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A NOTE ON DYADIC HARDY SPACES

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Dedicated to the 60th birthday of Professor Antal Járai

Abstract. The usual L^p-norms are trivially invariant with respect to multiplication by *Walsh* functions. The analogous question will be investigated in the dyadic Hardy space $\mathbf{H}$. We introduce an invariant subspace $\mathbf{H}_*$ of $\mathbf{H}$ in this sense and show some properties of $\mathbf{H}_*$. For example a function in $\mathbf{H}_*$ will be constructed the *Walsh–Fourier* series of which diverges in L^1-norm.

1. Introduction

Let w_n $(n \in \mathbf{N})$ be the *Walsh–Paley* system defined on the interval $[0, 1)$. It is well-known that $w_n = \prod_{k=0}^{\infty} r_k^{n_k}$, where r_k is the k-th *Rademacher* function $(k \in \mathbf{N})$ and $n = \sum_{k=0}^{\infty} n_k 2^k$ $(n_k = 0$ or 1 for all k's) is the dyadic representation of n. If $n = \sum_{k=0}^{\infty} n_k 2^k$, $m = \sum_{k=0}^{\infty} m_k 2^k \in \mathbf{N}$ then $w_n w_m = w_{n \oplus m}$, where the operation $\oplus$ is defined by

$$n \oplus m := \sum_{k=0}^{\infty} |n_k - m_k| 2^k.$$

Thus it is clear that

$$2^n \oplus m = 2^n + m \quad (n \in \mathbf{N},\ m = 0, \ldots, 2^n - 1),$$

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i.e. $r_n w_m = w_{2^n} w_m = w_{2^n+m}$. (For more details we refer to the book [1].) For $1 \leq p \leq \infty$ let $L^p := L^p[0,1)$ and let $\|.\|_p$ denote the usual *Lebesgue* space and norm. If $f \in L^1$, $n \in \mathbf{N}$ then let $S_n f$ be the n-th *Walsh–Fourier* partial sum of f, i.e. $S_n f = f * D_n$, where $D_n := \sum_{k=0}^{n-1} w_k$ and $*$ stands for dyadic convolution. We remark that $r_n D_{2^n} = D_{2^{n+1}} - D_{2^n} \quad (n \in \mathbf{N})$. The next famous property of D_{2^n}'s plays an important role in the Walsh analysis:

(1)
$$D_{2^n}(x) = \begin{cases} 2^n & (0 \leq x < 2^{-n}) \\ 0 & (2^{-n} \leq x < 1). \end{cases}$$

Therefore

$$S_{2^n} f(x) = 2^n \int_{I_n(x)} f \qquad (x \in [0,1)).$$

Here $x \in I_n(x) := [j2^{-n}, (j+1)2^{-n})$ with a proper integer $j(x) = j = 0, \ldots, 2^n - 1$. Set $I_n := I_n(0)$.

We recall that

(2)
$$\sup_n \frac{\|D_n\|_1}{\log n} < \infty.$$

The dyadic maximal function f^* of $f \in L^1$ is defined as follows:

$$f^* := \sup_n |S_{2^n} f|.$$

Then for all $p > 1$ we have $\|f\|_p \leq \|f^*\|_p \leq C_p \|f\|_p$. (Here and later C_p, C will denote positive constants depending at most on p, although not always the same in different occurences.) The so-called dyadic *Hardy* space $\mathbf{H} := \mathbf{H}[0,1)$ is defined by means of the maximal function as follows:

$$\mathbf{H} := \{f \in L^1 : \|f\| := \|f^*\|_1 < \infty\}.$$

The atomic structure of $\mathbf{H}$ is very useful in many investigations. Namely, we call a function $a \in L^\infty$ (dyadic) atom if $\int_0^1 a = 0$ and there exists a dyadic interval $I_n(z) \quad (n \in \mathbf{N},\ z \in [0,1))$ such that $a(x) = 0 \quad (x \in [0,1) \setminus I_n(z))$ and $\|a\|_\infty \leq 2^n$. Let supp $a := I_n(z)$. The characterization of $\mathbf{H}$ by means of atoms reads as follows:

$$f \in \mathbf{H} \iff f = \sum_{k=0}^{\infty} \alpha_k a_k,$$

where all a_k's are atoms and the coefficients α_k's have the next property: $\sum_{k=0}^{\infty} |\alpha_k| < \infty$. Furthermore,

$$\|f\| \sim \inf \sum_{k=0}^{\infty} |\alpha_k|,$$

where the infimum is taken over all atomic representations $\sum_{k=0}^{\infty} \alpha_k a_k$ of f. (For the martingale theoretic background we refer to [4].)

