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THE MODIFIED JOINT OPTIMAL STRATEGY CONCEPT IN ZERO-SUM FUZZY MATRIX GAMES

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Abstract. The purpose of this paper is the formulation of modified joint strategy concept in zero-sum fuzzy matrix game by using possibility distributions approach.

1. Introduction

In many practical problems, the quantities can only be estimated. In the case when the quantities are coefficients of the zero-sum matrix games, they may be characterized by fuzzy numbers. In this paper we consider zero-sum matrix games with fuzzy payoffs and fuzzy goals. For any pair of strategies, a player receives a payoff represented as a quasi-triangular fuzzy number. For example, when a payoff matrix of a game is constructed by information from a competitive system, elements of the payoff matrix would be ambiguous if imprecision or vagueness exists in the information. The theory on fuzzy games

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has been developed by Aubin (1981) and Butnariu (1978). Recently new approaches have emerged for studying fuzzy matrix games (e.g. Bector and Chandra, 2005; Vijay et al, 2007; Cevikel and Ahlatcioglu, 2009; Yu and Larbani, 2009).

In practice, when one or more coefficients of the optimization problem have uncertain values, then the optimal value will be uncertain. In order to reach the always α-level of optimal value we must take an optimal decision. Although the optimal value is uncertain, the decision must be unambiguous. Therefore, the α-optimal solution set contains vectors of real numbers. The concepts of modified joint optimal solution and fuzzy optimal value defined by Makó (2006) do comply with the above presented requirements. These concepts are founded on the notion of joint optimal solution defined by Buckley (1995).

In this paper we aim at utilizing these concepts to define a new generalized model for a matrix game with fuzzy goals and fuzzy payoffs by using possibility distributions approach.

The paper is organized as follows. The basic notions and the elementary properties of the quasi-triangular fuzzy numbers are discussed in Sections 2 and 3. The concepts of modified joint optimal solution and fuzzy optimal value of the fuzzy linear programming problem are presented in Section 4. Section 5 is devoted to the formulation of the concepts of modified joint optimal strategies in the zero-sum fuzzy matrix game and in the last section an application is presented.

2. Preliminaries

The fuzzy set concept was introduced in mathematics by Menger in 1942 and reintroduced in the system theory by Zadeh in 1965. Zadeh introduced this notion to measure quantitatively the vague of the linguistic variable.

Definition 2.1. Let X be a set. A mapping $\mu : X \to [0,1]$ is called *membership function*, and the set $\bar{A} = \{(x, \mu(x)) : x \in X\}$ is called *fuzzy set* on X. The membership function of $\bar{A}$ is denoted by $\mu_{\bar{A}}$.

The collection of all fuzzy subsets of X will be denoted by $\mathcal{F}(X)$. We place a bar over a symbol if it represents a fuzzy set. If $\bar{A}$ is a fuzzy set of X, then $\mu_{\bar{A}}(x)$ represents the membership degree of x to A. The empty fuzzy set is denoted by $\bar{\emptyset}$, where $\mu_{\bar{\emptyset}}(x) = 0$ for all $x \in X$. The total fuzzy set is denoted by $\bar{X}$, where $\mu_{\bar{X}}(x) = 1$ for all $x \in X$.

Definition 2.2. Let X be a topological space. The $\alpha-$*level* of $\bar{A}$ is defined as

$$[\bar{A}]^{\alpha} = \begin{cases} \{x \in X : \mu_{\bar{A}}(x) \geq \alpha\} & \text{if } \alpha > 0, \\ cl\left(\text{supp}\bar{A}\right) & \text{if } \alpha = 0. \end{cases}$$

where $cl\left(\text{supp}\bar{A}\right)$ is the closure of the support of $\bar{A}$.

Definition 2.3. A fuzzy set $\bar{A}$ on vector space X is *convex*, if all α -levels are convex subsets of X, and it is *normal* if $[\bar{A}]^{1} \neq \emptyset$.

Definition 2.4. A convex, normal fuzzy set on the real line $\mathbb{R}$ with upper semicontinuous membership function will be called *fuzzy number*.

