that waiting time is uniformly (resp. exponentially) distributed. We also give some constructions in which the direction-dependence is present. In the summary, we investigate further research directions. Throughout the article, we will use facts from elementary probability theory without further reference. These can be found in standard text books, e.g. [3, 6].

2. Two simple cases

We consider some simple cases where the passenger has a choice between two journeys. The strategy for minimizing the expected travel time is not always trivial. For the exact analysis of the situation, assumptions on the waiting times need to be made, which we defer to the following two sections. Now we only set up the scenarios and give an informal analysis of the travel times for illustration purposes.

2.1. Slow and fast journeys between two points

In scenario 1 (see Figure 1), the passenger travels from point A to point B and two lines are available for this: a slow line l_1 and a fast line l_2. The only kind of on-line information that the choice is influenced by is which bus line arrives at stop A earlier. Informally, is it worth waiting for the faster bus if the slower bus arrives first? We can also ask if the answer can be different at A and in the other direction at B.

It turns out that the one special case of the question has already been analyzed in [4]. A formerly known similar scenario is the "waiting versus walking" problem [1, 2, 5] where the question is whether it is worth waiting for a bus if the destination is within walking distance.

In our case, intuitively it is clear that if the average remaining waiting time for the fast bus is smaller than the difference between the travel times of the two lines then it is worth waiting for the faster bus when the slower bus arrives first. We give the precise analysis of the situation in two important cases in the

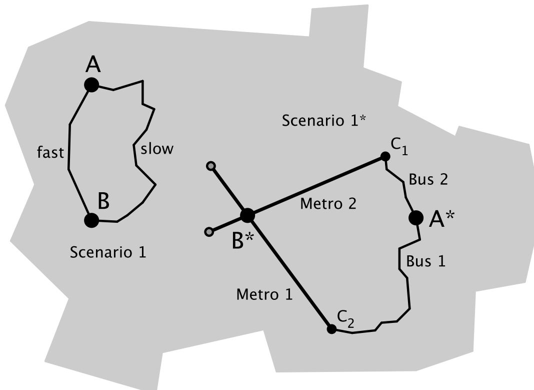

Figure 1. Scenarios 1 and 1*. There are two possibilities to go from A to B. The question is if it is better to wait for the faster possibility or we should take the line that arrives first.

following two sections. If we make the symmetry assumption that the waiting times at A and B are identically distributed and that the journeys last equally long in the two directions, then there is no difference in the strategy or the expected travel time in the two directions.

In scenario 1* (see Figure 1), there is again a choice between two journeys, but this time, both journeys contain a change. The slower journey consists of slower bus 1 from A to C_1, then metro 1 to B. The faster journey consists of faster bus 2 from A to C_2, then metro 2 to B. We assume that at B, the metro lines share the same platform (so we can decide to take the first arriving line).

The analysis will be similar to scenario 1 but this time, waiting times at A and B can be different. Thus the decision of waiting or taking the slower journey can be different in the two directions.

2.2. Near-parallel lines with one change

In scenario 2, the situation is similar: the passenger wants to get from A to B and also has two possibilities but this time a part of the two journeys is a common track of some line. The situation is depicted in Figure 2. Point A is situated far from the city center and there are two bus lines (bus 1 and bus 2 in the figure) connecting A to a frequently running metro line at points C_1 and C_2, respectively.

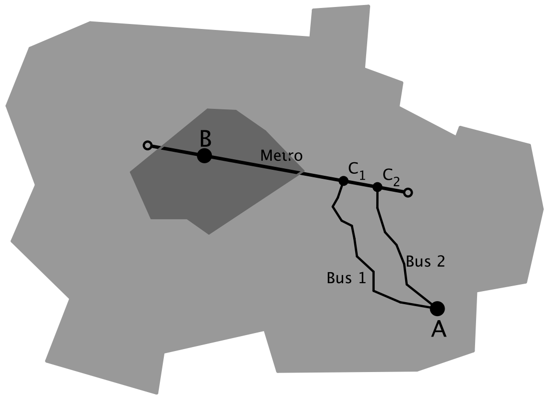

Figure 2. Scenario 2: near-parallel lines connect A to a metro line along which the destination B in the city center is situated. When going from A to B, we take the first bus that arrives and change to the metro line at C_1 or C_2. When going from B to A, we have to decide where to get off the metro possibly without knowing whether bus 1 or bus 2 will start earlier.

