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CESÀRO-SUMMABILITY OF HIGHER-DIMENSIONAL FOURIER SERIES

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Abstract. The triangular and cubic Cesàro summability of higher dimensional Fourier series is investigated. It is proved that the maximal operator of the Cesàro means of a d-dimensional Fourier series is bounded from the Hardy space $H_p(\mathbb{T}^d)$ to $L_p(\mathbb{T}^d)$ for all $d/(d+\alpha\wedge 1) < p \leq \infty$ and, consequently, is of weak type (1,1). As a consequence we obtain that the Cesàro means of a function $f \in L_p(\mathbb{T}^d)$ converge a.e. and in L_p-norm $(1 \leq p < \infty)$ to f. Moreover, we prove for the endpoint $p = d/(d+\alpha\wedge 1)$ that the maximal operator is bounded from $H_p(\mathbb{T}^d)$ to the weak $L_p(\mathbb{T}^d)$ space.

1. Introduction

The well known Carleson's theorem says that

$$(1)\qquad s_k f(x) := \sum_{j\in\mathbb{Z}^d, |j|\leq k} \widehat{f}(j) e^{\imath j\cdot x} \to f(x) \qquad \text{for a.e. } x \in \mathbb{T} \text{ as } k \to \infty$$

if $f \in L_p(\mathbb{T})$ $(1 < p < \infty)$ (see Carleson [5] and Grafakos [8]). This is false for $p = 1$. However, the Fejér and Cesàro summability means $\sigma_n f$ of f converge to

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f almost everywhere if $f \in L_1(\mathbb{T})$ (see Zygmund [24], Butzer and Nessel [4]). Recall that the Fejér means are defined by

$$(2) \qquad \sigma_n f(x) := \sum_{j \in \mathbb{Z}^d, |j| \le n} \left(1 - \frac{|j|}{n}\right) \widehat{f}(j) e^{\imath j \cdot x} = \frac{1}{n} \sum_{k=0}^{n-1} s_k f(x).$$

In this paper we generalize these results for higher dimensions. We consider the triangular and cubic summability by replacing $|\cdot|$ by $\|\cdot\|_1$ or $\|\cdot\|_\infty$ in (1) and (2). It is known that he analogue of Carleson's theorem holds (see Fefferman [7] and Grafakos [8]). We generalize the Fejér and Cesàro means and prove that the means $\sigma_n f \to f$ in L_p-norm if $f \in L_p(\mathbb{T}^d)$ $(1 \le p < \infty)$ and uniformly if $f \in C(\mathbb{T}^d)$. Next we obtain that the maximal operator σ_* is bounded from the Hardy space $H_p(\mathbb{T}^d)$ to $L_p(\mathbb{T}^d)$ for all $d/(d+\alpha \wedge 1) < p \le \infty$. This implies by interpolation that σ_* is of weak type (1,1). As a consequence we get the a.e. convergence of $\sigma_n f$ to f, whenever $f \in L_1(\mathbb{T}^d)$. Moreover, we prove for the critical index $p = d/(d+\alpha \wedge 1)$ that σ_* is bounded from $H_p(\mathbb{T}^d)$ to the weak $L_p(\mathbb{T}^d)$ space. This paper was the base of my talk given at the 9th Joint Conference on Mathematics and Computer Science, February 2012, in Siófok, Hungary.

2. The Dirichlet kernel functions

Let us fix $d \ge 1$, $d \in \mathbb{N}$. For a set $\mathbb{Y} \ne \emptyset$ let $\mathbb{Y}^d$ be its Cartesian product $\mathbb{Y} \times \ldots \times \mathbb{Y}$ taken with itself d-times. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$ set

$$u \cdot x := \sum_{k=1}^{d} u_k x_k, \ \|x\|_q := \left(\sum_{k=1}^{d} |x_k|^q\right)^{1/q} \ (0 < q < \infty), \ \|x\|_\infty := \sup_{1 \le k \le d} |x_k|.$$

We briefly write $L_p(\mathbb{T}^d)$ instead of the $L_p(\mathbb{T}^d, \lambda)$ space equipped with the norm (or quasi-norm)

$$\|f\|_p := \begin{cases} \left(\int_{\mathbb{T}^d} |f|^p \, d\lambda\right)^{1/p}, & 0 < p < \infty; \\ \sup_{\mathbb{T}^d} |f|, & p = \infty, \end{cases}$$

where $\mathbb{T} := [-\pi, \pi]$ is the torus and λ is the Lebesgue measure. The *weak* L_p

space, $L_{p,\infty}(\mathbb{T}^d)$ $(0<p<\infty)$ consists of all measurable functions f for which

$$\|f\|_{p,\infty} := \sup_{\rho>0} \rho\lambda(|f|>\rho)^{1/p} < \infty.$$

Note that $L_{p,\infty}(\mathbb{T}^d)$ is a quasi-normed space (see Bergh and Löfström [3]). It is easy to see that for each $0<p<\infty$,

$$L_p(\mathbb{T}^d) \subset L_{p,\infty}(\mathbb{T}^d) \qquad \text{and} \qquad \|\cdot\|_{p,\infty} \le \|\cdot\|_p.$$

The space of continuous functions with the supremum norm is denoted by $C(\mathbb{T}^d)$.

For a distribution f the nth *Fourier coefficient* is defined by $\widehat{f}(n) := f(e_{-n})$, where $e_n(x) := e^{\imath n\cdot x}$ $(n \in \mathbb{Z}^d)$ (see e.g. Edwards [6, p. 67]). If f is an integrable function then

$$\widehat{f}(n) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-\imath n\cdot x}\, dx, \qquad (\imath = \sqrt{-1}).$$

In this paper we generalize the partial sums (1) and summability means (2) for multi-dimensional functions by replacing the $|\cdot|$ by $\|\cdot\|_q$. Here we consider the cases $q=1$ (see also Berens, Li and Xu [1, 2, 21], Weisz [17, 18]) and $q=\infty$ (Marcinkiewicz [9], Zhizhiashvili [23] and Weisz [16, 19]).

For $f \in L_1(\mathbb{T}^d)$ the nth *ℓ_q-partial sum* $s_n^q f$ $(n \in \mathbb{N})$ is given by

$$s_n^q f(x) := \sum_{k\in\mathbb{Z}^d, \|k\|_q \le n} \widehat{f}(k) e^{\imath k\cdot x} = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-u) D_n^q(u)\, du,$$

where

$$D_n^q(u) := \sum_{k\in\mathbb{Z}^d, \|k\|_q \le n} e^{\imath k\cdot u}$$

is the *ℓ_q-Dirichlet kernel*. The partial sums are called *triangular* if $q=1$ and *cubic* if $q=\infty$ (see Figure 1).

