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FUNCTIONAL EQUATIONS RELATED TO HOMOGRAPHIC FUNCTIONS

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Dedicated to the sixtieth birthday of Professor Antal Járai

Abstract. A functional equation in two variables related to homographic functions is introduced. The solutions are established with the aid of some results on functional equations in a single variable. A conjecture on a general solution is presented.

1. Introduction

We consider the functional equation

$$\frac{\alpha\left(\frac{3x+y}{4}\right)-\alpha(x)}{\alpha\left(\frac{x+y}{2}\right)-\alpha(x)}\left(3-2\frac{\alpha\left(\frac{x+y}{2}\right)-\alpha(x)}{\alpha(y)-\alpha(x)}\right)=1,$$

in two variables where the unknown function α is continuous and strictly monotonic in a real interval. It is easy to verify that any homographic function is a solution. In section 2 we present some motivation. In section 3 we show that this equation is a consequence of a more complicated functional equation in three variables $(*)$ appearing in connection with the problem of existence of discontinuous Jensen affine functions in the sense of Beckenbach with respect

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to the two parameter family of functions $\{b\alpha + c : b, c \in \mathbb{R}\}$, and related to the invariance of double ratios of four points.

In section 4, applying an M. Laczkovich theorem [4], we prove that if a continuous function satisfies this equation in any interval (a_0, ∞) then it is a homographic function.

In section 5, assuming some local regularity conditions, we consider some related functional equations in a single variable. A possible application of the celebrated regularity theorems of A. Járai [1] is mentioned.

2. Some motivations

In order to present a problem leading to the considered equation, take a continuous and strictly monotonic function α defined on an interval I and consider a two parameter family of functions defined by

$$\mathcal{F}_\alpha := \{b\alpha + c : a, b \in \mathbb{R}\}.$$

The family $\mathcal{F}_\alpha$ has the property: for every two points $(x_1, y_1), (x_2, y_2) \in I \times \mathbb{R}$, $x_1 \neq x_2$, there is a unique function $b\alpha + c$ in $\mathcal{F}_\alpha$ such that

$$b\alpha(x_1) + c = y_1, \qquad b\alpha(x_2) + c = y_2;$$

more precisely, the real numbers

$$b = \frac{y_1 - y_2}{\alpha(x_1) - \alpha(x_2)}, \qquad c = \frac{\alpha(x_1)y_2 - \alpha(x_2)y_1}{\alpha(x_1) - \alpha(x_2)}$$

are uniquely determined. Following a more general idea due to Beckenbach, we say that a function $f : I \to \mathbb{R}$ is *convex with respect the family* $\mathcal{F}_\alpha$, briefly, $\mathcal{F}_\alpha$*-convex*, if for all $x_1, x_2 \in I$, $x_1 < x_2$, we have

$$f(x) \leq b\alpha(x) + c, \qquad x_1 < x < x_2,$$

where

$$b = \frac{f(x_1) - f(x_2)}{\alpha(x_1) - \alpha(x_2)}, \qquad c = \frac{\alpha(x_1)f(x_2) - \alpha(x_2)f(x_1)}{\alpha(x_1) - \alpha(x_2)},$$

$\mathcal{F}_\alpha$*-concave*, if the reversed inequality is satisfied, and $\mathcal{F}_\alpha$*-affine* if it is both $\mathcal{F}_\alpha$-convex and $\mathcal{F}_\alpha$-concave.

Note that a function f is $\mathcal{F}_\alpha$-affine iff $f \in \mathcal{F}_\alpha$.

Adopting the idea of Jensen, we say that a function $f : I \to \mathbb{R}$ is *Jensen $\mathcal{F}_\alpha$-convex* if, for all $x_1, x_2 \in I$,

$$f\left(\frac{x_1+x_2}{2}\right) \le b\alpha\left(\frac{x_1+x_2}{2}\right) + c,$$

where b and c are given by the above formula; *Jensen $\mathcal{F}_\alpha$-concave* if the reverse inequality is satisfied, and *Jensen $\mathcal{F}_\alpha$-affine* if it is both *Jensen $\mathcal{F}_\alpha$-convex* and *Jensen $\mathcal{F}_\alpha$-concave,* that is if, for all $x_1, x_2 \in I$,

$$f\left(\frac{x_1+x_2}{2}\right) = \frac{f(x_1)-f(x_2)}{\alpha(x_1)-\alpha(x_2)}\alpha\left(\frac{x_1+x_2}{2}\right) + \frac{\alpha(x_1)f(x_2)-\alpha(x_2)f(x_1)}{\alpha(x_1)-\alpha(x_2)}$$

or, equivalently

$$f\left(\frac{x_1+x_2}{2}\right) = \frac{\alpha\left(\frac{x_1+x_2}{2}\right)-\alpha(x_2)}{\alpha(x_1)-\alpha(x_2)}f(x_1) + \frac{\alpha(x_1)-\alpha\left(\frac{x_1+x_2}{2}\right)}{\alpha(x_1)-\alpha(x_2)}f(x_2).$$