For example the functions $r_n D_{2^n}$ $(n \in \mathbf{N})$ are trivially atoms by (1). Thus

$$f := \sum_{n=0}^{\infty} \alpha_n r_{\nu_n} D_{2^{\nu_n}} \tag{3}$$

belongs to $\mathbf{H}$ if $\sum_{k=0}^{\infty} |\alpha_n| < \infty$ and the indices $\nu_0 < \nu_1 < \dots$ are choosen arbitrarily. Moreover, $\|f\|_1 \le \|f\| \le \sum_{n=0}^{\infty} |\alpha_n|$.

It is not hard to see that the partial sums $S_{2^n} a$ $(n \in \mathbf{N})$ remain atoms if $a \in L^\infty$ is an atom. Indeed, if supp $a = I_N(z)$ $(N \in \mathbf{N},\ z \in [0,1))$ and $x \in [0,1) \setminus I_N(z)$ then for all $n \in \mathbf{N}$ the intervals $I_n(x)$ and $I_N(z)$ are disjoint or $I_n(x) \cap I_N(z) = I_N(z)$. Thus

$$|S_{2^n} a(x)| = \left| 2^n \int_{I_n(x)} a \right| = \left| 2^n \int_{I_n(x) \cap I_N(z)} a \right| \le \left| 2^n \int_{I_N(z)} a \right| = \left| 2^n \int_0^1 a \right| = 0,$$

thus $S_{2^n} a(x) = 0$. Furthermore, $\|S_{2^n} a\|_\infty \le \|a\|_\infty \le 2^N$, i.e. supp $S_{2^n} a = I_N(z)$ and $\int_0^1 S_{2^n} a = \int_0^1 a = 0$.

Therefore if $f = \sum_{k=0}^{\infty} \alpha_k a_k$ is an atomic representation of $f \in \mathbf{H}$ then $S_{2^n} f = \sum_{k=0}^{\infty} \alpha_k S_{2^n} a$ $(n \in \mathbf{N})$ is an atomic representation of $S_{2^n} f$. This means that $\|S_{2^n} f\| \le \sum_{k=0}^{\infty} |\alpha_k|$, i.e. $\|S_{2^n} f\| \le \|f\|$. (The last inequality follows also from the obvious estimation $(S_{2^n} f)^* \le f^*$.)

We remark that $\mathbf{H}$ can be defined also in another way. To this end let $f \in L^1$ and

$$Qf := \left(\sum_{n=-1}^{\infty} (\delta_n f)^2 \right)^{1/2}$$

be its quadratic variation, where $\delta_{-1} f := \int_0^1 f$, $\delta_n f := S_{2^{n+1}} f - S_{2^n} f = f * (r_n D_{2^n})$ $(n \in \mathbf{N})$. Then

$$\|f\| \sim \|Qf\|_1 \text{ , ill. } \|f\|_p \sim \|Qf\|_p \qquad (1 < p < \infty).$$

If $f \in L^1, n \in \mathbf{N}$ and $k = 0, \dots, 2^n - 1$, then w_k is constant on $I_n(x)$ $(x \in [0,1))$, consequently $w_k(x) \int_{I_n(x)} f = \int_{I_n(x)} (f w_k)$. This means that $w_k S_{2^n} f = S_{2^n}(f w_k)$. Furthermore, if $2^n \le k \in \mathbf{N}$ is arbitrary then let us write $k = \sum_{j=0}^{N} k_j 2^j$ (with some $\mathbf{N} \ni N \ge n$). It is clear that

$$\delta_j(w_k S_{2^n} f) = \begin{cases} 0 & (j \ne N) \\ w_k S_{2^n} f & (j = N) \end{cases} \qquad (j \in \mathbf{N}).$$

From this it follows that $Q(w_k S_{2^n} f) = |S_{2^n} f|$, i.e. for all $k \in \mathbf{N}$ we have

$$\|w_k S_{2^n} f\| = \|S_{2^n}(f w_k)\| \ \ (k < 2^n) \quad \text{and}$$
$$\|w_k S_{2^n} f\| \le C \|S_{2^n} f\|_1 \ \ (k \ge 2^n). \tag{4}$$