It is known that if $q=1,\infty$ and $f \in L_p(\mathbb{T}^d)$ for some $1<p<\infty$ then

$$\|s_n^q f\|_p \le C_p \|f\|_p \qquad (n \in \mathbb{N})$$

and

$$\lim_{n\to\infty} s_n^q f = f \qquad \text{in } L_p\text{-norm}.$$

The analogue of Carleson's theorem holds in higher dimensions for the triangular and cubic partial sums (see Fefferman [7] and Grafakos [8]), i.e.

$$\lim_{n\to\infty} s_n^q f = f \qquad \text{a.e.},$$

whenever $q=1,\infty$ and $f \in L_p(\mathbb{T}^d)$ for some $1<p<\infty$.

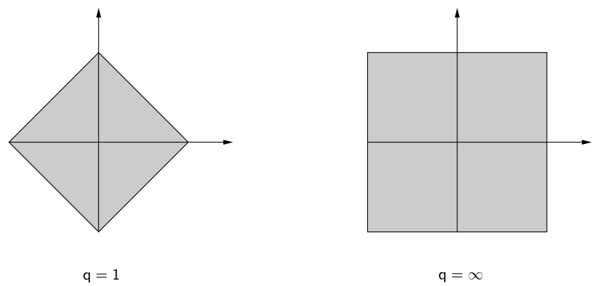

Figure 1. Regions of the ℓ_q-partial sums for $d = 2$.

For $k \in \mathbb{N}, \alpha \neq -1, -2, \ldots$ let

$$A_k^\alpha := \binom{k+\alpha}{k} = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+k)}{k!}.$$

It is known (see Zygmund [24, p. 77]) that

$$A_k^\alpha = \sum_{i=0}^{k} A_{k-i}^{\alpha-1}, \qquad A_k^\alpha - A_{k-1}^\alpha = A_k^{\alpha-1} \tag{3}$$

and

$$A_k^\alpha = O(k^\alpha) \qquad (k \in \mathbb{N}). \tag{4}$$

We define the *ℓ_q-Fejér and Cesàro means* of an integrable function $f \in$ $\in L_1(\mathbb{T}^d)$ by

$$\sigma_n^q f(x) := \sum_{k \in \mathbb{Z}^d, \|k\|_q \le n} \left(1 - \frac{\|k\|_q}{n}\right) \widehat{f}(k) e^{\imath k \cdot x} = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-u) K_n^q(u)\, du,$$

and

$$\sigma_n^{q,\alpha} f(x) : = \frac{1}{A_{n-1}^\alpha} \sum_{k \in \mathbb{Z}^d, \|k\|_q \le n} A_{n-1-\|k\|_q}^\alpha \widehat{f}(k) e^{\imath k \cdot x} =$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-u) K_n^{q,\alpha}(u)\, du,$$

where

$$K_n^q(u) := \sum_{k\in\mathbb{Z}^d, \|k\|_q\le n} \left(1-\frac{\|k\|_q}{n}\right) e^{\imath k\cdot u}$$

and

$$K_n^{q,\alpha}(u) := \frac{1}{A_{n-1}^\alpha} \sum_{k\in\mathbb{Z}^d, \|k\|_q\le n} A_{n-1-\|k\|_q}^\alpha e^{\imath k\cdot u}$$

are the ℓ_q-*Fejér- and Cesàro kernels*. The cubic summability (when $q=\infty$) is also called *Marcinkiewicz summability*. By (3)

$$K_n^{q,\alpha}(u) = \frac{1}{A_{n-1}^\alpha} \sum_{\|k\|_q\le n} \sum_{j=\|k\|_q}^{n-1} A_{n-1-j}^{\alpha-1} e^{\imath k\cdot u} = \frac{1}{A_{n-1}^\alpha} \sum_{j=0}^{n-1} A_{n-1-j}^{\alpha-1} D_j^q(u).$$

Hence

$$\sigma_n^{q,\alpha} f(x) = \frac{1}{A_{n-1}^\alpha} \sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1} s_k^q f(x).$$

Since $A_{n-1}^1 = n$ and $A_{n-1}^0 = 1$, if $\alpha = 1$, we get back the Fejér means. Observe that

$$|K_n^{q,\alpha}| \le Cn^d \qquad (n\in\mathbb{N}^d). \tag{5}$$

Now we characterize the Dirichlet kernel functions D_n^q. It is easy to see that in the one-dimensional case both kernel functions are the same and

$$D_n^q(u) = \frac{\sin((n+1/2)u)}{\sin(u/2)}$$

(see e.g. Grafakos [8]).

The situation is much more complicated in higher dimensions. First let us consider the case $q=1$. The nth divided difference of a function f at the (pairwise distinct) knots $x_1,\ldots,x_n\in\mathbb{R}$ is introduced inductively as

$$[x_1]f := f(x_1), \qquad [x_1,\ldots,x_n]f := \frac{[x_1,\ldots,x_{n-1}]f - [x_2,\ldots,x_n]f}{x_1-x_n}. \tag{6}$$

It was proved by Berens and Xu [2, 21] that

$$D_n^1(x) = [\cos x_1,\ldots,\cos x_d]G_n, \qquad (x\in\mathbb{T}^d),$$

where

$$G_n(\cos t) := (-1)^{[(d-1)/2]} 2\cos(t/2)(\sin t)^{d-2} \mathrm{soc}\,(n+1/2)t$$

and

$$\operatorname{soc} t := \begin{cases} \cos t, & \text{if } d \text{ is even;} \\ \sin t, & \text{if } d \text{ is odd.} \end{cases}$$

We point out the result for $d = 2$:

$$D_n^1(x) = 2\frac{\cos(x_1/2)\cos((n+1/2)x_1) - \cos(x_2/2)\cos((n+1/2)x_2)}{\cos x_1 - \cos x_2}.$$

The cubic Dirichlet kernels ($q = \infty$) can be given by

$$D_n^\infty(x) = \prod_{i=1}^d D_n^\infty(x_i) = \prod_{i=1}^d \frac{\sin((n+1/2)x_i)}{\sin(x_i/2)}.$$

Since the kernel functions are very different for $q = 1, \infty$, the proofs of the next results differ substantially from each other.

3. Norm convergence of the Cesàro summability

To obtain norm convergence of the Cesàro means we have to estimate first the Cesàro kernel functions. The one-dimensional case is again much more simpler because

$$|K_n^{q,\alpha}(u)| \le C \min(n, n^{-\alpha}u^{-\alpha-1}) \qquad (n \in \mathbb{N},\ u \ne 0,\ 0 < \alpha \le 1)$$

(see Zygmund [24]). Some of the next results are already known, for $q = \infty$, $d \in \mathbb{N}$ see [19], for $q = 1, d = 2$ see [20] and for $q = 1, \alpha = 1, d \ge 3$ see [18]. So we will focus on the case $q = 1, 0 < \alpha < \infty, d \ge 3$. We will only sketch the proofs and give the differences to the case $q = 1, \alpha = 1, d \ge 3$. $q = 2$ is also investigated in the literature, see e.g. Stein and Weiss [14], Grafakos [8] or Weisz [16].

