Annales Univ. Sci. Budapest., Sect. Comp. 35 (2011) 217–222

CONTINUOUS MAPS ON MATRICES TRANSFORMING GEOMETRIC MEAN TO ARITHMETIC MEAN

Lajos Molnár (Debrecen, Hungary)

Dedicated to Professor Antal Járai
on the occasion of his sixtieth birthday

Abstract. In this paper we determine the general form of continuous maps between the spaces of all positive definite and all self-adjoint matrices which transform geometric mean to arithmetic mean or the other way round.

In the papers [6, 7] we determined the structure of all bijective maps on the space of all positive semidefinite operators on a complex Hilbert space which preserve the geometric mean, or the harmonic mean, or the arithmetic mean of operators in the sense of Ando [1, 3]. In this short note we consider a related question. The logarithmic function is a continuous function from the set $\mathbb{R}_+$ of all positive real numbers to $\mathbb{R}$ that transforms geometric mean to arithmetic mean. Similarly, the exponential function is a continuous function from $\mathbb{R}$ to $\mathbb{R}_+$ that transforms arithmetic mean to geometric mean. Here we investigate the structure of maps between the spaces of all positive definite and all self-adjoint matrices with the analogous transformation properties.

Let us begin with the necessary definitions. For a given complex Hilbert space H, denote by $\mathcal{S}(H)$ and $\mathcal{P}(H)$ the spaces of all bounded self-adjoint and

1991 Mathematics Subject Classification: Primary 47B49, Secondary 47A64.

Key words and phrases: Positive definite operators and matrices, geometric mean, arithmetic mean, transformers.

The author was supported by the Hungarian Scientific Research Fund (OTKA) K81166 NK81402, and by the TÁMOP 4.2.1./B-09/1/KONV-2010-0007 project implemented through the New Hungary Development Plan co-financed by the European Social Fund and the European Regional Development Fund.

all bounded positive definite (i.e., invertible bounded positive semidefinite) operators on H, respectively. The geometric mean of $A, B \in \mathcal{P}(H)$ in Ando's sense is defined by

$$A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

We remark that Ando defined the geometric mean for all positive semidefinite operators, but in this note we consider only positive definite operators. The most important properties of the operation # are listed below. Let A, B, C, D be positive semidefinite operators on H.

(i) If $A \leq C$ and $B \leq D$, then $A\#B \leq C\#D$.

(ii) (Transfer property) We have $S(A\#B)S^* = (SAS^*)\#(SBS^*)$ for every invertible bounded linear operator S on H.

(iii) Suppose $A_1 \geq A_2 \geq \ldots \geq 0$, $B_1 \geq B_2 \geq \ldots \geq 0$ and $A_n \to A$, $B_n \to B$ strongly. Then we have that $A_n\#B_n \to A\#B$ strongly.

(iv) $A\#B = B\#A$.

The arithmetic mean of $A, B \in \mathcal{S}(H)$ is defined in the natural way, i.e., by $(A+B)/2$. For a finite dimensional Hilbert space H, our first result describes those continuous maps from $\mathcal{P}(H)$ to $\mathcal{S}(H)$ which transform geometric mean to arithmetic mean.

Theorem 1. *Assume* $2 \leq \dim H < \infty$. *Let* $\phi : \mathcal{P}(H) \to \mathcal{S}(H)$ *be a continuous map satisfying*

$$\phi(A\#B) = \frac{\phi(A) + \phi(B)}{2} \tag{1}$$

for all $A, B \in \mathcal{P}(H)$. *Then there are* $J, K \in \mathcal{S}(H)$ *such that* ϕ *is of the form*

$$\phi(A) = (\log(\det A))J + K, \quad A \in \mathcal{P}(H).$$

Proof. Considering the map $\phi(.) - \phi(I)$ we may and do assume that $\phi(I) = 0$. Inserting $B = I$ into the equality (1) we obtain that $\phi(\sqrt{A}) = \phi(A)/2$. Moreover, we compute

$$0 = \phi(I) = \phi(A\#A^{-1}) = (1/2)(\phi(A) + \phi(A^{-1}))$$

which implies $\phi(A^{-1}) = -\phi(A)$ for every $A \in \mathcal{P}(H)$. For any $A, B, T \in \mathcal{P}(H)$, using the uniqueness of the square root in $\mathcal{P}(H)$, it is easy to check that $T = A^{-1}\#B$ holds if and only if $TAT = B$. From