The *Walsh–Paley* system doesn't form a basis in L^1. Moreover, there exists $f \in \mathbf{H}$ such that

$$\sup_n \|S_n f\|_1 = \infty.$$

However (see [3]), if $f \in \mathbf{H}$ then

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{\|S_k f\|_1}{k} \to \|f\| \qquad (n \to \infty),$$

or equivalently

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{\|f - S_k f\|_1}{k} \to 0 \qquad (n \to \infty).$$

For the sake of the completeness and in order to demonstrate the usefulness of the atomic structure we sketch some examples. Namely we take the function given by (3). If $l_n = 0, 1, \ldots, 2^{\nu_n} - 1 \ \ (n \in \mathbf{N})$ then

$$\|S_{2^{\nu_n} + l_n} f - S_{2^{\nu_n}} f\|_1 = |\alpha_n| \|D_{l_n}\|_1. \tag{$*$}$$

It is well-known that $k_n \in \{0, 1, \ldots, 2^{\nu_n} - 1\}$ can be choosen so that

$$\|D_{k_n}\|_1 \ge C \nu_n \qquad (n \in \mathbf{N})$$

holds. Then we get

$$\|S_{2^{\nu_n} + k_n} f - S_{2^{\nu_n}} f\|_1 \ge C |\alpha_n| \nu_n \qquad (n \in \mathbf{N}).$$

If $\sup_n |\alpha_n| \nu_n = \infty$ then $\|S_{2^n} f\|_1 \le \sum_{k=0}^{\infty} |\alpha_k| < \infty$ implies $\sup_n \|S_n f\|_1 = \infty$. It is obvious that $\alpha_n := 2^{-n}, \nu_n := 2^{n^2} \quad (n \in \mathbf{N})$ are suitable. (We remark that $\inf_n |\alpha_n| \nu_n > 0$ is trivially sufficient for the $\|.\|_1$ divergence of the *Walsh–Fourier* series of f.)

If $f \in \mathbf{H}$ is given by (3) then $\|S_n f - f\|_1 \to 0 \quad (n \to \infty)$ if and only if $\nu_n \alpha_n \to 0 \quad (n \to \infty)$. Indeed, if $l_n := k_n$'s are as above then $C \nu_n |\alpha_n| \le$ $\le |\alpha_n| \|D_{k_n}\|_1$ and $(*)$ proves necessity. It is known that $\|S_{2^n} g - g\|_1 \to 0$ $(n \to \infty)$ for all $g \in L^1$. Therefore (see (2)) $\|D_{l_n}\|_1 \le C \log l_n \le C \nu_n$ and $\nu_n \alpha_n \to 0 \ \ (n \to \infty)$ together with $(*)$ imply the $\|.\|_1$ convergence of the series (3).

Finally, we cite an example $f \in L^1 \backslash H$ such that $\|S_n f - f\|_1 \to 0 \quad (n \to \infty)$. To this end we take a special function $f := \sum_{n=0}^{\infty} \alpha_n r_n D_{2^n}$ in (3) such that the coefficients α_n form a null-sequence of bounded variation, i.e. $\sum_{n=0}^{\infty} |\alpha_n - -\alpha_{n+1}| < \infty$. It is well-known that this assumption on the coefficients implies the $\|.\|_1$-convergence of the series in question. Indeed, for all $n, m \in \mathbf{N}$, $n < m$ it follows by (1) that

$$\left\| \sum_{k=n}^{m} \alpha_k r_k D_{2^k} \right\|_1 = \left\| \sum_{k=n}^{m} \alpha_k (D_{2^{k+1}} - D_{2^k}) \right\|_1 =$$

$$= \left\| \sum_{k=n+1}^{m} (\alpha_{k-1} - \alpha_k) D_{2^k} + \alpha_m D_{2^m} - \alpha_n D_{2^n} \right\|_1 \leq$$

$$\leq \sum_{k=n+1}^{m} |\alpha_{k-1} - \alpha_k| \|D_{2^k}\|_1 + |\alpha_m| \|D_{2^m}\|_1 + |\alpha_n| \|D_{2^n}\|_1 =$$

$$= \sum_{k=n+1}^{m} |\alpha_{k-1} - \alpha_k| + |\alpha_m| + |\alpha_n| \to 0 \quad (n, m \to \infty).$$