If we apply the inductive definition of the divided difference in (6) to D_n^1, then in the denominator we have to choose the factors from the following table:

$$\begin{array}{lllll}
\cos x_1 - \cos x_d & & & & \\
\cos x_1 - \cos x_{d-1} & \cos x_2 - \cos x_d & & & \\
\quad\dots & & & & \\
\cos x_1 - \cos x_{d-k+1} & \cos x_2 - \cos x_{d-k+2} & \dots & \cos x_k - \cos x_d & \\
\quad\dots & & & & \\
\cos x_1 - \cos x_2 & \cos x_2 - \cos x_3 & \dots & & \cos x_{d-1} - \cos x_d.
\end{array}$$

Observe that the kth row contains k terms and the differences of the indices in the kth row is equal to $d-k$, more precisely, if $\cos x_{i_k} - \cos x_{j_k}$ is in the kth row, then $j_k - i_k = d-k$. We choose exactly one factor from each row. First we choose $\cos x_1 - \cos x_d$ and then from the second row $\cos x_1 - \cos x_{d-1}$ or $\cos x_2 - \cos x_d$. If we have chosen the $(k-1)$th factor from the $(k-1)$th row, say $\cos x_j - \cos x_{j+d-k+1}$, then we have to choose the next one from the kth row which is below the $(k-1)$th factor (it is equal to $\cos x_j - \cos x_{j+d-k}$) or the right neighbor (it is equal to $\cos x_{j+1} - \cos x_{j+d-k+1}$). More exactly, we introduce a set $\mathcal{I}$ of sequences of integer pairs $((i_n, j_n); n = 1, \ldots, d-1)$. Let $i_1 = 1$, $j_1 = d$, (i_n) is non-decreasing and (j_n) is non-increasing. If (i_n, j_n) is given then let $i_{n+1} = i_n$ and $j_{n+1} = j_n - 1$ or $i_{n+1} = i_n + 1$ and $j_{n+1} = j_n$. If the sequence (i_n, j_n) has these properties then we say that it is in $\mathcal{I}$. Observe that the difference $\cos x_{i_k} - \cos x_{j_k}$ is in the kth row of the table $(k = 1, \ldots, d-1)$. So the factors we have just chosen can be written as $\prod_{l=1}^{d-1} (\cos x_{i_l} - \cos x_{j_l})$. In other words,

$$
\begin{aligned}
&D_n^1(x) = \\
&= \sum_{(i_l,j_l)\in\mathcal{I}} (-1)^{i_{d-1}-1} \prod_{l=1}^{d-2} (\cos x_{i_l} - \cos x_{j_l})^{-1} [\cos x_{i_{d-1}}, \cos x_{j_{d-1}}] G_n = \\
&= (-1)^{i_{d-1}-1} \prod_{l=1}^{d-1} (\cos x_{i_l} - \cos x_{j_l})^{-1} (G_n(\cos x_{i_{d-1}}) - G_n(\cos x_{j_{d-1}})) =: \\
&=: D^1_{n,(i_l,j_l)}(x).
\end{aligned} \tag{7}
$$

Then

$$
\begin{aligned}
K_n^{1,\alpha}(x) &= \sum_{(i_l,j_l)\in\mathcal{I}} \frac{(-1)^{i_{d-1}-1}}{A_{n-1}^{\alpha}} \prod_{l=1}^{d-1} (\cos x_{i_l} - \cos x_{j_l})^{-1} \times \\
&\quad \times \sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1} (G_k(\cos x_{i_{d-1}}) - G_k(\cos x_{j_{d-1}})) =: \\
&=: \sum_{(i_l,j_l)\in\mathcal{I}} K^{1,\alpha}_{n,(i_l,j_l)}(x).
\end{aligned}
$$

We may suppose that $\pi > x_1 > x_2 > \ldots > x_d > 0$.

Lemma 1. *Suppose that* $q=1, 0<\alpha\leq 1, d\geq 3$. *For all* $0<\beta<\frac{\alpha+1}{d-1}$,

$$|K^{1,\alpha}{}_{n,(i_l,j_l)}(x)| \leq$$

$$\leq \frac{C}{n^\alpha}\prod_{l=1}^{d-1}(x_{i_l}-x_{j_l})^{-1-\beta}x_{j_{d-1}}^{\beta(d-1)-\alpha-1}\mathbf{1}_{\{x_{j_{d-1}}\leq\pi/2\}}+$$

$$+\frac{C}{n}\prod_{l=1}^{d-1}(x_{i_l}-x_{j_l})^{-1-\beta}x_{j_{d-1}}^{\beta(d-1)-2}\mathbf{1}_{\{x_{j_{d-1}}\leq\pi/2\}}+ \tag{8}$$

$$+\frac{C}{n^\alpha}\prod_{l=1}^{d-1}(x_{i_l}-x_{j_l})^{-1-\beta}(\pi-x_{i_{d-1}})^{\beta(d-1)-\alpha-1}\mathbf{1}_{\{x_{j_{d-1}}>\pi/2\}}+$$

$$+\frac{C}{n}\prod_{l=1}^{d-1}(x_{i_l}-x_{j_l})^{-1-\beta}(\pi-x_{i_{d-1}})^{\beta(d-1)-2}\mathbf{1}_{\{x_{j_{d-1}}>\pi/2\}}.$$

Proof. Using the formula

$$\left|\sum_{k=0}^{n-1}A_{n-1-k}^{\alpha-1}\mathrm{soc}\,((k+1/2)u)\right|\leq\frac{C}{(\sin(u/2))^\alpha}+\frac{Cn^{\alpha-1}}{\sin(u/2)}$$

for $0<\alpha\leq 1$ (see Zygmund [24, I. p. 94]) and (4), we conclude

$$|K^{1,\alpha}_{n,(i_l,j_l)}(x)|\leq$$

$$\leq \frac{C}{n^\alpha}\prod_{l=1}^{d-1}\frac{(\sin x_{i_{d-1}})^{d-2}(\sin(x_{i_{d-1}}/2))^{-\alpha}+(\sin x_{j_{d-1}})^{d-2}(\sin(x_{j_{d-1}}/2))^{-\alpha}}{\sin((x_{i_l}-x_{j_l})/2)\sin((x_{i_l}+x_{j_l})/2)}+$$

$$+\frac{C}{n}\prod_{l=1}^{d-1}\frac{(\sin x_{i_{d-1}})^{d-2}(\sin(x_{i_{d-1}}/2))^{-1}+(\sin x_{j_{d-1}})^{d-2}(\sin(x_{j_{d-1}}/2))^{-1}}{\sin((x_{i_l}-x_{j_l})/2)\sin((x_{i_l}+x_{j_l})/2)}.$$