For $\alpha := \mathrm{id}|_I$ one gets the classical notions of convex, concave, affine and Jensen convex, Jensen concave and Jensen affine functions. It is known since Hamel that there are discontinuous Jensen affine functions and that every Jensen affine function $f : I \to \mathbb{R}$ is of the form $f(x) = A(x)+a$, $x \in I$, where A is an additive function and $a \in \mathbb{R}$ which, in general, does not belong to $\mathcal{F}_\alpha$. In this context a natural question arises: determine all functions $\alpha : I \to \mathbb{R}$ which admit the discontinuous Jensen $\mathcal{F}_\alpha$-affine functions.

In [7] it was shown that this problem leads to the following, quite complicated, functional equation of three variables

$$(*)\qquad \begin{aligned} &\frac{\alpha\left(\frac{x+2y+z}{4}\right)-\alpha(y)}{\alpha\left(\frac{x+z}{2}\right)-\alpha(y)}\cdot\frac{\alpha\left(\frac{x+z}{2}\right)-\alpha(z)}{\alpha(x)-\alpha(z)} = \\ &= \frac{\alpha\left(\frac{x+2y+z}{4}\right)-\alpha\left(\frac{y+z}{2}\right)}{\alpha\left(\frac{x+y}{2}\right)-\alpha\left(\frac{y+z}{2}\right)}\cdot\frac{\alpha\left(\frac{x+y}{2}\right)-\alpha(y)}{\alpha(x)-\alpha(y)}\end{aligned}$$

for all $x, y, z \in I$, $(x+z-2y)(x-z)(x-y) \neq 0$.

Note that this equation can be written as the equality of the following two double ratios:

$$\begin{aligned} &\frac{\alpha\left(\frac{x+2y+z}{4}\right)-\alpha(y)}{\alpha\left(\frac{x+z}{2}\right)-\alpha(y)} : \frac{\alpha\left(\frac{x+2y+z}{4}\right)-\alpha\left(\frac{y+z}{2}\right)}{\alpha\left(\frac{x+y}{2}\right)-\alpha\left(\frac{y+z}{2}\right)} = \\ &= \frac{\alpha\left(\frac{x+y}{2}\right)-\alpha(y)}{\alpha(x)-\alpha(y)} : \frac{\alpha\left(\frac{x+z}{2}\right)-\alpha(z)}{\alpha(x)-\alpha(z)}.\end{aligned}$$

Taking into account that for all admissible $x, y, z \in I$,

$$\frac{\frac{x+2y+z}{4} - y}{\frac{x+z}{2} - y} : \frac{\frac{x+2y+z}{4} - \frac{y+z}{2}}{\frac{x+y}{2} - \frac{y+z}{2}} = \frac{1}{4} = \frac{\frac{x+y}{2} - y}{x-y} : \frac{\frac{x+z}{2} - z}{x-z}$$

we conclude that any homographic function α satisfies equation $(*)$.

In [7] it was proved that a continuous and monotonic function satisfies $(*)$ if, and only if α is any homographic function. This fact implies that a family $\mathcal{F}_\alpha$ admits discontinuous Jensen affine functions in the Beckenbach sense iff α is a homographic function. In [7], as an application, an answer to a more general question posed by Zs. Páles [8] is given.

3. A functional equation related to equation (∗)

We prove the following

Theorem 1. *Let $I \subset \mathbb{R}$ be an interval. If a continuous function $\alpha : I \to \mathbb{R}$ satisfies equation $(*)$, then it is strictly monotonic and*

$$(1) \qquad \frac{\alpha\left(\frac{3x+y}{4}\right) - \alpha(x)}{\alpha\left(\frac{x+y}{2}\right) - \alpha(x)} \left(3 - 2\frac{\alpha\left(\frac{x+y}{2}\right) - \alpha(x)}{\alpha(y) - \alpha(x)}\right) = 1, \qquad x, y \in I,\ x \neq y.$$