Triangular norms and co-norms were introduced by Menger (1942) and studied first by Schweizer and Sklar (1961, 1963, 1983) to model distances in probabilistic metric spaces. In fuzzy sets theory triangular norms and co-norms are extensively used to model logical connections *and* and *or*. An important result is the following:

Theorem 2.1 (Ling, 1965). *Every Archimedean t-norm T can be represented by a continuous and decreasing function $g : [0,1] \to [0,+\infty]$ with $g(1) = 0$ and*

$$T(x,y) = g^{[-1]}(g(x) + g(y)),$$

where

$$g^{[-1]}(x) = \begin{cases} g^{-1}(x) & \text{if} \quad 0 \leq x < g(0), \\ 0 & \text{if} \quad x \geq g(0). \end{cases}$$

If g_1 and g_2 are generator functions of T, then there exists $c > 0$ such that $g_1 = cg_2$.

In order to use fuzzy sets and relations in any intelligent system we must be able to perform arithmetic operations. In fuzzy theory the extension of arithmetic operations to fuzzy sets was formulated by Zadeh in 1975.

If T is a t-norm and "$*$" is a binary operation on $\mathbb{R}$, then "$*$" can be extended to fuzzy quantities in the sense of the generalized extension principle of Zadeh (Fuller, 1998).

Definition 2.5. Let $\bar{A}$ and $\bar{B}$ be two fuzzy numbers. Then the membership function of fuzzy set $\bar{A} * \bar{B} \in \mathcal{F}(\mathbb{R})$ is

$$\mu_{\bar{A}*\bar{B}}(y) = \sup\{T(\mu_{\bar{A}}(x_1), \mu_{\bar{B}}(x_2)) : x_1 * x_2 = y\}, \tag{2.1}$$

for all $y \in \mathbb{R}$.

If we replace "$*$" by operations "$+$", "$-$","$\cdot$", or "$/$", then we get the membership functions of sum, difference, product or fraction.

The fuzzy numbers can also be considered as possibility distributions (Zadeh, 1978). If $\bar{A}$ is a fuzzy number and x a real number, then $\mu_{\bar{A}}(x)$ can be interpreted as the degree of possibility of the statement "x is in $\bar{A}$", namely $Pos\left(\bar{A}=x\right)=\mu_{\bar{A}}(x)$ for all $x\in\mathbb{R}$.

Let $\bar{A}$ and $\bar{B}$ be fuzzy numbers. The degree of possibility that the proposition "$\bar{A}$ is less than or equal to $\bar{B}$" is true will be denoted by $\mathrm{Pos}\left(\bar{A}\le\bar{B}\right)$ and defined by the generalized extension principle of Zadeh as

$$Pos\left(\bar{A}\le\bar{B}\right)=\sup_{x\le y}T\left(\mu_{\bar{A}}(x),\mu_{\bar{B}}(y)\right).$$

3. Quasi-triangular fuzzy numbers

Let $p\in[1,+\infty]$ and let $g:[0,1]\to[0,\infty]$ be a continuous, strictly decreasing function with the boundary properties $g(1)=0$ and $\lim_{t\to0}g(t)=g_0\le\infty$. The quasi-triangular fuzzy number is defined by using the Archimedean t-norm:

$$T_{gp}(x,y)=g^{[-1]}\left(\left(g^p(x)+g^p(y)\right)^{\frac{1}{p}}\right). \tag{3.1}$$

Definition 3.1 (Kovács, 1992)**.** The *set of quasi-triangular fuzzy numbers* is

$$\begin{aligned}\mathcal{N}_g=\Big\{\bar{A}\in\mathcal{F}(\mathbb{R})\ &:\ \text{there are } a\in\mathbb{R}, d>0 \text{ such that}\\ &\mu_{\bar{A}}(x)=g^{[-1]}\left(\frac{|x-a|}{d}\right)\ \text{for all } x\in\mathbb{R}\Big\}\bigcup\\ &\big\{\bar{A}\in\mathcal{F}(\mathbb{R})\ :\ \text{there is } a\in\mathbb{R} \text{ such that}\\ &\ \mu_{\bar{A}}(x)=\chi_{\{a\}}(x)\ \text{ for all } x\in\mathbb{R}\big\},\end{aligned} \tag{3.2}$$

where χ_A is the characteristic function of the set A. The elements of $\mathcal{N}_g$ will be called *quasi-triangular fuzzy numbers* generated by g with center a and spread d and we will denote them with $\langle a,d\rangle$.