If $x_{j_{d-1}}\leq\pi/2$ then $(x_{i_l}+x_{j_l})/2\leq 3\pi/4$ and so

$$|K^{1,\alpha}_{n,(i_l,j_l)}(x)| \leq \frac{C}{n^\alpha}\prod_{l=1}^{d-1}(x_{i_l}-x_{j_l})^{-1}(x_{i_l}+x_{j_l})^{-1}(x_{i_{d-1}}^{d-\alpha-2}+x_{j_{d-1}}^{d-\alpha-2})+$$

$$+\frac{C}{n}\prod_{l=1}^{d-1}(x_{i_l}-x_{j_l})^{-1}(x_{i_l}+x_{j_l})^{-1}(x_{i_{d-1}}^{d-3}+x_{j_{d-1}}^{d-3}).$$

Since $x_{i_l} + x_{j_l} > x_{i_l} - x_{j_l}$ and $x_{i_l} + x_{j_l} > x_{i_{d-1}} > x_{j_{d-1}}$ we can see that

$$|K^{1,\alpha}_{n,(i_l,j_l)}(x)| \le$$

$$\le \frac{C}{n^\alpha}\prod_{l=1}^{d-1}(x_{i_l} - x_{j_l})^{-1-\beta}(x_{i_{d-1}}^{d-\alpha-2+(\beta-1)(d-1)} + x_{j_{d-1}}^{d-\alpha-2+(\beta-1)(d-1)}) +$$

$$+ \frac{C}{n}\prod_{l=1}^{d-1}(x_{i_l} - x_{j_l})^{-1-\beta}(x_{i_{d-1}}^{d-3+(\beta-1)(d-1)} + x_{j_{d-1}}^{d-3+(\beta-1)(d-1)}) \le$$

$$\le \frac{C}{n^\alpha}\prod_{l=1}^{d-1}(x_{i_l} - x_{j_l})^{-1-\beta}x_{j_{d-1}}^{\beta(d-1)-\alpha-1} + \frac{C}{n}\prod_{l=1}^{d-1}(x_{i_l} - x_{j_l})^{-1-\beta}x_{j_{d-1}}^{\beta(d-1)-2}$$

for all $0 < \beta < \frac{\alpha+1}{d-1}$.

If $x_{j_{d-1}} > \pi/2$ then $(x_{i_l} + x_{j_l})/2 > \pi/4$ and

$$|K^{1,\alpha}_{n,(i_l,j_l)}(x)| \le \frac{C}{n^\alpha}\prod_{l=1}^{d-1}(x_{i_l} - x_{j_l})^{-1}\times$$

$$\times\ (2\pi - x_{i_l} - x_{j_l})^{-1}\Big((\pi - x_{i_{d-1}})^{d-\alpha-2} + (\pi - x_{j_{d-1}})^{d-\alpha-2}\Big) +$$

$$+ \frac{C}{n}\prod_{l=1}^{d-1}(x_{i_l} - x_{j_l})^{-1}(2\pi - x_{i_l} - x_{j_l})^{-1}((\pi - x_{i_{d-1}})^{d-3} + (\pi - x_{j_{d-1}})^{d-3}).$$

Observe that $2\pi - x_{i_l} - x_{j_l} > x_{i_l} - x_{j_l}$ and $2\pi - x_{i_l} - x_{j_l} > \pi - x_{j_l} > \pi - x_{j_{d-1}} > > \pi - x_{i_{d-1}}$. Thus

$$|K^{1,\alpha}_{n,(i_l,j_l)}(x)| \le \frac{C}{n^\alpha}\prod_{l=1}^{d-1}(x_{i_l} - x_{j_l})^{-1-\beta}\times$$

$$\times\Big((\pi - x_{i_{d-1}})^{d-\alpha-2+(\beta-1)(d-1)} + (\pi - x_{j_{d-1}})^{d-\alpha-2+(\beta-1)(d-1)}\Big) +$$

$$+ \frac{C}{n}\prod_{l=1}^{d-1}(x_{i_l} - x_{j_l})^{-1-\beta}\times$$

$$\times\Big((\pi - x_{i_{d-1}})^{d-3+(\beta-1)(d-1)} + (\pi - x_{j_{d-1}})^{d-3+(\beta-1)(d-1)}\Big) \le$$

$$\le \frac{C}{n^\alpha}\prod_{l=1}^{d-1}(x_{i_l} - x_{j_l})^{-1-\beta}(\pi - x_{i_{d-1}})^{\beta(d-1)-\alpha-1} +$$

$$+ \frac{C}{n}\prod_{l=1}^{d-1}(x_{i_l} - x_{j_l})^{-1-\beta}(\pi - x_{i_{d-1}})^{\beta(d-1)-2},$$

if $0 < \beta < \frac{\alpha+1}{d-1}$. ■

Lemma 2. *Suppose that* $q=1, 0<\alpha\leq 1, d\geq 3$. *For all* $0<\beta<\frac{\alpha+1}{d-2}$,

$$
\begin{aligned}
&|K^{1,\alpha}_{n,(i_l,j_l)}(x)| \leq \\
&\leq Cn^{1-\alpha}\prod_{l=1}^{d-2}(x_{i_l}-x_{j_l})^{-1-\beta}x_{j_{d-1}}^{\beta(d-2)-\alpha-1}\mathbf{1}_{\{x_{j_{d-1}}\leq\pi/2\}}+\\
&\quad + C\prod_{l=1}^{d-2}(x_{i_l}-x_{j_l})^{-1-\beta}x_{j_{d-1}}^{\beta(d-2)-2}\mathbf{1}_{\{x_{j_{d-1}}\leq\pi/2\}}+\\
&\quad + Cn^{1-\alpha}\prod_{l=1}^{d-2}(x_{i_l}-x_{j_l})^{-1-\beta}(\pi-x_{i_{d-1}})^{\beta(d-2)-\alpha-1}\mathbf{1}_{\{x_{j_{d-1}}>\pi/2\}}+\\
&\quad + C\prod_{l=1}^{d-2}(x_{i_l}-x_{j_l})^{-1-\beta}(\pi-x_{i_{d-1}})^{\beta(d-2)-2}\mathbf{1}_{\{x_{j_{d-1}}>\pi/2\}}.
\end{aligned}
\tag{9}
$$