Therefore $f \in L^1$. Furthermore, if $2^{-k-1} \leq x < 2^{-k} \quad (k \in \mathbf{N})$ then

$$Qf(x) = \sqrt{\sum_{n=0}^{\infty} \alpha_n^2 D_{2^n}^2(x)} = \sqrt{\sum_{n=0}^{k} \alpha_n^2 2^{2n}} \geq |\alpha_k| 2^k,$$

and

$$\|Qf\|_1 \geq \sum_{k=0}^{\infty} \int_{2^{-k-1}}^{2^{-k}} Qf \geq \sum_{k=0}^{\infty} \int_{2^{-k-1}}^{2^{-k}} |\alpha_k| 2^k = \frac{1}{2} \sum_{k=0}^{\infty} |\alpha_k|.$$

This means that $\|f\| = \infty$ if $\sum_{k=0}^{\infty} |\alpha_k| = \infty$. Now, we prove the $\|.\|_1$ convergence of the sequence $S_n f$. To this end let $1 \leq n \in \mathbf{N}$ and $m_n = 0, ..., 2^n - 1$. Then by (2) we have

$$\|S_{2^n + m_n} f - S_{2^n} f\|_1 = \|\alpha_n r_n D_{m_n}\|_1 = |\alpha_n| \|D_{m_n}\|_1 \leq C |\alpha_n| \log m_n \leq Cn |\alpha_n|.$$

Hence $n\alpha_n \to 0 \quad (n \to \infty)$ is implies to $\|S_{2^n + m_n} f - S_{2^n} f\|_1 \to 0 \quad (n \to \infty)$. Since $\|S_{2^n} f - f\|_1 \to 0 \quad (n \to \infty)$ we get $\|S_n f - f\|_1 \to 0 \quad (n \to \infty)$. A simple calculation shows that the sequence

$$\alpha_n := \frac{1}{(n+2) \log (n+2)} \qquad (n \in \mathbf{N})$$

satisfies all of the conditions above. By means of similar observations it can be proved that the assumption $\sum_{n=0}^{\infty} |\alpha_n| < \infty$ in (3) is necessary to $f \in \mathbf{H}$ in the general case as well.

2. Results

It is clear that for all $f \in L^p$ $(1 \le p \le \infty)$ and $n \in \mathbf{N}$ we have $fw_n \in L^p$ and $\|fw_n\|_p = \|f\|_p$. The situation in the case of $\mathbf{H}$ is more complicated. For example if we take the atoms $f_n := r_n D_{2^n} \in \mathbf{H}$ $(n \in \mathbf{N})$ then $\|f_n\| = 1$ and

$$\|r_n f_n\| = \|D_{2^n}\| = \|D_{2^n}^*\|_1 = \left\| \max_{k \le n} D_{2^k} \right\|_1 ,$$

where by (1)

$$\max_{k \le n} D_{2^k}(x) = \begin{cases} 2^k & (2^{-k-1} \le x < 2^{-k},\ k = 0, \dots, n-1) \\ 2^n & (0 \le x < 2^{-n}). \end{cases}$$

From this it follows immediately that $\|D_{2^n}\| = \dfrac{n+2}{2}$, i.e.

$$\|r_n f_n\| = \|w_{2^n} f_n\| = \frac{n+2}{2} \|f_n\|.$$

First we prove that an analogous relation holds in general.

Theorem 1. *Let $k \in \mathbf{N}$. Then there exists a constant C_k such that for all $f \in \mathbf{H}$ the product fw_k belongs to $\mathbf{H}$ and $\|fw_k\| \le C_k \|f\|$.*

Our example above shows that $C_{2^n} \ge \dfrac{n+2}{2}$ $(n \in \mathbf{N})$, i.e. $\sup_k C_k = \infty$. Since all *Walsh* functions are final products of *Rademacher* functions, we need to prove Theorem 1 only for $k = 2^n$ $(n \in \mathbf{N})$.