Proof. Equation $(*)$ implies that α is one-to-one. The continuity of α implies that it is strictly monotonic. By the continuity of α, letting $x \to y$ in $(*)$, we infer that, for every $y \in I$, the limit

$$(2) \qquad \varphi(y) := \lim_{x \to y} \frac{\alpha\left(\frac{x+y}{2}\right) - \alpha(y)}{\alpha(x) - \alpha(y)}$$

exists and, for all $y \neq z$,

$$(3) \qquad \frac{\alpha\left(\frac{3y+z}{4}\right) - \alpha(y)}{\alpha\left(\frac{y+z}{2}\right) - \alpha(y)} \frac{\alpha\left(\frac{y+z}{2}\right) - \alpha(z)}{\alpha(y) - \alpha(z)} = \frac{\alpha\left(\frac{3y+z}{4}\right) - \alpha\left(\frac{y+z}{2}\right)}{\alpha(y) - \alpha\left(\frac{y+z}{2}\right)} \varphi(y).$$

Similarly, letting $y \to x$ in $(*)$, we infer that, for every $x \in I$, the limit

$$(4) \qquad \psi(x) := \lim_{y \to x} \frac{\alpha\left(\frac{x+y}{2}\right) - \alpha(y)}{\alpha(x) - \alpha(y)}$$

exists and, for all $x \neq z$,

$$\frac{\alpha\left(\frac{3x+z}{4}\right)-\alpha\left(x\right)}{\alpha\left(\frac{x+z}{2}\right)-\alpha\left(x\right)}\frac{\alpha\left(\frac{x+z}{2}\right)-\alpha\left(z\right)}{\alpha\left(x\right)-\alpha\left(z\right)}=\frac{\alpha\left(\frac{3x+z}{4}\right)-\alpha\left(\frac{x+z}{2}\right)}{\alpha\left(x\right)-\alpha\left(\frac{x+z}{2}\right)}\psi(x).$$

Thus

$$\psi=\varphi. \tag{5}$$

Hence, letting $x \to z$ in $(*)$, making use of the definitions of φ and ψ and the identity

$$\alpha\left(\frac{x+2y+z}{4}\right)=\alpha\left(\frac{\frac{x+y}{2}+\frac{y+z}{2}}{2}\right)$$

we get

$$\frac{\alpha\left(\frac{y+z}{2}\right)-\alpha\left(y\right)}{\alpha\left(z\right)-\alpha\left(y\right)}\varphi(z)=\varphi\left(\frac{y+z}{2}\right)\frac{\alpha\left(\frac{z+y}{2}\right)-\alpha\left(y\right)}{\alpha\left(z\right)-\alpha\left(y\right)},$$

for $y \neq z$, whence

$$\varphi(z)=\varphi\left(\frac{y+z}{2}\right), \qquad y\neq z,$$

and, consequently, φ is a constant function in I.

Letting $x \to y$ in the identity

$$\frac{\alpha\left(\frac{x+y}{2}\right)-\alpha\left(y\right)}{\alpha\left(x\right)-\alpha\left(y\right)}+\frac{\alpha(x)-\alpha\left(\frac{x+y}{2}\right)}{\alpha\left(x\right)-\alpha\left(y\right)}=1$$

and making use of (2), (4) we get $\varphi+\psi=1$, whence by (5),

$$\varphi=\frac{1}{2}.$$

Now, from (3), we get

$$\frac{\alpha\left(\frac{3y+z}{4}\right)-\alpha\left(y\right)}{\alpha\left(\frac{y+z}{2}\right)-\alpha\left(y\right)}\frac{\alpha\left(\frac{y+z}{2}\right)-\alpha\left(z\right)}{\alpha\left(y\right)-\alpha\left(z\right)}=\frac{1}{2}\frac{\alpha\left(\frac{3y+z}{4}\right)-\alpha\left(\frac{y+z}{2}\right)}{\alpha\left(y\right)-\alpha\left(\frac{y+z}{2}\right)}$$

for $y \neq z$. Since

$$\frac{\alpha\left(\frac{3y+z}{4}\right)-\alpha\left(\frac{y+z}{2}\right)}{\alpha\left(y\right)-\alpha\left(\frac{y+z}{2}\right)}=1-\frac{\alpha\left(\frac{3y+z}{4}\right)-\alpha\left(y\right)}{\alpha\left(\frac{y+z}{2}\right)-\alpha(y)}$$

we get

$$\frac{\alpha\left(\frac{3y+z}{4}\right)-\alpha\left(y\right)}{\alpha\left(\frac{y+z}{2}\right)-\alpha\left(y\right)}\frac{\alpha\left(\frac{y+z}{2}\right)-\alpha\left(z\right)}{\alpha\left(y\right)-\alpha\left(z\right)}=\frac{1}{2}\left(1-\frac{\alpha\left(\frac{3y+z}{4}\right)-\alpha\left(y\right)}{\alpha\left(\frac{y+z}{2}\right)-\alpha(y)}\right)$$

that is, for $y \neq z$,

$$\frac{\alpha\left(\frac{3y+z}{4}\right)-\alpha(y)}{\alpha\left(\frac{y+z}{2}\right)-\alpha(y)}\left(\frac{\alpha\left(\frac{y+z}{2}\right)-\alpha(z)}{\alpha(y)-\alpha(z)}+\frac{1}{2}\right)=\frac{1}{2}.$$