Proof. Lagrange theorem and (7) imply that there exists $x_{i_{d-1}}>\xi>x_{j_{d-1}}$, such that

$$D^1_{k,(i_l,j_l)}(x)=(-1)^{i_{d-1}-1}\prod_{l=1}^{d-1}(\cos x_{i_l}-\cos x_{j_l})^{-1}H'_k(\xi)(x_{i_{d-1}}-x_{j_{d-1}}),$$

where

$$H_k(t)=(-1)^{[(d-1)/2]}2\cos(t/2)(\sin t)^{d-2}\mathrm{soc}\,(k+1/2)t.$$

Then

$$
\begin{aligned}
&|K^{1,\alpha}_{n,(i_l,j_l)}(x)| \leq \\
&\leq C\prod_{l=1}^{d-1}\frac{(\sin\xi)^{d-2}+n(\sin\xi)^{d-2}}{n^\alpha\sin((x_{i_l}-x_{j_l})/2)\sin((x_{i_l}+x_{j_l})/2)(\sin(\xi/2))^\alpha}(x_{i_{d-1}}-x_{j_{d-1}})+\\
&\quad + C\prod_{l=1}^{d-1}\frac{(\sin\xi)^{d-2}+n(\sin\xi)^{d-2}}{n\sin((x_{i_l}-x_{j_l})/2)\sin((x_{i_l}+x_{j_l})/2)\sin(\xi/2)}(x_{i_{d-1}}-x_{j_{d-1}})+\\
&\quad + C\prod_{l=1}^{d-1}\frac{n^\alpha(\sin\xi)^{d-3}}{n^\alpha\sin((x_{i_l}-x_{j_l})/2)\sin((x_{i_l}+x_{j_l})/2)}(x_{i_{d-1}}-x_{j_{d-1}}).
\end{aligned}
$$

In the last step we have used that $|\sum_{k=0}^{n-1}A^{\alpha-1}_{n-1-k}\mathrm{soc}\,((k+1/2)t)|\leq n^\alpha$. In case $x_{j_{d-1}} \leq \pi/2$,

$$
\begin{aligned}
&|K^{1,\alpha}_{n,(i_l,j_l)}(x)| \leq \\
&\leq Cn^{1-\alpha} \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1} (x_{i_l} + x_{j_l})^{-1} (x_{i_{d-1}} - x_{j_{d-1}}) \xi^{d-\alpha-2} + \\
&\quad + C \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1} (x_{i_l} + x_{j_l})^{-1} (x_{i_{d-1}} - x_{j_{d-1}}) \xi^{d-3} \leq \\
&\leq Cn^{1-\alpha} \prod_{l=1}^{d-2} (x_{i_l} - x_{j_l})^{-1-\beta} \xi^{d-\alpha-3+(\beta-1)(d-2)} + \\
&\quad + C \prod_{l=1}^{d-2} (x_{i_l} - x_{j_l})^{-1-\beta} \xi^{d-4+(\beta-1)(d-2)} \leq \\
&\leq Cn^{1-\alpha} \prod_{l=1}^{d-2} (x_{i_l} - x_{j_l})^{-1-\beta} x_{j_{d-1}}^{\beta(d-2)-\alpha-1} + C \prod_{l=1}^{d-2} (x_{i_l} - x_{j_l})^{-1-\beta} x_{j_{d-1}}^{\beta(d-2)-2}
\end{aligned}
$$

for all $0 < \beta < \frac{\alpha+1}{d-2}$. The case $x_{j_{d-1}} > \pi/2$ can be handled similarly. ■

Lemma 3. *Suppose that* $q = 1, 0 < \alpha \leq 1, d \geq 3$. *For all* $0 < \beta < \frac{\alpha+1}{d-1}$,

$$
\begin{aligned}
&|K^{1,\alpha}_{n,(i_l,j_l)}(x)| \leq \\
&\leq C \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1-\beta} x_{j_{d-1}}^{\beta(d-1)-\alpha-1} (x_{i_{d-1}} - x_{j_{d-1}})^{\alpha} \mathbf{1}_{\{x_{j_{d-1}} \leq \pi/2\}} + \\
&\quad + C \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1-\beta} x_{j_{d-1}}^{\beta(d-1)-2} (x_{i_{d-1}} - x_{j_{d-1}}) \mathbf{1}_{\{x_{j_{d-1}} \leq \pi/2\}} + \\
&\quad + C \prod_{l=1}^{d-2} (x_{i_l} - x_{j_l})^{-1-\beta} x_{j_{d-1}}^{\beta(d-2)-\alpha-1} (x_{i_{d-1}} - x_{j_{d-1}})^{\alpha-1} \mathbf{1}_{\{x_{j_{d-1}} \leq \pi/2\}} + \\
&\quad + C \prod_{l=1}^{d-2} (x_{i_l} - x_{j_l})^{-1-\beta} x_{j_{d-1}}^{\beta(d-2)-2} \mathbf{1}_{\{x_{j_{d-1}} \leq \pi/2\}} + \\
&\quad + C \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1-\beta} (\pi - x_{i_{d-1}})^{\beta(d-1)-\alpha-1} (x_{i_{d-1}} - x_{j_{d-1}})^{\alpha} \mathbf{1}_{\{x_{j_{d-1}} > \pi/2\}} + \\
&\quad + C \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1-\beta} (\pi - x_{i_{d-1}})^{\beta(d-1)-2} (x_{i_{d-1}} - x_{j_{d-1}}) \mathbf{1}_{\{x_{j_{d-1}} > \pi/2\}} +
\end{aligned}
$$

$$+ C \prod_{l=1}^{d-2} (x_{i_l} - x_{j_l})^{-1-\beta} (\pi - x_{i_{d-1}})^{\beta(d-2)-\alpha-1} (x_{i_{d-1}} - x_{j_{d-1}})^{\alpha-1} \mathbf{1}_{\{x_{j_{d-1}} > \pi/2\}} +$$

$$+ C \prod_{l=1}^{d-2} (x_{i_l} - x_{j_l})^{-1-\beta} (\pi - x_{i_{d-1}})^{\beta(d-2)-2} \mathbf{1}_{\{x_{j_{d-1}} > \pi/2\}} =:$$

$$\text{(10)} \qquad =: \sum_{m=1}^{8} K^{1,\alpha}_{(i_l,j_l),m}(x).$$

Proof. The result follows from

$$|K^{1,\alpha}_{n,(i_l,j_l)}(x)| \leq |K^{1,\alpha}_{n,(i_l,j_l)}(x)| \mathbf{1}_{n > (x_{i_{d-1}} - x_{j_{d-1}})^{-1}} + |K^{1,\alpha}_{n,(i_l,j_l)}(x)| \mathbf{1}_{n \leq (x_{i_{d-1}} - x_{j_{d-1}})^{-1}}$$

and (8) and (9). ■

The next lemma can be proved as Lemmas 1 and 2.