From relation (2.1) we have that, if $p\in[1,+\infty)$, then the T_{gp}-*sum* of $\bar{A}$ and $\bar{B}$ is

$$\mu_{\bar{A}+\bar{B}}(z)=\sup_{x+y=z}\left[g^{[-1]}\left(\left[g^p\left(\mu_{\bar{A}}(x)\right)+g^p\left(\mu_{\bar{B}}(y)\right)\right]^{\frac{1}{p}}\right)\right],$$

and if $p = +\infty$, then the T_{gp}-*sum* of $\bar{A}$ and $\bar{B}$ is

$$\mu_{\bar{A}+\bar{B}}(z) = \sup_{x+y=z} \min\{\mu_{\bar{A}}(x), \mu_{\bar{B}}(y)\},$$

for all $z \in \mathbb{R}$.

Keresztfalvi and Kovács (1992) proved the formula (3.3) for the T_{gp}-sum of quasi-triangular fuzzy numbers.

Theorem 3.1. *Let $p \in [1, +\infty]$. If $\bar{A} = \langle a, d\rangle$ and $\bar{B} = \langle b, e\rangle$ are quasi-triangular fuzzy numbers, then also $\bar{A} + \bar{B}$ is a quasi-triangular fuzzy number, and*

$$\bar{A} + \bar{B} = \left\langle a + b, (d^q + e^q)^{\frac{1}{q}} \right\rangle, \tag{3.3}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proposition 3.2 (Makó, 2006 (2))**.** *Let $p \in [1, +\infty]$. If $\bar{A} = \langle a, d\rangle$ and $\bar{B} = \langle b, e\rangle$ are quasi-triangular fuzzy numbers, then*

$$Pos\,(\langle a, d\rangle \le \langle b, e\rangle) = \begin{cases} 1 & \text{if} \quad a \le b, \\ g^{[-1]}\left(\frac{a-b}{(d^q+e^q)^{1/q}}\right) & \text{if} \quad a > b. \end{cases} \tag{3.4}$$

We know that $\langle a, d\rangle + \langle a, d\rangle =< 2a, 2^{\frac{1}{q}} d >$. We generalize this property as follows.

Definition 3.2. For all $\langle a, d\rangle \in \mathcal{N}_g$ and for all $\lambda \in \mathbb{R}_+$ the *scalar multiplication* $\lambda \langle a, d\rangle$ is defined by

$$\lambda \langle a, d\rangle = \left\langle \lambda a, \lambda^{1/q} d \right\rangle.$$

The shortage that not every quasi-triangular fuzzy number has an additive inverse related to the t-norm-based addition, only the ones with spreads zero, can be solved if the set $\mathcal{N}_g$ is included isomorphically in an extended set. Makó (2006 (2)) proves that this extended set forms a vector space with respect to t-norm-based addition and scalar multiplication.

Remark 3.1 (Makó, 2006 (2))**.** 1. If $\langle a, d\rangle \in \mathcal{N}_g$ and $d > 0$, then

$$[\langle a, d\rangle]^\alpha = [\lambda a - dg(\alpha), a + dg(\alpha)]$$

and if $d = 0$, then $[\langle a, d\rangle]^\alpha = \{a\}$, for all $\alpha \in [0, 1]$.

2. If $\langle a_i, d_i\rangle, \langle m, p\rangle \in \mathcal{N}_g$ and $\lambda_i \geq 0$ for all $i = 1, ..., n$ then

$$Pos\left(\sum_{i=1}^{n} \lambda_i \langle a_i, d_i\rangle \leq \langle m, p\rangle\right) \geq \alpha \iff \tag{3.5}$$

$$\iff \sum_{i=1}^{n} \lambda_i a_i \leq m + g(\alpha)\left(p^q + \sum_{i=1}^{n} d_i^q \lambda_i\right)^{1/q}$$

and

$$Pos\left(\sum_{i=1}^{n} \lambda_i \langle a_i, d_i\rangle \geq \langle m, p\rangle\right) \geq \alpha \iff \tag{3.6}$$

$$\iff \sum_{i=1}^{n} \lambda_i a_i \geq m - g(\alpha)\left(p^q + \sum_{i=1}^{n} d_i^q \lambda_i\right)^{1/q}$$

for all $\alpha \in [0, 1]$.