Since

$$\frac{\alpha\left(\frac{y+z}{2}\right)-\alpha(z)}{\alpha(y)-\alpha(z)}=1-\frac{\alpha(y)-\alpha\left(\frac{y+z}{2}\right)}{\alpha(y)-\alpha(z)}$$

we get, for all $y, z \in I$, $y \neq z$,

$$\frac{\alpha\left(\frac{3y+z}{4}\right)-\alpha(y)}{\alpha\left(\frac{y+z}{2}\right)-\alpha(y)}\left(\frac{3}{2}-\frac{\alpha\left(\frac{y+z}{2}\right)-\alpha(y)}{\alpha(z)-\alpha(y)}\right)=\frac{1}{2}\,,$$

which was to be shown. ■

Remark 1. Let $A : \mathbb{R} \to \mathbb{R}$ be an arbitrary additive function and $a, b, c, d \in \in \mathbb{R}$ be such that $ad - bc \neq 0$. Then it is easy to check that the function α given by

$$\alpha(x) := \frac{aA(x)+b}{cA(x)+d}$$

is a solution of equation (1) (as well as of equation $(*)$).

Remark 2. A function $\alpha : I \to \mathbb{R}$ satisfies equation (1) iff so does the function $h \circ \alpha$, where h is an arbitrary nonconstant homographic function.

Remark 3. Let $k, m, p, q \in \mathbb{R}$, $kp \neq 0$ be arbitrarily fixed. A function $\alpha : I \to \mathbb{R}$ satisfies equation (1) iff the function $\beta(x) = k\alpha(px+q)+m$ satisfies equation (1) with α replaced by β and the interval I replaced by $J := \{x \in \mathbb{R} : px+q \in I\}$.

Remark 4. Interchanging x and y in (1) and then eliminating $\alpha\left(\frac{y+z}{2}\right)$ from both equations we obtain the functional equation

$$[\alpha(x)-\alpha(x)]\left[\alpha\left(\frac{3x+y}{4}\right)-\alpha\left(\frac{x+3y}{4}\right)\right]=$$

$$=8\left[\alpha(y)-\alpha\left(\frac{3x+y}{4}\right)\right]\left[\alpha\left(\frac{3x+y}{4}\right)-\alpha(x)\right],$$

$$x, y \in I,$$

which can be written in the form

$$8\alpha(x)\,\alpha(y)+\alpha(x)\,\alpha\left(\frac{3x+y}{4}\right)+\alpha(y)\,\alpha\left(\frac{x+3y}{4}\right)+$$

$$+8\alpha\left(\frac{3x+y}{4}\right)\alpha\left(\frac{x+3y}{4}\right)=9\alpha(x)\,\alpha\left(\frac{x+3y}{4}\right)+9\alpha(y)\,\alpha\left(\frac{3x+y}{4}\right),$$

whence

$$\frac{8\alpha(x)\,\alpha(y)}{\alpha\left(\frac{x+3y}{4}\right)\alpha\left(\frac{3x+y}{4}\right)}+\frac{\alpha(x)-9\alpha(y)}{\alpha\left(\frac{x+3y}{4}\right)}+\frac{\alpha(y)-9\alpha(x)}{\alpha\left(\frac{3x+y}{4}\right)}+8=0.$$

Remark 5. Interchanging x and y in (1) we obtain the simultaneous system of functional equations

$$\alpha\left(\frac{3x+y}{4}\right)=\frac{\alpha\left(\frac{x+y}{2}\right)[3\alpha(x)-\alpha(y)]-2\alpha(x)\,\alpha(y)}{2\alpha\left(\frac{x+y}{2}\right)+\alpha(x)-3\alpha(y)}$$

$$\alpha\left(\frac{x+3y}{4}\right)=\frac{\alpha\left(\frac{x+y}{2}\right)[3\alpha(y)-\alpha(x)]-2\alpha(x)\,\alpha(y)}{2\alpha\left(\frac{x+y}{2}\right)+\alpha(y)-3\alpha(x)},$$

which can be iterated.

4. Main result

In this section we need the following result which is a special case of M. Laczkovich theorem [4].

Lemma 1. (M. Laczkovich [4]) *Let p, q, A, B be positive and such that $\frac{\log p}{\log q}$ is irrational. If λ_1, λ_2 are the roots of the equation*

$$Ap^\lambda + Bq^\lambda = 1$$

then every nonnegative measurable solution $f : (0,\infty) \to (0,\infty)$ of the functional equation

$$f(x) = Af(px) + Bf(qx), \qquad x > 0,$$

is of the form

$$f(x) = rx^{\lambda_1} + sx^{\lambda_2}, \qquad x > 0.$$

Remark 6. If $A + B = 1$ then the condition of positivity of the solution can be replaced by a weaker condition of the boundedness below.