Lemma 4. *Suppose that* $q = 1, 0 < \alpha \leq 1, d \geq 3$. *If* $0 < \beta < \frac{\alpha+1}{d-1} \wedge \frac{d-2}{d-1}$ *then for all* $q = 1, \ldots, d$,

$$|\partial_q K^{1,\alpha}_{n,(i_l,j_l)}(x)| \leq$$

$$\leq \quad C n^{1-\alpha} \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1-\beta} x_{j_{d-1}}^{\beta(d-1)-\alpha-1} \mathbf{1}_{\{x_{j_{d-1}} \leq \pi/2\}} +$$

$$+ C \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1-\beta} x_{j_{d-1}}^{\beta(d-1)-2} \mathbf{1}_{\{x_{j_{d-1}} \leq \pi/2\}} +$$

$$+ C n^{1-\alpha} \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1-\beta} (\pi - x_{i_{d-1}})^{\beta(d-1)-\alpha-1} \mathbf{1}_{\{x_{j_{d-1}} > \pi/2\}} +$$

$$+ C \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1-\beta} (\pi - x_{i_{d-1}})^{\beta(d-1)-2} \mathbf{1}_{\{x_{j_{d-1}} > \pi/2\}}.$$

Now we verify that the L_1-norm of the kernel functions are uniformly bounded.

Theorem 1. *If* $q = 1, \infty$ *and* $0 < \alpha < \infty$ *then*

$$\int_{\mathbb{T}^d} |K^{q,\alpha}_n(x)| \, dx \leq C \qquad (n \in \mathbb{N}).$$

Proof. As we mentioned above we may suppose that $q = 1$, $0 < \alpha \le 1$, $d \ge 3$ and $\pi > x_1 > x_2 > \ldots > x_d > 0$. If $x_1 \le 16/n$ or $\pi - x_d \le 16/n$ then (5) implies

$$\int\limits_{\{16/n \ge x_1 > x_2 > \ldots > x_d > 0\}} |K_n^{1,\alpha}(x)|\,dx + \int\limits_{\{\pi > x_1 > x_2 > \ldots > x_d \ge \pi - 16/n\}} |K_n^{1,\alpha}(x)|\,dx \le C.$$

Hence it is enough to integrate over

$$\mathcal{S} := \{x \in \mathbb{T}^d : \pi > x_1 > x_2 > \ldots > x_d > 0, x_1 > 16/n, x_d < \pi - 16/n\}.$$

For a sequence $(i_l, j_l) \in \mathcal{I}$ let us define the set $\mathcal{S}_{(i_l,j_l),k}$ by

$$\mathcal{S}_{(i_l,j_l),k} := \begin{cases} x \in \mathcal{S} : x_{i_l} - x_{j_l} > \frac{4}{n}, l = 1, \ldots, k-1, x_{i_k} - x_{j_k} \le \frac{4}{n}, & \text{if } k < d; \\ x \in \mathcal{S} : x_{i_l} - x_{j_l} > \frac{4}{n}, l = 1, \ldots, d-1, & \text{if } k = d \end{cases}$$

and

$$\mathcal{S}_{(i_l,j_l),k,1} := \begin{cases} x \in \mathcal{S}_{(i_l,j_l),k} : x_{j_k} > \frac{4}{n},\ x_{j_{d-1}} \le \frac{\pi}{2}, & \text{if } k < d; \\ x \in \mathcal{S}_{(i_l,j_l),k} : x_{j_{d-1}} > \frac{4}{n},\ x_{j_{d-1}} \le \frac{\pi}{2}, & \text{if } k = d, \end{cases}$$

$$\mathcal{S}_{(i_l,j_l),k,2} := \begin{cases} x \in \mathcal{S}_{(i_l,j_l),k} : x_{j_k} \le \frac{4}{n},\ x_{j_{d-1}} \le \frac{\pi}{2}, & \text{if } k < d; \\ x \in \mathcal{S}_{(i_l,j_l),k} : x_{j_{d-1}} \le \frac{4}{n},\ x_{j_{d-1}} \le \frac{\pi}{2}, & \text{if } k = d, \end{cases}$$

$$\mathcal{S}_{(i_l,j_l),k,3} := \begin{cases} x \in \mathcal{S}_{(i_l,j_l),k} : \pi - x_{i_k} > \frac{4}{n},\ x_{j_{d-1}} > \frac{\pi}{2}, & \text{if } k < d; \\ x \in \mathcal{S}_{(i_l,j_l),k} : \pi - x_{i_{d-1}} > \frac{4}{n},\ x_{j_{d-1}} > \frac{\pi}{2}, & \text{if } k = d, \end{cases}$$

$$\mathcal{S}_{(i_l,j_l),k,4} := \begin{cases} x \in \mathcal{S}_{(i_l,j_l),k} : \pi - x_{i_k} \le \frac{4}{n},\ x_{j_{d-1}} > \frac{\pi}{2}, & \text{if } k < d; \\ x \in \mathcal{S}_{(i_l,j_l),k} : \pi - x_{i_{d-1}} \le \frac{4}{n},\ x_{j_{d-1}} > \frac{\pi}{2}, & \text{if } k = d. \end{cases}$$

Then

$$\int\limits_{\mathbb{T}^d} |K_n^{1,\alpha}(x)| \mathbf{1}_{\mathcal{S}}(x) dx \le \sum_{k=1}^{d} \sum_{m=1}^{4} \int\limits_{\mathbb{T}^d} |K_{n,(i_l,j_l)}^{1,\alpha}(x)| \mathbf{1}_{\mathcal{S}_{(i_l,j_l),k,m}(x)}\,dx.$$

Since the proof is similar to that for Fourier transforms (see Weisz [17]), we do not give a full version of the proof. We will consider the sets $\mathcal{S}_{(i_l,j_l),k,1}$, only. First let $1 \le k \le d-1$:

$$\sum_{k=1}^{d-1} \int\limits_{\mathbb{T}^d} |K_{n,(i_l,j_l)}^{1,\alpha}(x)| \mathbf{1}_{\mathcal{S}_{(i_l,j_l),k,1}}(x)\,dx \le \sum_{m=1}^{8} \sum_{k=1}^{d-1} \int\limits_{\mathbb{T}^d} K_{(i_l,j_l),m}^{1,\alpha}(x) \mathbf{1}_{\mathcal{S}_{(i_l,j_l),k,1}}(x)\,dx.$$