When going from A to B one has the choice of taking either bus 1 or bus 2 and one has the option of taking the bus that arrives first. In the backward direction we do not know which bus will have the shorter waiting time.

Scenario 2 was the situation that I observed in practice. At point A, the waiting time until either bus 1 or bus 2 arrives is rarely more than 10 minutes. In the other direction, after getting off the metro at either C_1 or C_2 waiting times for buses relatively often exceed 10 minutes. It seems that this kind of situation arises more often when A is in the suburban area and B in the center, hence the title of this paper.

3. Uniform waiting times

The most simple waiting time distribution one can think of is the uniform distribution. It seems plausible and has been observed [7] that in bus stops close to the dispatch area, the interval between consecutive bus arrivals is almost constant with small deviation from the average. Thus we may model the situation by saying that arriving at a random time instant one is faced with a waiting time $W_{l,v}$ uniformly distributed in the interval $[0, w]$ for some w.

In scenario 1 from A to B, let the waiting times be independent and uniform in the intervals $[0, w_1]$ and $[0, w_2]$, and let the expected travel times after getting on the bus be e_1 and e_2 for bus 1 and 2 respectively (we assume that these are known to the passenger). The strategy can be described by telling whether to get on or not when the first bus arrives. The decision should only depend on the time already spent by waiting and on the kind of bus arriving first. Formally, it is a function $f : [0, \min\{w_1, w_2\}] \times \{1, 2\} \to \{1, 2\}$ where $f(t, i)$ is i iff we should take bus i if it is the one arriving first after t time of waiting. Conditional on the condition that we have been waiting for time t, the (remaining) waiting time pair is uniformly distributed in the rectangle $[0, w_1 - t] \times [0, w_2 - t]$. If t is the time instant when bus 1 arrives, then the expected travel time from now on is e_1 for bus 1, and $e_2 + (w_2 - t)/2$ for bus 2, since we still have to wait in the latter case. If the bus arriving at time t is line 2, then the expected travel times are $e_1 + (w_1 - t)/2$ and e_2. The strategy is as follows:

$$f(t, i) = i \qquad \text{if and only if } 2(e_i - e_{\bar{i}}) < w_{\bar{i}} - t \quad .$$

Here $\overline{1} = 2$ and $\overline{2} = 1$. It is clear that we should take the faster line if it arrives first. The formula says that we should take the slower one if twice the time lost compared to the fast line is smaller than the worst-case time until the fast bus arrives. Note the dependence of the strategy on t: if we have waited long enough, we should wait some more because the remaining waiting time decreases.

The expected travel time including waiting for a passenger arriving at A at a random time instant can be calculated by integrating the expected time needed for the optimal strategy over the rectangle $[0, w_1] \times [0, w_2]$, where (t_1, t_2) in this rectangle represents a situation where bus i arrives t_i time later ($i = 1, 2$). Since the bus arriving first is 1 if and only if $t_1 < t_2$, the expected travel time is:

$$\int\limits_{t_1 < t_2, f(t_1,1)=1} t_1 + e_1 \,\mathrm{d}\mu + \int\limits_{t_1 < t_2, f(t_1,1)=2} t_1 + e_2 + (w_2 - t)/2 \,\mathrm{d}\mu +$$

$$+ \int\limits_{t_1 \geq t_2, f(t_2,2)=1} t_2 + e_1 + (w_1 - t)/2 \,\mathrm{d}\mu + \int\limits_{t_1 \geq t_2, f(t_2,2)=2} t_2 + e_1 \,\mathrm{d}\mu.$$

Here μ is the uniform probability measure on $[0, w_1] \times [0, w_2]$. If we assume that $e_2 < e_1$ then the third integral is not present since we always take the fast bus if it arrives first. We illustrate the regions over which the integrals are taken in Figure 3.

Clearly, if we make symmetry assumptions on the lines, then at A and B the analysis yields the same numbers. This is not true for scenario 1*, however.