In this case let $f = \sum_{k=0}^{\infty} \alpha_k a_k$ be an atomic representation of $f \in \mathbf{H}$. Then

$$\|fw_{2^n}\| = \|fr_n\| = \|(fr_n)^*\|_1 \le \left\| \sum_{k=0}^{\infty} |\alpha_k| (a_k r_n)^* \right\|_1 \le$$
$$\le \sum_{k=0}^{\infty} |\alpha_k| \|(a_k r_n)^*\|_1 = \sum_{k=0}^{\infty} |\alpha_k| \|a_k r_n\|.$$

If we can show that

$$(**) \qquad A_n := \sup_a \|ar_n\| < \infty$$

(where the supremum is taken over all atoms a), then

$$\|(fr_n)^*\|_1 \le A_n \sum_{k=0}^{\infty} |\alpha_k|,$$

i.e. $\|fr_n\| \le A_n\|f\|$.

Proof of the inequality $(**)$. Let a be an atom, $k \in \mathbf{N}, x \in [0,1)$. In the case $k > n$ the n-th *Rademacher* function r_n is constant on the interval $I_k(x)$ and thus

$$S_{2^k}(ar_n)(x) = 2^k \int_{I_k(x)} ar_n = 2^k r_n(x) \int_{I_k(x)} a.$$

Therefore

$$(ar_n)^* = \sup_k |S_{2^k}(ar_n)| \le \max_{k\le n} |S_{2^k}(ar_n)| + \sup_{k>n} |S_{2^k}a| \le$$
$$\le \max_{k\le n} |S_{2^k}(ar_n)| + \sup_k |S_{2^k}a| = \max_{k\le n} |S_{2^k}(ar_n)| + a^* =: (ar_n)^{**} + a^*.$$

From this it follows that

$$\|ar_n\| = \|(ar_n)^*\|_1 \le \|(ar_n)^{**}\|_1 + \|a^*\|_1 =$$
$$= \|(ar_n)^{**}\|_1 + \|a\| \le \|(ar_n)^{**}\|_1 + 1.$$

This means that it is enough to show only

$$\sup_a \|(ar_n)^{**}\|_1 < \infty$$

(where the supremum is taken over all atoms a).

To this end let a be an atom. For the sake of simplicity we assume that $\operatorname{supp} a = I_N$ (with some $N \in \mathbf{N}$). Then

$$\|(ar_n)^{**}\|_1 = \int_{I_N} (ar_n)^{**} + \int_{2^{-N}}^{1} (ar_n)^{**} =: J_1(a) + J_2(a).$$

Hence by means of the *Cauchy* inequality and the properties of atoms it follows that

$$J_1(a) \le \left(\int_{I_N} ((ar_n)^{**})^2\right)^{1/2} \cdot 2^{-N/2} \le 2^{-N/2}\|(ar_n)^{**}\|_2 \le C_2 2^{-N/2}\|ar_n\|_2 \le$$
$$\le C_2 2^{-N/2}\|a\|_\infty 2^{-N/2} \le C_2.$$

We will show that

$$\sup_a J_2(a) < \infty.$$

Indeed, if a is the atom as above and $n < N$, then $ar_n = a$, i.e.

$$J_2(a) \le \|(ar_n)^{**}\|_1 = \|\max_{k\le n} |S_{2^k} a|\|_1 \le \|a^*\|_1 = \|a\| \le 1.$$

Thus it can be assumed that $N \le n$. Let $k = 0, ..., n$ and $2^{-N} \le x < 1$. Then

$$S_{2^k}(ar_n)(x) = 2^k \int\limits_{I_k(x)} ar_n = 2^k \int\limits_{I_k(x)\cap I_N} ar_n,$$

where $I_k(x) \cap I_N \ne \emptyset$ exactly if $k \le N-1$ and $x < 2^{-k}$ (in this case $I_k(x) = I_k$ and $I_k(x) \cap I_N = I_N$). This means that with the notation $k_0(x) := \max\{k = = 0, ..., N-1 : x < 2^{-k}\}$ we get

$$(ar_n)^{**}(x) = \max_{k\le k_0(x)} |S_{2^k}(ar_n)(x)| = \max_{k\le k_0(x)} 2^k \left|\int\limits_{I_N} ar_n\right| \le$$

$$\le \max_{k\le k_0(x)} 2^k \|a\|_1 \le 2^{k_0(x)} \le \frac{1}{x}.$$

Summarizing the above facts it follows that

$$J_2(a) = \int\limits_{2^{-N}}^{1} (ar_n)^{**} \le \int\limits_{2^{-N}}^{1} \frac{dx}{x} \le C \log_2 2^N = CN \le Cn,$$