4. The modified joint optimal solution of fuzzy linear programming problem

The *fuzzy linear programming problem* (*FLP problem*) is

$$\begin{cases} Z = \bar{c}x \to \max, \\ \bar{A}_i x \underset{Pos}{\preceq} \bar{b}_i \quad i \in I, \ x \geq 0, \end{cases} \tag{4.1}$$

where $\bar{c} = (\bar{c}_1, \bar{c}_2, ..., \bar{c}_n)$ is a $1 \times n$ vector of fuzzy numbers, $\bar{b}_i$ are fuzzy numbers for all $i \in I = \{1, 2, ..., m\}$, $\bar{A}_i = (\bar{a}_{i,1}, \bar{a}_{i,2}, ..., \bar{a}_{i,n})$ is an $1 \times n$ vector of fuzzy numbers for any $i \in I$.

Definition 4.1 (Makó, 2006)**.** Let $\alpha \in [0, 1]$. The *α-feasible set of problem* (4.1) is defined as

$$\mathcal{H}_\alpha\left(\bar{A}, \bar{b}\right) = \left\{x \geq 0 : \text{Pos}\left(\bar{A}_i x \leq \bar{b}_i\right) \geq \alpha, \forall i \in I\ \right\}.$$

The *$\alpha-$optimal solution set of problem* (4.1) is denoted by $\mathcal{S}_\alpha\left(\bar{A}, \bar{b}, \bar{c}\right)$. If we look for the maximum of the objective function in (4.1), then $\mathcal{S}_\alpha\left(\bar{A}, \bar{b}, \bar{c}\right)$ is defined as

$$\mathcal{S}_\alpha\left(\bar{A}, \bar{b}, \bar{c}\right) = \left\{x \in \mathcal{H}_\alpha\left(\bar{A}, \bar{b}\right) \ : \ Pos\left(\bar{c}y \leq \bar{c}x\right) \geq \alpha, \ \forall y \in \mathcal{H}_\alpha\left(\bar{A}, \bar{b}\right)\right\}.$$

The fuzzy subset of $\mathbb{R}^n$ defined by its membership function

$$\mu_{\bar{X}}(x) = \sup\left\{\alpha \in [0,1] \,:\, x \in \mathcal{S}_\alpha\left(\bar{A}, \bar{b}, \bar{c}\right)\right\}$$

is called the *modified joint solution of the problem* (4.1) and is denoted by $\bar{X}$. Let

$$\Gamma_\alpha = \big\{cx \,:\, x \in \mathcal{S}_\alpha\left(\bar{A}, \bar{b}, \bar{c}\right) \text{ and } c = (c_1, c_2, ..., c_n)\,, \\ \text{where } \ c_j \in [\bar{c}_j]^\alpha\,, \forall j = 1, ..., n\big\}\,,$$

with $0 \le \alpha \le 1$.

The *fuzzy optimal value of the objective function* in the problem (4.1) is a fuzzy set on $\mathbb{R}$, defined by its membership function

(4.2)
$$\mu_{\bar{Z}}(t) = \begin{cases} \sup\{\alpha \in [0,1] \,:\, t \in \Gamma_\alpha\} & \text{if} \quad \exists \alpha \in (0,1] \text{ such that } t \in \Gamma_\alpha, \\ 0 & \text{else.} \end{cases}$$

Remark 4.1. The determination of the modified joint solution of problem (4.1) means that we determine the fuzzy optimal value of the objective function and at least one element of $\mathcal{S}_\alpha\left(\bar{A}, \bar{b}, \bar{c}\right)$ for all $\alpha \in [0,1]$ if $\mathcal{S}_\alpha\left(\bar{A}, \bar{b}, \bar{c}\right)$ is not empty.