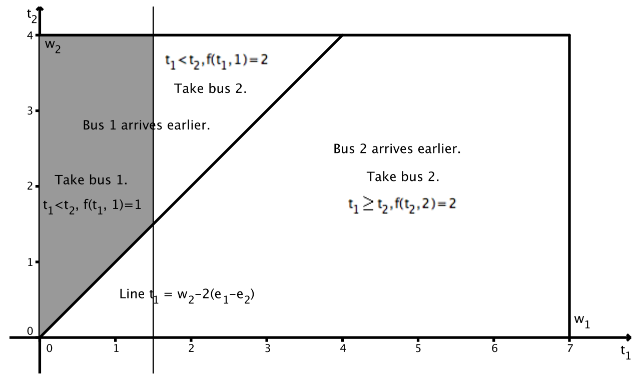

Figure 3. The three regions illustrate the strategy in the case of uniformly distributed waiting times. The x-axis represents the time to the next arrival of bus 1, the y-axis the time to the next bus 2. Bus 2 is faster.

The formulae apply to this case as well, but w_1, e_1, w_2 and e_2 might be different at A and B. We prove that this can result in asymmetry of the expected travel time by constructing an example. Suppose that in scenario 1*, bus 1 and 2 both have uniform waiting times in the interval $[0, 12]$ (in minutes) and both take 30 minutes to travel from A to C$_i$. Both metro lines come in every 6 minutes and the metro journey is 20 minutes long. By symmetry w.r.t to journey 1 and 2 (or the above analysis), the strategy is to get on the line that arrives first. From A to B we have $w_1 = w_2 = 12$, $e_1 = e_2 = 30 + 6/2 + 20$, including waiting for the metro at C$_i$. The expected length of the travel time from A to B including waiting is $12/3 + 30 + 6/2 + 20 = 57$ by the integral formula. The $12/3$ is the expected time before the first one of bus 1 or 2 arrives. In the opposite direction, the expected travel time is $6/3 + 20 + 12/2 + 30 = 58$. Informally, having an alternative between to infrequent lines (the bus lines) is slightly preferable to having an alternative between two frequent lines (the metro lines). Making available two parallel lines with identical waiting times reduces the waiting time at a stop by one third if waiting times are uniformly distributed and independent. The larger the waiting time was, the larger the gain.

In scenario 2, the analysis is similar. First notice that if one were to get from A to C$_1$, the latter being between B and C$_2$ on the metro line, the situation would become a degenerate special case of scenario 1*, with metro 1 replaced by an empty journey. The complete scenario 2 from A to B can also be handled similarly.

But in the opposite direction, we have to make a decision about getting off at C_1 or C_2 without any information on the bus lines. Let w_{Bi} be the maximum waiting time (both at A and C_i) and e_{Bi} the expected travel time between A and C_i for bus i and let e_{12} be the expected travel time (without waiting) between C_1 and C_2. The choice can be made in advance, by comparing $w_{B1}+e_{B1}+e_{12}$ with $w_{B2}+e_{B2}$. Let the waiting time for the metro be uniform in $[0, w_M]$ and the metro journey without waiting between B and C_1 have expected time e_M.

We give an example where asymmetry occurs. Let $w_{Bi} = 18$, $e_{B1} = 30$, $e_{B2} = 25$, $w_M = 2$, $e_{12} = 5$ and $e_M = 10$. From A to B there is no bias between the two journeys (since $e_{B1} = e_{12} + e_{B2}$), so we choose whichever bus comes first ($w_{Bi}/3$ waiting on average). The expected travel time from A to B is $18/3 + 30 + w_M/2 + e_M = 47$. In the opposite direction either strategy gives $w_M/2 + 10 + w_{Bi}/2 + 30 = 50$ minutes.

4. Exponential waiting times

Contrary to stops near the terminus, at stops farther away from the dispatch area, the distribution of the time between two consecutive arrivals spreads out to a larger interval. We will approximate the arrivals of the buses by a Poisson process (see [7]), which means that waiting times are described by an exponential distribution. The average waiting time is the same as at the terminus, so the parameter λ of the distribution is the reciprocal of the waiting time.

The main difference compared to the uniform distribution case is that waiting does not reduce the remaining waiting time. So the strategy should not depend on the time that we have already spent with waiting. Scenario 1 is analyzed in [4] but we give an analysis for completeness.