which proves Theorem 1. ■

Therefore it can be assumed that $\frac{n+2}{2} \le C_{2^n} \le C(n+1) \quad (n \in \mathbf{N})$. Furthermore, if $n = \sum_{j=0}^{\infty} n_j 2^j$ is the dyadic representation of $n \in \mathbf{N}$, then

$$\|fw_n\| \le \|f\| \prod_{j=0}^{\infty} C_{2^j}^{n_j} \le C^{|n|}[n]\|f\| \qquad (f \in \mathbf{H}),$$

where $|n| := \sum_{j=0}^{\infty} n_j$, and $[n] := \prod_{j=0}^{\infty} (j+1)^{n_j}$, and the above estimation cannot be improved. For example $|2^k| = 1$ and $[2^k] = k+1 \quad (k \in \mathbf{N})$.

Theorem 1 involves the next concept: if $f \in \mathbf{H}$ then let

$$\|f\|_* := \sup_n \|fw_n\|.$$

It follows immediately that $\|.\|_*$ is a norm, $\|.\| \leq \|.\|_*$ but (see the above remarks) $\|.\|_*, \|.\|$ are not equivalent. Moreover, it is not hard to construct $f \in \mathbf{H}$ such that $\|f\|_* = \infty$. Indeed, we take the function given in (3). Then for all $k \in \mathbf{N}$ we get

$$\|fr_{\nu_k}\| \geq |\alpha_k|\|D_{2^{\nu_k}}\| - \left\| \sum_{k \neq n=0}^{\infty} \alpha_n r_{\nu_k} r_{\nu_n} D_{2^{\nu_n}} \right\|.$$

It is clear that all products $r_{\nu_k} r_{\nu_n} D_{2^{\nu_n}} \quad (k \neq n \in \mathbf{N})$ are atoms, which implies

$$\left\| \sum_{k \neq n=0}^{\infty} \alpha_n r_{\nu_k} r_{\nu_n} D_{2^{\nu_n}} \right\| \leq \sum_{n=0}^{\infty} |\alpha_n| = q < \infty.$$

Then

$$\|f\|_* \geq \|fr_{\nu_k}\| \geq |\alpha_k|\|D_{2^{\nu_k}}\| - q = |\alpha_k| \frac{\nu_k + 2}{2} - q \to \infty \qquad (k \to \infty)$$

follows by means of a suitable choice of parameters.

F. Schipp (see [2]) introduced the following norms

$$\|f\|_{*p} := \| \sup_n Q(fw_n)\|_p \ , \ \ \|f\|^{*p} := \left\| \sup_{m,n} |S_{2^m}(fw_n)| \right\|_p$$
$$(f \in L^1, \ 1 \leq p < \infty),$$

and proved the non-trivial equivalence $\|f\|_{*p} \sim \|f\|_p \quad (1 < p < \infty)$. It is clear that these norms are shift invariant, i.e. for all $n \in \mathbf{N}$ the equalities $\|fw_n\|_{*p} = \|f\|_{*p}$, $\|fw_n\|^{*p} = \|f\|^{*p}$ hold. Furthermore, the inequality $\|.\|_* \leq \|.\|^{*1}$ follows immediately. Moreover, for all $k \in \mathbf{N}$ we get

$$\|fw_k\| \leq C\|Q(fw_k)\|_1 \leq C\| \sup_n Q(fw_n)\|_1 = C\|f\|_{*1},$$

i.e. $\|f\|_* \leq C\|f\|_{*1}$ holds, too. Schipp proved for $F := \sum_{n=0}^{\infty} 2^{-n/2} r_{2^n} D_{2^{2^n}}$ that $F \in \mathbf{H}$ but $\|F\|_{*1} = \infty$. (This example is a special case of (3).) Our example above along with $\|.\| \leq \|.\|_* \leq \|.\|^{*1}$ shows also the existence of $f \in \mathbf{H}$ such that $\|f\|^{*1} = \infty$. The question wheter the norm $\|.\|_{*1}$ and the norm $\|.\|^{*1}$ are equivalent or not remains open.