Lemma 2. *Let p, A be positive numbers and $p < 1$. If for some $\delta > 0$, a function $f : (0,\infty) \to \mathbb{R}$ is strictly increasing and positive in an interval $(0,\delta)$ and satisfies the functional equation*

$$f(x) = (1 + A)f(px) - Af(p^2x), \qquad x > 0,$$

then f is positive in $(0,\infty)$.

Proof. Suppose that f satisfies the assumptions of the lemma. Putting $\varphi(x) := f(x) - f(px)$ for $x > 0$ we get

$$\varphi(x) = f(x) - f(px) = A\left[f(px) - f(p^2x)\right] = A\varphi(px),$$

whence, by induction,

$$\varphi(x) = A^n\varphi(p^nx), \qquad n \in \mathbb{N}, \ \ x > 0.$$

Take an arbitrary $x > 0$. Since $p < 1$, there is an $n_0 \in \mathbb{N}$ such that $p^nx \in (0,\delta)$ for all $n \in \mathbb{N}$, $n \geq n_0$. Since f is increasing in $(0,\delta)$, we get

$$\varphi(x) = A^n\varphi(p^nx) = A^n\left[f(p^nx) - f(p^{n+1}x)\right] > 0, \qquad n \geq n_0,$$

whence

$$\varphi(x) > 0, \qquad x > 0,$$

and, consequently,

$$f(x) > f(px), \qquad x > 0.$$

Hence, by induction,

$$f(x) > f(p^nx), \qquad x > 0, \ \ n \in \mathbb{N}.$$

Since f is strictly increasing and positive in $(0,\delta)$, letting $n \to \infty$ we get $f(x) > 0$ for all $x > 0$ which was to be shown. ■

The main result reads as follows.

Theorem 2. *Let $a_0 \in \mathbb{R}$ be fixed. A continuous function $\alpha : (a_0,\infty) \to \mathbb{R}$ satisfies equation* (1) *if and only if, α is a homographic function, i.e.*

$$\alpha(x) = \frac{ax+b}{cx+d}, \qquad x > a_0,$$

for some $a, b, c, d \in \mathbb{R}$, $ad \neq bc$.

Proof. Suppose that a continuous function $\alpha : (a_0,\infty) \to \mathbb{R}$ satisfies equation (1). By (1) it must be strictly monotonic in (a_0,∞). Without loss of generality we can assume that α is strictly increasing. Take arbitrary $x_0 > 0$ and define $\beta : (0,\infty) \to \mathbb{R}$, $\beta(x) := \alpha(x+x_0) - \alpha(x_0)$. Of course β is continuous, strictly increasing, $\beta(0) = 0$ and, by Remarks 2 and 3, β satisfies equation (1), that is

$$\frac{\beta\left(\frac{3x+y}{4}\right) - \beta(x)}{\beta\left(\frac{x+y}{2}\right) - \beta(x)}\left(3 - 2\frac{\beta\left(\frac{x+y}{2}\right) - \beta(x)}{\beta(y) - \beta(x)}\right) = 1, \qquad x, y > 0, \ x \neq y.$$

Setting $y = 0$ we get

$$\frac{\beta\left(\frac{3x}{4}\right) - \beta(x)}{\beta\left(\frac{x}{2}\right) - \beta(x)} \left(3 - 2\frac{\beta\left(\frac{x}{2}\right) - \beta(x)}{-\beta(x)}\right) = 1, \qquad x > 0,$$

which, after simple calculation, can be written in the equivalent form

$$\frac{3}{\beta\left(\frac{3x}{4}\right)} = \frac{1}{\beta\left(\frac{x}{2}\right)} + \frac{2}{\beta(x)}, \qquad x > 0.$$

It follows that the function $f : (0, \infty) \to (0, \infty)$,

$$f(x) := \frac{1}{\beta(x)}, \qquad x > 0,$$

is decreasing and satisfies the functional equation

$$f(x) = \tfrac{1}{3} f\left(\tfrac{2}{3}x\right) + \tfrac{2}{3} f\left(\tfrac{4}{3}x\right), \qquad x > 0.$$