5. Zero-sum fuzzy matrix games

We begin this section with describing a crisp game. Let $A \in \mathbb{R}^{m\times n}$ be an $m \times n$ real matrix and

$$S^m = \left\{x \in \mathbb{R}^m_+ : \sum_{i=1}^m x_i = 1\right\}, \ S^n = \left\{y \in \mathbb{R}^n_+ : \sum_{j=1}^n y_j = 1\right\}.$$

By a crisp two person zero-sum matrix game G we mean the triplet $G = (S^m, S^n, A)$ where S^m and S^n are called the strategy space for Player I and Player II, respectively.

The matrix A is called the payoff matrix. For $x \in S^m$, $y \in S^n$, the scalar $x^t Ay$ is the payoff to Player I, and as the game G is zero-sum, the payoff to Player II is $-x^t Ay$.

Definition 5.1. The triplet $(x^*, y^*, \lambda) \in S^m \times S^n \times R$ is called a *solution of the game* G if

$$x^{*t}Ay \ge \lambda \quad (\forall)\, y \in S^n \text{ and } x^t Ay^* \le \lambda \quad (\forall)\, x \in S^m.$$

Here x^* and y^* are called the optimal strategy for Player I and Player II, respectively and λ is called the value of the game G.

Given a two person zero-sum matrix game G with $\max\limits_{i=\overline{1,m}} \min\limits_{j=\overline{1,n}} (a_{ij}) > 0$, it is customary to associate the following pair of primal-dual linear programming problems (LP) and (LD) with Player I and Player II, respectively:

$$LP : \begin{cases} z = \sum\limits_{i=1}^{m} u_i \to \min, \\ \sum\limits_{i=1}^{m} a_{ij} u_i \geq 1, \quad j = 1, ..., n, \\ u \geq 0. \end{cases} \qquad LD : \begin{cases} w = \sum\limits_{j=1}^{n} v_j \to \max, \\ \sum\limits_{j=1}^{n} a_{ij} v_j \leq 1, \quad i = 1, ..., m, \\ v \geq 0. \end{cases}$$

In this context the following theorems are well known (see for example Owen (1995)).

Theorem 5.1. *Every zero-sum matrix game G has a solution.*

Theorem 5.2. *If $\max\limits_{i=\overline{1,m}} \min\limits_{j=\overline{1,n}} (a_{ij}) > 0$ then $(x^*, y^*, \lambda) \in S^m \times S^n \times \mathbb{R}$ is a solution of the game G if and only if*

$$\begin{aligned} x^* &= (u_1^*/z_{\min}, u_2^*/z_{\min}, ..., u_m^*/z_{\min}), \\ y^* &= (v_1^*/w_{\max}, v_2^*/w_{\max}, ..., v_n^*/w_{\max}), \\ \lambda &= 1/z_{\min} = 1/w_{\max}, \end{aligned} \tag{5.1}$$

where $(u_1^*, u_2^*, ..., u_m^*)$ is an optimal solution and $z_{\min}$ is optimal value of the problem LP, and $(v_1^*, v_2^*, ..., v_n^*)$ is an optimal solution and $w_{\max}$ is optimal value of the problem LD.

Using the possibility distributions approach, we now propose a new model of a fuzzy matrix game. Let the payoff matrix be $\bar{A} = [\langle a_{ij}, d_{ij} \rangle]_{i=1,m; j=1,n}$ with entries as quasi-triangular fuzzy numbers. In the discussion to follow, we assume that the numbers $\bar{1}$ in FP and FD are vectors of quasi-triangular fuzzy numbers with spreads $(b_j)_{j=\overline{1,n}}$ and $(c_i)_{i=\overline{1,m}}$. A zero-sum fuzzy matrix game is a generalization of a zero-sum crisp matrix game G associated with the fuzzified version of problems LP and LD to Player I and Player II, respectively:

$$FP : \begin{cases} z = \sum\limits_{i=1}^{m} u_i \to \min, \\ \sum\limits_{i=1}^{m} u_i \langle a_{ij}, d_{ij} \rangle \underset{Pos}{\succeq} \langle 1, b_j \rangle, \\ j = 1, ..., n, \\ u \geq 0. \end{cases} \qquad FD : \begin{cases} w = \sum\limits_{j=1}^{n} v_j \to \max, \\ \sum\limits_{j=1}^{n} v_j \langle a_{ij}, d_{ij} \rangle \underset{Pos}{\preceq} \langle 1, c_i \rangle, \\ i = 1, ..., m, \\ v \geq 0. \end{cases}$$