Let us introduce the space $\mathbf{H}_*$ as follows:

$$\mathbf{H}_* := \{f \in H : \|f\|_* < \infty\}.$$

Then $\mathbf{H}_*$ is a proper subspace of $\mathbf{H}$. For all $n, k \in \mathbf{N}$ it is clear that $1 = \|w_n\| = \|w_{k\oplus n}\| = \|w_k w_n\|$, i.e. $\|w_n\|_* = 1$. Thus $w_n \in \mathbf{H}_*$ and therefore every *Walsh* polynomial (finite linear combination of *Walsh* functions) belongs to $\mathbf{H}_*$. Furthermore, if $f \in \mathbf{H}_*$ then

$$\|fw_n\|_* = \sup_k \|fw_n w_k\| = \sup_k \|fw_{n\oplus k}\| = \sup_j \|fw_j\| = \|f\|_*.$$

In other words the norm $\|.\|_*$ is also invariant with respect to multiplication by *Walsh* functions.

Above we remarked that there exists $f \in \mathbf{H}$ such that its *Walsh–Fourier* series diverges in $\|.\|_1$ norm. We show that this result can be sharpened. Namely, the next theorem holds:

Theorem 2. *There exists $f \in \mathbf{H}_*$ with $\|.\|_1$-divergent Walsh–Fourier series.*

Proof. We take the function $f := \sum_{n=0}^{\infty} \alpha_n r_{\nu_n} D_{2^{\nu_n}}$ from (3). It was shown above (see $(*)$) that $q := \sum_{n=0}^{\infty} |\alpha_n| < \infty$ and $\inf_n |\alpha_n|\nu_n > 0$ imply the $\|.\|_1$ divergence of the *Walsh–Fourier* series of f.

To the proof of $f \in H_*$ let $k = \sum_{j=0}^{\infty} k_j 2^j$ be the dyadic representation of $k \in \mathbf{N}$. Then $w_k = \prod_{j=0}^{\infty} r_j^{k_j}$. Taking into account that

$$w_k r_s D_{2^s} = \prod_{j=s}^{\infty} r_j^{k_j} r_s D_{2^s} \qquad (s \in \mathbf{N})$$

is obviously an atom, provided $k_s = 0$ or $k_s = 1$, but there is $j \geq s+1$ such that $k_j = 1$. Let $\mathbf{N}_s$ be the set of such k's. Then $k \in \mathbf{N}^s := \mathbf{N} \setminus \mathbf{N}_s$ iff $k = 2^s + \sum_{j=0}^{s-1} k_j 2^s$, i.e. $\mathbf{N}^s = \mathbf{N} \cap [2^s, 2^{s+1})$. In this case $w_k r_s D_{2^s} = D_{2^s}$.

If $k \notin \bigcup_{n=0}^{\infty} \mathbf{N}^{\nu_n}$, then

$$fw_k = \sum_{n=0}^{\infty} \alpha_n w_k r_{\nu_n} D_{2^{\nu_n}}$$

is an atomic representation of fw_k and so $\|fw_k\| \leq \sum_{n=0}^{\infty} |\alpha_n| = q$.

If $k \in \bigcup_{n=0}^{\infty} \mathbf{N}^{\nu_n}$, then there is a unique $m \in \mathbf{N}$ such that $k \in \mathbf{N}^{\nu_m}$:

$$fw_k = \alpha_m D_{2^{\nu_m}} + \sum_{m \neq n=0}^{\infty} \alpha_n w_k r_{\nu_n} D_{2^{\nu_n}} =: \alpha_m D_{2^{\nu_m}} + f_0.$$

The above observations lead to $\|f_0\| \le \sum_{n=0}^{\infty} |\alpha_n| = q < \infty$ and

$$\|fw_k\| \le |\alpha_m| \|D_{2^{\nu_m}}\| + \|f_0\| \le C|\alpha_m|\nu_m + q.$$

We see that the assumption $\sup_n |\alpha_n|\nu_n < \infty$ is sufficient to

$$\sup_k \|fw_k\| \le C \sup_n |\alpha_n|\nu_n + q < \infty.$$

In this case $f \in H_*$. For example if $\alpha_n := 2^{-n}, \nu_n := 2^n \quad (n \in \mathbf{N})$, then the function $f = \sum_{n=0}^{\infty} 2^{-n} r_{2^{2^n}} D_{2^{2^n}}$ proves Theorem 2. ■