Put $p = \frac{2}{3}$, $q = \frac{4}{3}$, $A = \frac{1}{3}$, $B = \frac{2}{3}$. Note that $\frac{\log p}{\log q}$ is irrational and the only solutions of the equation $Ap^\lambda + Bq^\lambda = 1$, that is

$$\frac{1}{3}\left(\frac{2}{3}\right)^\lambda + \frac{2}{3}\left(\frac{4}{3}\right)^\lambda = 1$$

are the numbers $\lambda_1 = 0$ and $\lambda_2 = -1$. By Lemma 1 there are $r, s \in \mathbb{R}$, such that

$$f(x) = rx^0 + sx^{-1} = r + \frac{s}{x}, \qquad x > 0.$$

Thus, by the definition of f,

$$\beta(x) = \frac{1}{f(x)} = \frac{x}{rx + s}, \qquad x > 0,$$

where, obviously, $s \neq 0$. Now the definition of β implies that

$$\alpha(x + x_0) = \alpha(x_0) + \frac{x}{rx + s}, \qquad x > 0.$$

It follows that α is a homographic function in the interval (x_0, ∞), i.e.

$$\alpha(x) = \frac{ax + b}{cx + d}, \qquad x > x_0,$$

for some $a, b, c, d \in \mathbb{R}$, $ad \neq bc$. Since $x_0 > a_0$ is arbitrarily chosen, the proof is completed. ■

5. Some related functional equations

Assume that a one-to-one function α satisfies equation (1) in the interval I. Take an $x_0 \in I$ and define a function β by

$$\beta(x) = \alpha(x + x_0) - \alpha(x_0), \qquad x \in J := I - x_0. \tag{6}$$

In view of Remark 3 the function β satisfies equation (1) in the interval J, i.e.

$$\frac{\beta\left(\frac{3x+y}{4}\right) - \beta(x)}{\beta\left(\frac{x+y}{2}\right) - \beta(x)} \left(3 - 2\frac{\beta\left(\frac{x+y}{2}\right) - \beta(x)}{\beta(y) - \beta(x)}\right) = 1, \qquad x, y \in J,\ x \neq y. \tag{7}$$

Since $\beta(0) = 0$, setting here $x = 0$ and then replacing y by x we get

$$\frac{\beta\left(\frac{x}{4}\right)}{\beta\left(\frac{x}{2}\right)} \left(3 - 2\frac{\beta\left(\frac{x}{2}\right)}{\beta(x)}\right) = 1, \qquad x \in J,\ x \neq 0.$$

It follows that $\varphi : J \to \mathbb{R}$ defined by

$$\varphi(x) := \frac{\beta\left(\frac{x}{2}\right)}{\beta(x)}\ , \qquad x \neq 0, \tag{8}$$

satisfies the functional equation

$$\varphi\left(\frac{x}{2}\right)[3 - 2\varphi(x)] = 1, \qquad x \in J,\ x \neq 0.$$

If the limit $\eta := \lim_{x\to 0} \varphi(x)$ exists then, obviously, $\eta \neq 0$. Setting $\varphi(0) := \eta$, we see that φ satisfies the functional equation

$$\varphi(x) = \frac{3}{2} - \frac{1}{2\varphi\left(\frac{x}{2}\right)}, \qquad x \in J. \tag{9}$$

Theorem 3. *Let $J \subset \mathbb{R}$ be an interval such that $0 \in J$. Suppose that $\varphi : J \to \mathbb{R}$ satisfies equation* (9)*. Then either $\varphi(0) = 1$ or $\varphi(0) = \frac{1}{2}$. Moreover,*

1. *if $\varphi(0) = 1$ and*
$$\varphi(x) = 1 + 0(x), \qquad x \to 0,$$
then φ satisfies (9) *iff $\varphi \equiv 1$ in J;*

2. *if $\varphi(0) = \frac{1}{2}$ and, for some $p \in \mathbb{R}$,*
$$\varphi(x) = \frac{1}{2} + px + 0(x^2), \qquad x \to 0,$$

then φ satisfies (9) *iff*

$$\varphi(x) = \frac{4px+1}{4px+2}, \qquad x \in J. \tag{10}$$

Proof. Setting $x = 0$ in (9) we get $\eta = \frac{3}{2} - \frac{1}{2\eta}$ for $\eta := \varphi(0)$, whence either $\eta = 1$ or $\eta = \frac{1}{2}$.

Putting $f(x) = \frac{x}{2}$ for $x \in J$ and $H(y) := \frac{3}{2} - \frac{1}{2y}$ for all $y \in \mathbb{R}$ we can write equation (9) in the form

$$\varphi(x) = H(\varphi[f(x)]), \qquad x \in J.$$

In the case when $\eta = 1$ we have $H'(\eta) = \frac{1}{2}$, whence, by the continuity of H' at the point $\eta = 1$ we infer that there exists a $\theta \in [\frac{1}{2}, 1)$ and $\delta > 0$ such that

$$|H(y_1) - H(y_2)| \leq \theta\,|y_1 - y_2| \tag{11}$$

for all $y \in (\eta - \delta, \eta + \delta)$. Since $0 \leq f(x) \leq sx$ for all $x \in J$ with $s = \frac{1}{2}$ and $s\theta < 1$, by applying a general uniqueness theorem [5, Theorem 1] (cf. also [4], p. 200-201), we conclude that there exists at most one continuous solution φ such that $\varphi(0) = 1$. Since the constant function $\varphi \equiv 1$ satisfies equation (9), the first part of the theorem is proved.