By formulas (3.5) and (3.6)we have that for any $\alpha \in (0,1]$ the α-feasible set, α-optimal solution set of problem FP and the α-optimal strategy set for Player I are:

(5.2)

$$\mathcal{H}1_\alpha\left(\bar{A},\bar{1}\right)=\left\{u\geq 0:\,\sum_{i=1}^{m}a_{ij}u_i\geq 1-g\left(\alpha\right)\left(b_j^q+\sum_{i=1}^{m}d_{ij}^q u_i\right)^{1/q},j=1,...,n\right\},$$

$$(5.3)\quad \mathcal{S}1_\alpha\left(\bar{A},\bar{1},1\right)=\left\{u\in\mathcal{H}1_\alpha\left(\bar{A},\bar{1}\right)\,:\,\sum_{i=1}^{m}u_i\leq\sum_{i=1}^{m}u_i',\ \forall u'\in\mathcal{H}1_\alpha\left(\bar{A},\bar{1}\right)\right\},$$

$$(5.4)\qquad \mathcal{P}1_\alpha=\left\{x=u/\sum_{i=1}^{m}u_i\,:\,u\in\mathcal{S}1_\alpha\left(\bar{A},\bar{1},1\right)\right\}.$$

The membership functions of modified joint optimal strategy and optimal solution of fuzzy matrix game for Player I are:

$$(5.5)\qquad \begin{aligned}\mu_{\bar{\mathcal{P}}1}\left(x\right)&=\sup\left\{\alpha\in[0,1]\,:\,x\in\mathcal{P}1_\alpha\right\},\\ \mu_{\bar{z}}\left(t\right)&=\sup\left\{\alpha\in[0,1]\,:\,t\in\Gamma1_\alpha\right\},\end{aligned}$$

where

$$\Gamma1_\alpha=\left\{1/\sum_{i=1}^{m}u_i\,:\,u\in\mathcal{S}1_\alpha\left(\bar{A},\bar{1},1\right)\right\}.$$

Similarly, for any $\alpha \in (0,1]$ the α-feasible set, α-optimal solution set of problem FD and the α-optimal strategy set for Player II are:

(5.6)

$$\mathcal{H}2_\alpha\left(\bar{A},\bar{1}\right)=\left\{v\geq 0:\,\sum_{j=1}^{n}a_{ij}v_j\leq 1+g\left(\alpha\right)\left(c_i^q+\sum_{j=1}^{n}d_{ij}^q v_j\right)^{1/q},\quad i=1,...,m\right\},$$

$$(5.7)\quad \mathcal{S}2_\alpha\left(\bar{A},\bar{1},1\right)=\left\{v\in\mathcal{H}2_\alpha\left(\bar{A},\bar{1}\right)\,:\,\sum_{j=1}^{n}v_j\geq\sum_{j=1}^{n}v_j',\ \forall v'\in\mathcal{H}2_\alpha\left(\bar{A},\bar{1}\right)\right\},$$

$$(5.8)\qquad \mathcal{P}2_\alpha=\left\{y=v/\sum_{j=1}^{n}v_j\,:\,v\in\mathcal{S}2_\alpha\left(\bar{A},\bar{1},1\right)\right\}.$$

The membership functions of modified joint optimal strategy and optimal solution of fuzzy matrix game for Player II are:

$$\mu_{\bar{\mathcal{P}}2}(y) = \sup\{\alpha \in [0,1] \,:\, y \in \mathcal{P}2_\alpha\}, \qquad (5.9)$$
$$\mu_{\bar{w}}(t) = \sup\{\alpha \in [0,1] \,:\, t \in \Gamma 2_\alpha\},$$

where

$$\Gamma 2_\alpha = \left\{ 1/\sum_{j=1}^{n} v_j \,:\, v \in \mathcal{S}2_\alpha\left(\bar{A}, \bar{1}, 1\right) \right\}.$$