Since $x_{i_{d-1}} - x_{j_{d-1}} \le x_{i_l} - x_{j_l}$, (10) implies

$$\int_{\mathbb{T}^d} K^{1,\alpha}_{(i_l,j_l),1}(x) \mathbf{1}_{\mathcal{S}_{(i_l,j_l),k,1}}(x)\, dx \le$$

$$\le C \int_{\mathbb{T}^d} \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1-\beta} x_{j_{d-1}}^{\beta(d-1)-\alpha-1} (x_{i_{d-1}} - x_{j_{d-1}})^{\alpha} \mathbf{1}_{\mathcal{S}_{(i_l,j_l),k,1}}(x)\, dx \le$$

$$\le C \int_{\mathbb{T}^d} \prod_{l=1}^{k-1} (x_{i_l} - x_{j_l})^{-1-\beta} \prod_{l=k}^{d-1} (x_{i_l} - x_{j_l})^{-1-\beta+\alpha/(d-k)} x_{j_{d-1}}^{\beta(d-1)-\alpha-1} \times$$

$$\times \mathbf{1}_{\mathcal{S}_{(i_l,j_l),k,1}}(x)\, dx.$$

First we choose the indices $j_{d-1} (= i'_d)$, $i_{d-1} (= i'_{d-1})$ and then i_{d-2} if $i_{d-2} \ne \ne i_{d-1}$ or j_{d-2} if $j_{d-2} \ne j_{d-1}$. (Exactly one case of these two cases is satisfied.) If we repeat this process then we get an injective sequence $(i'_l, l = 1, \ldots, d)$. We integrate the term $x_{i_1} - x_{j_1}$ in $x_{i'_1}$, the term $x_{i_2} - x_{j_2}$ in $x_{i'_2}$, …, and finally the term $x_{i_{d-1}} - x_{j_{d-1}}$ in $x_{i'_{d-1}}$ and $x_{j_{d-1}}$ in $x_{i'_d}$. Since $x_{i_l} - x_{j_l} > 4/n$ $(l = 1, \ldots, k-1)$, $x_{i_l} - x_{j_l} \le 4/n$ $(l = k, \ldots, d-1)$, $x_{j_{d-1}} \ge x_{j_k} > 4/n$ and we can choose β such that $\beta < \alpha/(d-1)$, we have

$$\int_{\mathbb{T}^d} K^{1,\alpha}_{(i_l,j_l),1}(x) \mathbf{1}_{\mathcal{S}_{(i_l,j_l),k,1}}(x)\, dx \le$$

$$\le C \prod_{l=1}^{k-1} (1/n)^{-\beta} \prod_{l=k}^{d-1} (1/n)^{-\beta+\alpha/(d-k)} (1/n)^{\beta(d-1)-\alpha} \le C.$$

The kernel function $K^{1,\alpha}_{(i_l,j_l),2}$ can be handles in the same way, because it is exactly $K^{1,\alpha}_{(i_l,j_l),1}$ with $\alpha = 1$.

Similarly, if $\beta < \alpha/(d-1)$ then

$$\int_{\mathbb{T}^d} K^{1,\alpha}_{(i_l,j_l),3}(x) \mathbf{1}_{\mathcal{S}_{(i_l,j_l),k,1}}(x)\, dx \le$$

$$\le C \int_{\mathbb{T}^d} \prod_{l=1}^{k-1} (x_{i_l} - x_{j_l})^{-1-\beta} \prod_{l=k}^{d-2} (x_{i_l} - x_{j_l})^{-1-\beta+\alpha/(d-k)} \times$$

$$\times (x_{i_{d-1}} - x_{j_{d-1}})^{\alpha/(d-k)-1} x_{j_{d-1}}^{\beta(d-2)-\alpha-1} \mathbf{1}_{\mathcal{S}_{(i_l,j_l),k,1}}(x)\, dx \le$$

$$\le C \prod_{l=1}^{k-1} (1/n)^{-\beta} \prod_{l=k}^{d-2} (1/n)^{-\beta+\alpha/(d-k)} (1/n)^{\alpha/(d-k)} (1/n)^{\beta(d-2)-\alpha} \le C,$$

and the same holds for the kernels $K^{\theta}_{(i_l,j_l),m}(x)$, $m = 4, \ldots, 8$.

For the dth summand we use (8) to obtain

$$\begin{aligned}
\int_{\mathbb{T}^d} |K_{n,(i_l,j_l)}^{1,\alpha}(x)| \mathbf{1}_{\mathcal{S}_{(i_l,j_l),d,1}}(x)\,dx \le \\
\le \quad & Cn^{-\alpha} \int_{\mathbb{T}^d} \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1-\beta} x_{j_{d-1}}^{\beta(d-1)-\alpha-1} \mathbf{1}_{\mathcal{S}_{(i_l,j_l),d,1}}(x)\,dx + \\
& + Cn^{-1} \int_{\mathbb{T}^d} \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1-\beta} x_{j_{d-1}}^{\beta(d-1)-2} \mathbf{1}_{\mathcal{S}_{(i_l,j_l),d,1}}(x)\,dx \le \\
\le \quad & Cn^{-\alpha} \prod_{l=1}^{d-1} (1/n)^{-\beta} (1/n)^{\beta(d-1)-\alpha} + Cn^{-1} \prod_{l=1}^{d-1} (1/n)^{-\beta} (1/n)^{\beta(d-1)-1} \le \\
\le \quad & C,
\end{aligned}$$

if $\beta < \alpha/(d-1)$, which proves the theorem. ■

Theorem 2. *Suppose that $q = 1, \infty$ and $0 < \alpha < \infty$. If B denotes either $L_p(\mathbb{T}^d)$ $(1 \le p < \infty)$ or $C(\mathbb{T}^d)$ then*

$$\|\sigma_n^{q,\alpha} f\|_B \le C_p \|f\|_B \qquad (n \in \mathbb{N})$$

and

$$\lim_{n\to\infty} \sigma_n^{q,\alpha} f = f \qquad \text{in } B\text{-norm for all } f \in B.$$

Proof. Observe that

$$\|\sigma_n^{q,\alpha} f\|_B \le \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \|f(\cdot - u)\|_B K_n^{q,\alpha}(u)\,du = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \|f\|_B K_n^{q,\alpha}(u)\,du.$$

Since the trigonometric polynomials are dense in B, the theorem follows from Theorem 1 and from Banach-Steinhaus theorem. ■

4. Almost everywhere convergence of the Cesàro summability

We can extend the definition of the *Cesàro means* to distributions by

$$\sigma_n^{q,\alpha} f := f * K_n^{q,\alpha} \qquad (n \in \mathbb{N}).$$

To investigate the almost everywhere convergence we introduce the *maximal operator*

$$\sigma_*^{q,\alpha} f := \sup_{n\geq 1} |\sigma_n^{q,\alpha} f|$$

and the Hardy spaces. The *Hardy space* $H_p(\mathbb{T}^d)$ $(0 < p \leq \infty)$ consists of all distributions f for which

$$\|f\|_{H_p} := \|\sup_{0<t} |f * P_t^d|\|_p < \infty,$$

where

$$P_t^d(x) := \sum_{m\in\mathbb{Z}^d} e^{-t\|m\|_2} e^{\imath m\cdot x} \qquad (x \in \mathbb{T}^d,\ t > 0)$$

is the d-dimensional *Poisson kernel.* In the one-dimensional case we get back the usual Poisson kernel

$$P_t(x) := P_t^1(x) = \sum_{k=-\infty}^{\infty} r^{|k|} e^{\imath kx} = \frac{1-r^2}{1+r^2-2r\cos x} \qquad (x \in \mathbb{T}),$$

where $r = e^{-t}$. It is known that the Hardy spaces $H_p(\mathbb{T}^d)$ are equivalent to the $L_p(\mathbb{T}^d)$ spaces when $1 < p \leq \infty$ (see e.g. Stein [12] or Weisz [16]).