In the case when $\eta = \frac{1}{2}$ we have $H'(\eta) = 2$. By the continuity of H' there exists $\theta \in [2, 4)$ and $\delta > 0$ such that (11) is fulfilled for all $y \in (\eta - \delta, \eta + \delta)$ and $s^2\theta = \frac{1}{2} < 1$. Since the function (10) is a solution of (9) and

$$\varphi(x) = \frac{1}{2} + px - \frac{4px^2}{4px+1} = \frac{1}{2} + px + 0(x^2), \qquad x \to 0,$$

the uniqueness of φ follows from the already cited theorem in [5]. This completes the proof. ■

Now applying this result we prove

Theorem 4. *Let $I \subset \mathbb{R}$ be an interval. Suppose that the function $\alpha : I \to \mathbb{R}$ satisfies equation* (1). *If for some $x_0 \in I$ there exists the limit*

$$\eta := \lim_{x \to 0} \frac{\alpha\left(\frac{x}{2} + x_0\right) - \alpha(x_0)}{\alpha\left(x + x_0\right) - \alpha(x_0)},$$

then $\eta = \frac{1}{2}$. If moreover, for some $p \in \mathbb{R}$,

$$\frac{\alpha\left(\frac{x}{2} + x_0\right) - \alpha(x_0)}{\alpha\left(x + x_0\right) - \alpha(x_0)} = \frac{1}{2} + px + 0(x^2), \qquad x \to 0,$$

and α is continuous at least at one point $x_1 \in I$, $x_1 \neq x_0$, then

$$\alpha(x) = \frac{ax+b}{cx+d}, \qquad x \in I,$$

for some $a, b, c, d \in \mathbb{R}$, $ad \neq bc$.

Proof. Suppose that $\alpha : I \to \mathbb{R}$ satisfies equation (1). Take an $x_0 \in I$, put $J := I - x_0$ and define the function $\beta : J \to \mathbb{R}$ by (6). By Remark 3, β satisfies equation (7). According to what we have observed at the beginning of this section, the function φ defined by (8) satisfies equation (9) and

$$\varphi(x) := \frac{\alpha\left(\frac{x}{2} + x_0\right) - \alpha(x_0)}{\alpha\left(x + x_0\right) - \alpha(x_0)} \qquad x \in J.$$

By the first statement of Theorem 3 either $\eta = 1$ or $\eta = \frac{1}{2}$. Assume first that $\eta = 1$. Then

$$\frac{\beta\left(\frac{x}{2}\right)}{\beta\left(x\right)} = 1, \qquad x \in J,$$

would imply that β and, consequently α, would be constant function. This is a contradiction, as every function satisfying (1) must be one-to-one.

Consider the case when $\eta = \frac{1}{2}$. Now from Theorem 3 we get

$$\frac{\beta\left(\frac{x}{2}\right)}{\beta\left(x\right)} = \frac{4px+1}{4px+2}, \qquad x \in J,$$

or equivalently, setting $q := 4p$,

$$\beta\left(\frac{x}{2}\right) = \frac{qx+1}{qx+2}\beta\left(x\right), \qquad x \in J, \tag{12}$$

for some $q \in \mathbb{R}$, $q \neq 0$, which can be written in the form

$$\left(\frac{x}{2} + 1\right)\beta\left(\frac{x}{2}\right) = \frac{1}{2}\left(x+1\right)\beta\left(x\right), \qquad x \in J. \tag{13}$$

Setting $y = 0$ in (1) we get

$$\frac{\beta\left(\frac{3x}{4}\right) - \beta\left(x\right)}{\beta\left(\frac{x}{2}\right) - \beta\left(x\right)}\left(3 + 2\frac{\beta\left(\frac{x}{2}\right) - \beta\left(x\right)}{\beta\left(x\right)}\right) = 1, \qquad x \in J,\ x \neq 0.$$

Applying here (12) we obtain

$$\frac{\beta\left(\frac{3x}{4}\right) - \beta\left(x\right)}{\frac{qx+1}{qx+2}\beta\left(x\right) - \beta\left(x\right)}\left(3 + 2\frac{\frac{qx+1}{qx+2}\beta\left(x\right) - \beta\left(x\right)}{\beta\left(x\right)}\right) = 1, \qquad x \in J,\ x \neq 0,$$

which reduces to the equation

$$\left(q\frac{3}{4}x+1\right)\beta\left(\frac{3}{4}x\right)=\frac{3}{4}\left(qx+1\right)\beta\left(x\right), \qquad x\in J. \tag{14}$$