If $f \in \mathbf{H}$ then $Qf \in L^1$, i.e. $Qf = \left(\sum_{k=-1}^{\infty} (\delta_k f)^2\right)^{1/2} < \infty$ a.e. Thus $\left(\sum_{k=n}^{\infty} (\delta_k f)^2\right)^{1/2} \to 0 \quad (n \to \infty)$ a.e. and we get by *Lebesgue*'s theorem that

$$\|f - S_{2^n} f\| \le C\|Q(f - S_{2^n})\|_1 = C \left\| \left(\sum_{k=n}^{\infty} (\delta_k f)^2 \right)^{1/2} \right\|_1 \to 0 \qquad (n \to \infty).$$

However, this last convergence property doesn't hold true if the norm $\|.\|$ will be replaced by $\|.\|_*$. Indeed, taking the function $f \in \mathbf{H}_*$ from the proof of Theorem 2 we get analogously that

$$\|f - S_{2^{\nu_n}} f\|_* = \left\| \sum_{k=n}^{\infty} \alpha_k r_{\nu_k} D_{2^{\nu_k}} \right\|_* \ge C \inf_{k \ge n} |\alpha_k|\nu_k - q \qquad (n \in \mathbf{N}).$$

Let $\alpha_k := 2^{-k}, \nu_k := 2^{k+s} \ (k \in \mathbf{N})$, where $s \in \mathbf{N}$ is defined by $2^s C > 2$. Then $q = \sum_{k=0}^{\infty} |\alpha_k| = 2$ and $\|f - S_{2^{\nu_n}} f\|_* \ge 2^s C - 2 \quad (n \in \mathbf{N})$, i.e. $\|f - S_{2^n} f\|_*$ doesn't tend to zero if $n \to \infty$.

We recall that $\|S_{2^n} f\|_1 \le \|f\|_1 \ (f \in L^1)$, $\|S_{2^n} f\| \le \|f\| \ (f \in \mathbf{H}, n \in \mathbf{N})$. Applying (4) it is not hard to prove that an analogous inequality holds if we replace the norm $\|.\|$ by $\|.\|_*$. Indeed,

$$\|S_{2^n} f\|_* = \sup_k \|w_k S_{2^n} f\| = \max\left\{ \sup_{k<2^n} \|w_k S_{2^n} f\|, \sup_{k \ge 2^n} \|w_k S_{2^n} f\| \right\} \le$$

$$\le \max\left\{ \sup_{k<2^n} \|fw_k\|, C\|S_{2^n} f\|_1 \right\} \le \max\left\{ \sup_k \|fw_k\|, C\|f\|_1 \right\} \le C\|f\|_*.$$

Hence if $f \in L^1$ then

$$\|f\|_* = \sup_n \|(fw_n)^*\|_1 = \sup_n \|\sup_m |S_{2^m}(fw_n)|\|_1.$$

Let $p > 1$ and $f \in L^p$. Then for arbitrary $n \in \mathbf{N}$ we can write

$$\|fw_n\| = \|(fw_n)^*\|_1 \leq \|(fw_n)^*\|_p \leq C_p\|fw_n\|_p = C_p\|f\|_p,$$

i.e. $\|f\|_* \leq C_p\|f\|_p$. Thus $L^p \subset H_*$. In other words $\bigcup_{p>1} L^p \subset \mathbf{H}_*$. We will show that the next statement holds:

Theorem 3. $H_* \setminus \left(\bigcup_{p>1} L^p\right) \neq \emptyset$.

Proof. Let $1 < p < \infty$ and take the function $f = \sum_{n=0}^{\infty} 2^{-n} r_{2^{2^n}} D_{2^{2^n}} =:$ $=: \sum_{n=0}^{\infty} \alpha_n r_{\nu_n} D_{2^{\nu_n}}$ as in the proof of Theorem 2. Then $f \in H_*$. On the other hand

$$\|f\|_p^p \geq C_p\|Qf\|_p^p \geq C_p \left\| \sqrt{\sum_{n=0}^{\infty} \alpha_n^2 D_{2^{\nu_n}}^2} \right\|_p^p \geq C_p \sum_{k=0}^{\infty} \int_{2^{-\nu_k-1}}^{2^{-\nu_k}} \left(\sum_{n=0}^{k} \alpha_n^2 D_{2^{\nu_n}}^2 \right)^{p/2} =$$

$$= C_p \sum_{k=0}^{\infty} 2^{-\nu_k} \left(\sum_{n=0}^{k} \alpha_n^2 2^{2\nu_n} \right)^{p/2} \geq C_p \sum_{k=0}^{\infty} \alpha_k^p 2^{(p-1)\nu_k} = \infty. \blacksquare$$

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