Theorem 3. *If* $q = 1, \infty$, $0 < \alpha < \infty$ *and* $d/(d + \alpha \wedge 1) < p \leq \infty$ *then*

$$\|\sigma_*^{q,\alpha} f\|_p \leq C_p \|f\|_{H_p} \qquad (f \in H_p(\mathbb{T}^d)) \tag{11}$$

and for $f \in H_{d/(d+\alpha\wedge 1)}(\mathbb{T}^d)$,

$$\|\sigma_*^{q,\alpha} f\|_{d/(d+\alpha\wedge 1),\infty} = \sup_{\rho>0} \rho\lambda(\sigma_*^{q,\alpha} f > \rho)^{(d+\alpha\wedge 1)/d} \leq C\|f\|_{H_{d/(d+\alpha\wedge 1)}}. \tag{12}$$

Since this theorem can be proved similarly to [17, 15], we omit the proof. For Fejér and Riesz means it was proved by the author [18, 17, 19, 15] (for Fourier transforms and $q = \infty$ see also Oswald [11]).

If p is smaller or equal to the critical index then this theorem is not true (Oswald [11], Stein, Taibleson and Weiss [13]).

Theorem 4. *If* $q = \infty$ *and* $\alpha = 1$ *then* (11) *does not hold for* $0 < p \leq$ $\leq d/(d + 1)$ *and* (12) *for* $0 < p < d/(d + 1)$.

Marcinkiewicz [9] verified for two-dimensional Fourier series that the cubic (i.e. $q = \infty$) Fejér means of a function $f \in L\log L(\mathbb{T}^2)$ converge a.e. to f as $n \to \infty$. Later Zhizhiashvili [22, 23] extended this result to all $f \in L_1(\mathbb{T}^2)$, Oswald [11] to Fourier transform and Riesz means and the author [19] to higher dimensions. The same result for $q = 1$ can be found in Weisz [18, 17] (see also Berens, Li and Xu [1]).

Corollary 1. *Suppose that* $q = 1, \infty$ *and* $0 < \alpha < \infty$. *If* $f \in L_1(\mathbb{T}^d)$ *then*

$$\sup_{\rho>0} \rho\, \lambda(\sigma_*^{q,\alpha} f > \rho) \le C\|f\|_1.$$

Corollary 2. *Suppose that* $q = 1, \infty$ *and* $0 < \alpha < \infty$. *If* $f \in L_1(\mathbb{T}^d)$ *then*

$$\lim_{n\to\infty} \sigma_n^{q,\alpha} f = f \qquad a.e.$$

Proof. Since the trigonometric polynomials are dense in $L_1(\mathbb{T}^d)$, the result follows from Corollary 2 and the usual density argument due to Marcinkiewicz and Zygmund [10]. ■

References

[1] **Berens, H., Z. Li, and Y. Xu,** On l_1 Riesz summability of the inverse Fourier integral, *Indag. Mathem.*, **12** (2001), 41–53.

[2] **Berens, H. and Y. Xu,** Fejér means for multivariate Fourier series, *Math. Z.*, **221** (1996), 449–465.

[3] **Bergh, J. and J. Löfström,** *Interpolation Spaces, an Introduction*, Springer, Berlin, 1976.

[4] **Butzer, P.L. and R. J. Nessel,** *Fourier Analysis and Approximation*, Birkhäuser Verlag, Basel, 1971.

[5] **Carleson, L.,** On convergence and growth of partial sums of Fourier series, *Acta Math.*, **116** (1966), 135–157.

[6] **Edwards, R.E.,** *Fourier Series, A Modern Introduction*, volume 2, Springer, Berlin, 1982.

[7] **Fefferman, C.,** On the convergence of multiple Fourier series, *Bull. Amer. Math. Soc.*, **77** (1971), 744–745.

[8] **Grafakos, L.,** *Classical and Modern Fourier Analysis*, Pearson Education, New Jersey, 2004.

[9] **Marcinkiewicz, J.,** Sur une méthode remarquable de sommation des séries doubles de Fourier, *Ann. Scuola Norm. Sup. Pisa*, **8** (1939), 149–160.

[10] **Marcinkiewicz, J. and A. Zygmund,** On the summability of double Fourier series, *Fund. Math.*, **32** (1939), 122–132.

[11] **Oswald, P.,** On Marcinkiewicz–Riesz summability of Fourier integrals in Hardy spaces, *Math. Nachr.*, **133** (1987), 173–187.

[12] **Stein, E.M.,** *Harmonic Analysis: Real-variable Methods, Orthogonality and Oscillatory Integrals*, Princeton Univ. Press, Princeton, N.J., 1993.

[13] **Stein, E.M., M. H. Taibleson, and G. Weiss,** Weak type estimates for maximal operators on certain H^p classes, *Rend. Circ. Mat. Palermo, Suppl.*, **1** (1981), 81–97.

[14] **Stein, E.M. and G. Weiss,** *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, Princeton, N.J., 1971.

[15] **Weisz, F.,** Weak type inequalities for the ℓ_1-summability of higher dimensional Fourier transforms, preprint.

[16] **Weisz, F.,** *Summability of Multi-dimensional Fourier Series and Hardy Spaces*, Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.

[17] **Weisz, F.,** ℓ_1-summability of d-dimensional Fourier transforms, *Constr. Approx.*, **34** (2011), 421–452.

[18] **Weisz, F.,** ℓ_1-summability of higher-dimensional Fourier series, *J. Approx. Theory*, **163** (2011), 99–116.

[19] **Weisz, F.,** Marcinkiewicz-summability of more-dimensional Fourier transforms and Fourier series, *J. Math. Anal. Appl.*, **379** (2011), 910–929.

[20] **Weisz, F.,** Triangular Cesàro summability of two-dimensional Fourier series, *Acta Math. Hungar.*, **132** (2011), 27–41.

[21] **Xu, Y.,** Christoffel functions and Fourier series for multivariate orthogonal polynomials, *J. Appr. Theory*, **82** (1995), 205–239.

[22] **Zhizhiashvili, L.,** A generalization of a theorem of Marcinkiewicz, *Math. USSR, Izvestija*, **2** (1968), 1065–1075, (in Russian).

[23] **Zhizhiashvili, L.,** *Trigonometric Fourier Series and their Conjugates*, Kluwer Academic Publishers, Dordrecht, 1996.

[24] **Zygmund, A.,** *Trigonometric Series*, Cambridge Press, London, 3rd edition, 2002.

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