$$\phi(T) = (1/2)(\phi(A^{-1}) + \phi(B)) = (1/2)(\phi(B) - \phi(A))$$

we obtain $\phi(B) = 2\phi(T) + \phi(A)$. Therefore, we have

$$\phi(TAT) = 2\phi(T) + \phi(A)$$

for any $A, T \in \mathcal{P}(H)$. Pick an arbitrary $X \in \mathcal{S}(H)$ and consider the functional $\varphi_X : A \mapsto \exp(\mathrm{tr}[\phi(A)X])$ on $\mathcal{P}(H)$. It is easy to see that $\varphi_X : \mathcal{P}(H) \to \mathbb{R}$ is a continuous function satisfying

$$\varphi_X(TAT) = \varphi_X(T)\varphi_X(A)\varphi_X(A)$$

for all $A, T \in \mathcal{P}(H)$. In [4, Theorem 2] the structure of such functions has been completely described. It follows from that result that there is a real number c_X such that $\varphi_X(A) = (\det A)^{c_X}$ $(A \in \mathcal{P}(H))$. Therefore, we have

$$\mathrm{tr}[\phi(A)X] = c_X \log(\det A)$$

for all $A \in \mathcal{P}(H)$. It follows from that formula that $c_X \in \mathbb{R}$ depends linearly on X, i.e., $X \mapsto c_X$ is a linear functional on $\mathcal{S}(H)$. By Riesz representation theorem it follows that there is a $J \in \mathcal{S}(H)$ such that $c_X = \mathrm{tr}[XJ]$ for every $X \in \mathcal{S}(H)$. Hence we obtain that

$$\mathrm{tr}[\phi(A)X] = c_X \log(\det A) = \mathrm{tr}[\log(\det A))JX]$$

holds for all $A \in \mathcal{P}(H)$ and $X \in \mathcal{S}(H)$. This gives us that

$$\phi(A) = (\log(\det A))J$$

for every $A \in \mathcal{P}(H)$ and the statement of the theorem follows. ■

Remark 1. One may ask what happens in the infinite dimensional case, i.e., when $\dim H = \infty$. The answer to that question is that ϕ is necessarily constant. In order to see this, just as above, applying the simple and apparent reduction $\phi(I) = 0$, one can follow the first part of the proof to check that for every vector $x \in H$, the continuous functional $\varphi_x : A \mapsto \exp(\langle \phi(A)x, x \rangle)$ maps $\mathcal{P}(H)$ into the set of all positive real numbers and satisfies

$$\varphi_x(TAT) = \varphi_x(T)\varphi_x(A)\varphi_x(A)$$

for all $A, T \in \mathcal{P}(H)$. Lemma in [5] states that then φ_x is necessarily identically 1. This gives us that $\langle \phi(A)x, x \rangle = 0$ for all $x \in H$ and $A \in \mathcal{P}(H)$ which implies $\phi \equiv 0$.

In our second result we consider the reverse problem. We describe the form of all continuous maps from $\mathcal{S}(H)$ to $\mathcal{P}(H)$ which transform arithmetic mean to geometric mean.

Theorem 2. *Assume* $2 \leq \dim H < \infty$. *Let* $\phi : \mathcal{S}(H) \to \mathcal{P}(H)$ *be a continuous map satisfying*

$$\phi\left(\frac{A+B}{2}\right) = \phi(A)\#\phi(B) \tag{2}$$

for all $A, B \in \mathcal{S}(H)$. *Then there are a* $T \in \mathcal{P}(H)$, *a collection of mutually orthogonal rank-one projections* P_i *on* H *and a collection of self-adjoint operators* $J_i \in \mathcal{S}(H)$, $i = 1, \ldots, \dim H$ *such that* ϕ *is of the form*

$$\phi(A) = T\left(\sum_{i=1}^{\dim H} (\exp(\mathrm{tr}[AJ_i]))P_i\right)T, \quad A \in \mathcal{S}(H).$$

Proof. Using the transfer property we see that considering the transformation $\phi(0)^{-1/2}\phi(.)\phi(0)^{-1/2}$ we may and hence do assume that $\phi(0) = I$. Inserting $B = 0$ into (2) we obtain $\phi(A/2) = \sqrt{\phi(A)}$. We next have

$$I = \phi(0) = \phi(A)\#\phi(-A).$$

It requires easy computation to deduce from this equality that $\phi(-A) = \phi(A)^{-1}$. Setting $T = (A + (-B))/2$ we infer

$$\begin{aligned}\phi(T) = \phi(-B)\#\phi(A) = \phi(B)^{-1}\#\phi(A) \\ = \phi(B)^{-1/2}(\phi(B)^{1/2}\phi(A)\phi(B)^{1/2})^{1/2}\phi(B)^{-1/2}.\end{aligned}$$

Multiplying both sides by $\phi(B)^{1/2}$ and taking square, we deduce

$$\phi(B)^{1/2}\phi(T)\phi(B)\phi(T)\phi(B)^{1/2} = \phi(B)^{1/2}\phi(A)\phi(B)^{1/2}.$$