By (13) and (14) the function $\gamma : J \to \mathbb{R}$ defined by

$$\gamma(x)=\left(qx+1\right)\beta\left(x\right), \qquad x\in J,$$

the simultaneous system of functional equations

$$\gamma\left(\frac{x}{2}\right)=\frac{1}{2}\gamma\left(x\right), \qquad \gamma\left(\frac{3}{4}x\right)=\frac{3}{4}\gamma\left(x\right), \qquad x\in J.$$

It is easy to show (taking into account that $\gamma(0)=0$), that the function γ can be uniquely extended to the function satisfying this system of equations, respectively in $[0,\infty)$ or $(-\infty,0]$ or in $\mathbb{R}$ depending on whether x_0 is the left end point of I, the right endpoint of I or the interior point of I. Assume for instance that x_0 is the left end point of I and, for convenience, denote this extension by γ. Since $(\log\frac{1}{2})/(\log\frac{3}{4})$ is irrational and γ is continuous at a point in the interval $(0,\infty)$, we infer that (cf. [6]),

$$\gamma(x)=\gamma(1)x, \qquad x\geq 0.$$

By the definition of γ we get

$$\beta(x)=\frac{\gamma(1)x}{qx+1}, \qquad x\in J,$$

whence, by the definition of β we get the result. In the case when x_0 is the right end point of I the argument is analogous. In the case when x_0 is an interior point of I, then, according to the previous cases, α must be a homographic function at least at one of the intervals $I\cap[x_0,\infty)$ and $I\cap(-\infty,x_0]$. In this case equation (1) easily implies that α is a homographic function in the interval I. This completes the proof. ■

For the discussion the question if the regularity conditions assumed in Theorems 3 and 4 can be omitted consider

Remark 7. Equation (1) is equivalent to the functional equation

$$\alpha(y)=\frac{\alpha\left(x\right)\left[3\alpha\left(\frac{x+y}{2}\right)-\alpha\left(\frac{3x+y}{4}\right)\right]-2\alpha\left(\frac{x+y}{2}\right)\alpha\left(\frac{3x+y}{4}\right)}{2\alpha\left(x\right)+\alpha\left(\frac{x+y}{2}\right)-3\alpha\left(\frac{3x+y}{4}\right)} \tag{15}$$

$$x,y\in I, \quad x\neq y.$$

Proof. Assume that α is one-to-one and satisfies equation (1). From (1), for all $x, y \in I, x \neq y$, we have

$$\alpha(y)\left[2\alpha(x)+\alpha\left(\frac{x+y}{2}\right)-3\alpha\left(\frac{3x+y}{4}\right)\right]=$$

$$=\alpha(x)\left[3\alpha\left(\frac{x+y}{2}\right)-\alpha\left(\frac{3x+y}{4}\right)\right]-2\alpha\left(\frac{x+y}{2}\right)\alpha\left(\frac{3x+y}{4}\right).$$

Suppose that $2\alpha(x)+\alpha\left(\frac{x+y}{2}\right)-3\alpha\left(\frac{3x+y}{4}\right)=0$, that is

$$\alpha\left(\frac{3x+y}{4}\right)=\frac{2}{3}\alpha(x)+\frac{1}{3}\alpha\left(\frac{x+y}{2}\right)$$

for some $x, y \in I, x \neq y$. Setting this to the right-hand side of the above equality we get $\left[\alpha(x)-\alpha\left(\frac{x+y}{2}\right)\right]^2=0$, whence $y = x$, as α is one-to-one. Thus equation (1) implies (15). The converse implication is obvious. ■

Remark 8. Thus equation (15) is of the form

$$\alpha(y)=h\left(\alpha(x),\alpha\left(\frac{x+y}{2}\right)\alpha\left(\frac{3x+y}{4}\right)\right),$$

where

$$h(z_1,z_2,z_3)=\frac{z_1z_3-3z_1z_2+2z_2z_3}{3z_3-z_2+2z_1}$$

and one could try to employ the celebrated regularity theory due to Antal Járai [1] by the assumption that the unknown function α is monotonic, so it is a.e. differentiable. To get its differentiability one could apply Theorem 17.6 in [1], and then, to get higher regularity, Theorem 15.2. At this background a question arises if the lack of regularity of h at the points (z_1, z_2, z_3) such that $3z_3 - z_2 + 2z_1 = 0$ is a serious difficulty.

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