In scenario 1, let λ_1 and λ_2 be the parameters of the waiting times, $e_1 \geq e_2$ the expected travel times (without waiting) for bus 1 and 2 respectively. At any time instant t before the buses arrive, the expected travel time including waiting is $1/\lambda_i + e_i$ for bus i, independently of t. Thus if the slower bus 1 arrives first, we take it iff $e_1 < e_2 + 1/\lambda_2$. If $e_1 - e_2 < 1/\lambda_1$, then we take the first bus that arrives, otherwise we always take bus 2. There is only a real choice in the former case for which we calculate the average waiting times. This can be done similarly to the above integrals, but now only two terms remain. The overall travel time is then calculated as follows.

The probability that bus 1 arrives before bus 2 is $\frac{\lambda_1}{\lambda_1+\lambda_2}$. The expected time for the first bus to arrive is $\frac{1}{\lambda_1+\lambda_2}$. Thus the expected travel time is the time for waiting plus the time for actual travel:

$$\frac{1}{\lambda_1+\lambda_2}+\frac{\lambda_1}{\lambda_1+\lambda_2}\cdot e_1+\frac{\lambda_2}{\lambda_1+\lambda_2}\cdot e_2=\frac{1+\lambda_1 e_1+\lambda_2 e_2}{\lambda_1+\lambda_2}.$$

If we assume in scenario 1 that the parameters at A and B are identical, then there is no asymmetry.

In scenario 1*, we construct an example similar to the uniform case, but leave the details to the reader. We let every aspect of bus 1 be identical to that of bus 2, and similarly for the metro lines. If the waiting times for the buses have parameter λ, strictly smaller than the waiting time parameter μ for both metro lines, then the average overall waiting time is $\frac{1}{2\lambda}+\frac{1}{\mu}$ from A to B, and $\frac{1}{2\mu}+\frac{1}{\lambda}$ from B to A, which is greater. Again, parallelism on a less frequent line has larger benefits. This time, introducing two parallel lines reduces waiting to half the time at that stop.

For an asymmetry in scenario 2, we let actual travel times be equal along the two journeys and in either direction. Let the parameter for the waiting time of the metro be μ, and that of both buses be λ. From A to B we wait an expected amount of $\frac{1}{2\lambda}+\frac{1}{\mu}$, and in the opposite direction $\frac{1}{\mu}+\frac{1}{\lambda}$, which is clearly greater.

5. Summary and further work

We described some situations with asymmetry between the two directions even in the absence of asymmetries of any bus line frequencies or travel times of single bus lines. Continuing this work, other waiting time distributions and mixed cases (e.g. when the waiting time for bus 1 obeys a uniform distribution but bus 2 arrivals are a Poisson process) should also be considered.

A further research possibility is to build a computer model of Budapest public transport with (V, E), the $X_{l,e}$ and the dispatch process at the termini fed into it, and see how $W_{l,v}$ and the optimal travel times are distributed. One could then investigate between which points the asymmetry occurs, or more generally, what can be said about the distribution of travel times of optimal journeys and which choice should be taken. We made several reasonable simplifying assumptions during the analysis and the model could be used to justify or reject them. One thing that can clearly fail in practice is the assumption of arriving at a "random time instant": knowing the schedule (and the time) is valuable information which helps in taking decisions about journeys. Another important factor might be that near-parallel lines are often coordinated: the arrival times are interleaved and waiting times are not independent.

The passenger's strategies may be generalized: instead of minimizing the expected travel time, one might want to minimize the probability that the travel time exceeds a specified time length. Or we can imagine a bet between two friends at a bus stop on getting to some other place sooner: what is the optimal strategy for maximizing the probability of winning the bet.

Finally, although I was unable to locate similar problems for routing in the networking literature, there can be situations where some choice made during path planning occurs in one direction and not in the other, resulting in a similar time-irreversibility of the "journey". It would be interesting to know if there are similar scenarios in some other context outside traffic modeling.

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P. Burcsi
Department of Computer Algebra
Faculty of Informatics
Eötvös Loránd University
Pázmány P. sétány 1/C.
H-1117 Budapest
Hungary
peter.burcsi@compalg.inf.elte.hu