Let $q = 1$ (or $p = \infty$). According to formulas (5.2) and (5.7), for the determination of the α-optimal solution sets for a given α, we have to solve the following linear programming problems (FLP) and (FLD) for Player I and Player II, respectively.

$$(5.10) \qquad FLP(\alpha) : \begin{cases} z = \sum_{i=1}^{m} u_i \to \min, \\ \sum_{i=1}^{m} \left(a_{ij} + g(\alpha)\, d_{ij}\right) u_i \geq 1 - g(\alpha)\, b_j, \quad j = 1, ..., n, \\ u \geq 0, \end{cases}$$

$$(5.11) \qquad FLD(\alpha) : \begin{cases} w = \sum_{j=1}^{n} v_j \to \max, \\ \sum_{j=1}^{n} \left(a_{ij} - g(\alpha)\, d_{ij}\right) v_j \leq 1 + g(\alpha)\, c_i, \quad i = 1, ..., m, \\ v \geq 0, \end{cases}$$

6. Application

Two television networks are battling for viewer shares (approximately 100 million viewers). They make their programming decisions independently and simultaneously. Each network can show sports, comedy or western. Network 2 has a programming advantage in sports and Network 1 has it in comedy and western. The possible outcomes of Network 1 are represented in a table, where the numbers are the approximate values of viewers in millions:

Network 1	Network 2		
	sports	comedy	western
sports	35	15	60
comedy	32	58	50
western	38	14	70

In this application we consider that $q = 1$, and $g : (0,1] \to [0,\infty)$ is a function given by $g(t) = \sqrt{-2\ln t}$. Then the membership function of quasi-triangular fuzzy numbers $\langle a,d\rangle$ is

$$\mu(t) = e^{-\frac{(t-a)^2}{2d^2}} \quad \text{if } d>0, \quad \text{and}$$
$$\mu(t) = \begin{cases} 1 & \text{if } t=a, \\ 0 & \text{if } t\neq a, \end{cases} \quad \text{if } d=0.$$

In many situations people are not able to characterize numerical data precisely. For example, people use terms like: "approximately 35" or "nearly 35". These examples may be characterized by fuzzy numbers. Here we consider that the quasi-triangular fuzzy number is $\bar{1} = \langle 1, 1/100\rangle$ and all elements of payoff matrix are

$$\bar{A} = \begin{bmatrix} \langle 35,5\rangle & \langle 15,5\rangle & \langle 60,10\rangle \\ \langle 32,5\rangle & \langle 58,5\rangle & \langle 50,5\rangle \\ \langle 38,7\rangle & \langle 14,3\rangle & \langle 70,5\rangle \end{bmatrix}.$$

We determine the α-optimal solution sets for a given α by using the linear programming problems (5.10) and (5.11):

$$FLP(\alpha):$$
$$\begin{cases} z = u_1+u_2+u_3 \to \min, \\ (35+5g(\alpha))\,u_1 + (32+5g(\alpha))\,u_2 + (38+7g(\alpha))\,u_3 \geq 1 - \frac{1}{100}g(\alpha) \\ (15+5g(\alpha))\,u_1 + (58+5g(\alpha))\,u_2 + (14+3g(\alpha))\,u_3 \geq 1 - \frac{1}{100}g(\alpha) \\ (60+10g(\alpha))\,u_1 + (50+5g(\alpha))\,u_2 + (70+5g(\alpha))\,u_3 \geq 1 - \frac{1}{100}g(\alpha) \\ u_1,u_2,u_3 \geq 0. \end{cases}$$

$$FLD(\alpha):$$
$$\begin{cases} w = v_1+v_2+v_3 \to \max, \\ (35-5g(\alpha))\,v_1 + (15-5g(\alpha))\,v_2 + (60-10g(\alpha))\,v_3 \leq 1 + \frac{1}{100}g(\alpha) \\ (32-5g(\alpha))\,v_1 + (58-5g(\alpha))\,v_2 + (50-5g(\alpha))\,v_3 \leq 1 + \frac{1}{100}g(\alpha) \\ (38-7g(\alpha))\,v_1 + (14-3g(\alpha))\,v_2 + (70-5g(\alpha))\,v_3 \leq 1 + \frac{1}{100}g(\alpha) \\ v_1,v_2,v_3 \geq 0. \end{cases}$$

The α-optimal strategies and optimal solutions are shown in Figure 1 and Figure 2.