Again, multiplying both sides by $\phi(B)^{-1/2}$ we obtain $\phi(T)\phi(B)\phi(T) = \phi(A) = \phi(2T + B)$. It follows that

$$\phi(T)\phi(B)\phi(T) = \phi(2T + B)$$

for every $B, T \in \mathcal{S}(H)$. Since $\phi(T)^{1/2} = \phi(T/2)$, we infer

$$\phi(T)^{1/2}\phi(B)\phi(T)^{1/2} = \phi(T + B) = \phi(B + T) = \phi(B)^{1/2}\phi(T)\phi(B)^{1/2}.$$

We learn from [2, Corollary 3] that for any $C, D \in \mathcal{P}(H)$ we have $C^{1/2}DC^{1/2} = D^{1/2}CD^{1/2}$ if and only if $CD = DC$. Therefore, it follows that the range of ϕ is commutative. Let us now identify the operators in $\mathcal{P}(H)$ with $n \times n$ matrices, where $n = \dim H$. By its commutativity, the range of ϕ is simultaneously diagonisable by some unitary matrix U. Considering the transformation $U^*\phi(.)U$ we may and do assume that $\phi(A) = \mathrm{diag}[\phi_1(A), \ldots, \phi_n(A)]$

($A \in \mathcal{S}(H)$), where ϕ_i maps $\mathcal{S}(H)$ into the set of all positive real numbers and satisfies $\phi_i((A+B)/2) = \sqrt{\phi_i(A)\phi_i(B)}$ for every $A, B \in \mathcal{S}(H)$ and $i = 1, \dots, n$. Using continuity and $\phi(0) = I$, it is easy to see that $\log \phi_i$ is a linear functional on $\mathcal{S}(H)$. Therefore, for every $i = 1, \dots, n$ we have $J_i \in \mathcal{S}(H)$ such that $\log(\phi_i(A)) = \mathrm{tr}[AJ_i]$ implying $\phi_i(A) = \exp(\mathrm{tr}[AJ_i])$ for all $A \in \mathcal{S}(H)$. Consequently, we obtain

$$\phi(A) = \mathrm{diag}[\exp(\mathrm{tr}[AJ_1]), \dots, \exp(\mathrm{tr}[AH_n])]$$

for all $A \in \mathcal{S}(H)$, and the proof can be completed in an easy way. ■

Remark 2. As for the case $\dim H = \infty$, we note that for any $T \in \mathcal{P}(H)$, any collection $P_1, \dots, P_n$ of mutually orthogonal projections with sum I and any collection $J_1, \dots, J_n$ of self-adjoint trace-class operators on H, the formula

$$\phi(A) = T\left(\sum_{i=1}^{n} (\exp(\mathrm{tr}[AJ_i]))P_i\right)T, \quad A \in \mathcal{S}(H) \tag{3}$$

defines a continuous map from $\mathcal{S}(H)$ to $\mathcal{P}(H)$ which transforms arithmetic mean to geometric mean. With some more effort and refining the continuity assumption on the transformations, one could obtain a result which would show that a "continuous analogue" of the formula (3) (i.e., with integral in the place of the sum) describes the general form of continuous maps from $\mathcal{S}(H)$ to $\mathcal{P}(H)$ that transform arithmetic mean to geometric mean. However, we do not present the precise details here.

References

[1] **Ando, T.,** *Topics on Operator Inequalities,* Mimeographed lecture notes, Hokkaido University, Sapporo, 1978.

[2] **Gudder, S. and G. Nagy,** Sequentially independent effects, *Proc. Amer. Math. Soc.,* **130** (2002), 1125–1130.

[3] **Kubo, F. and T. Ando,** Means of positive linear operators, *Math. Ann.,* **246** (1980), 205–224.

[4] **Molnár, L.,** A remark to the Kochen-Specker theorem and some characterizations of the determinant on sets of Hermitian matrices, *Proc. Amer. Math. Soc.,* **134** (2006), 2839–2848.

[5] **Molnár, L.,** Non-linear Jordan triple automorphisms of sets of self-adjoint matrices and operators, *Studia Math.,* **173** (2006), 39–48.

[6] **Molnár, L.,** Maps preserving the geometric mean of positive operators, *Proc. Amer. Math. Soc.,* **137** (2009), 1763–1770.

[7] **Molnár, L.,** Maps preserving the harmonic mean or the parallel sum of positive operators, *Linear Algebra Appl.,* **430** (2009), 3058–3065.

L. Molnár
Institute of Mathematics
University of Debrecen
P.O. Box 12
H-4010 Debrecen
Hungary
molnarl@science.unideb.hu
http://www.math.unideb.hu/~molnarl/