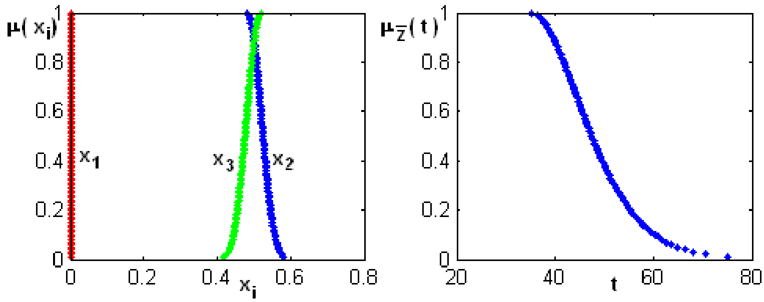

Figure 1. The membership functions of optimal strategies and optimal solution for Player I.

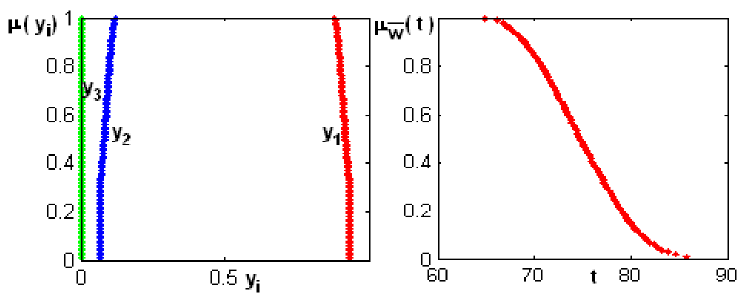

Figure 2. The membership functions of optimal strategies and optimal solution for Player II.

In the following table we provide the α-optimal strategies and α-optimal solutions for certain sample values of α.

	Player I		Player II	
α	$x^*(\alpha)$	$\bar{z}(\alpha)$	$y^*(\alpha)$	$\bar{w}(\alpha)$
0.1	$(0, 0.55, 0.44)$	60.22	$(0.935, 0.065, 0)$	81.0918
0.2	$(0, 0.54, 0.46)$	55.24	$(0.934, 0.066, 0)$	79.0361
0.3	$(0, 0.54, 0.46)$	52.04	$(0.934, 0.066, 0)$	77.5265
0.4	$(0, 0.53, 0.47)$	49.56	$(0.926, 0.073, 0)$	76.0859
0.5	$(0, 0.52, 0.48)$	47.44	$(0.919, 0.081, 0)$	74.7656
0.6	$(0, 0.51, 0.49)$	45.51	$(0.913, 0.087, 0)$	73.4833
0.7	$(0, 0.51, 0.49)$	43.65	$(0.907, 0.093, 0)$	72.1695
0.8	$(0, 0.50, 0.50)$	41.75	$(0.901, 0.099, 0)$	70.7332
0.9	$(0, 0.49, 0.51)$	39.58	$(0.894, 0.106, 0)$	68.9760
1	$(0, 0.48, 0.52)$	35.12	$(0.880, 0.120, 0)$	64.8800

Let $x \in S^m$ and $y \in S^n$. By using the formulas of modified joint optimal strategies (5.5) and (5.9) we can determine the possibility that x and y are optimal strategies for Player I and Player II, respectively. For example, if $x = (0, 0.52, 0.48)$ and $y = (0.89, 0.11, 0)$, then the possibility that x is an optimal strategy for Player I is $\mu_{\bar{\mathcal{P}}1}(x) = 0.5410$ and that y is an optimal strategy for Player II is $\mu_{\bar{\mathcal{P}}2}(y) = 0.911$. In this case 46.6365 is the optimal solution of game for Player I with possibility $\mu_{\bar{z}}(46.6365) = 0.5410$ and 68.7420 is the optimal solution of game for Player II with possibility $\mu_{\bar{w}}(68.7420) = = 0.911$.

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