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SOME REMARKS ON THE CARMICHAEL AND ON THE EULER'S φ FUNCTION

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Dedicated to my friend, Professor Antal Járai on his 60th anniversary

Abstract. Several theorems on the iterates of the Carmichael and on the Euler's φ function is presented, some of them without proof.

1. Introduction

We shall formulate several in my opinion new theorems on the divisors of the Carmichael and Euler's totient function.

Some of them can be proved by direct application of sieve theorems. We omit the proof of them. We shall prove only Theorem 6, 10, 11, 12.

1.1. Notations. $\mathcal{P}$ = set of primes; p, π with and without suffixes always denote prime numbers; $\pi(x) = \#\{p \leq x\}$, $\pi(x,k,l) = \#\{p \leq x,\ p \equiv l \pmod{k}\}$.

$\lambda(n)$ = Carmichael function defined for p^α by

$$\lambda(p^\alpha) = \begin{cases} p^{\alpha-1}(p-1), & \text{if } p \geq 3, \text{ or } \alpha \leq 2, \\ 2^{\alpha-2}, & \text{if } p = 2 \text{ and } \alpha \geq 3, \end{cases}$$

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and for $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ $(p_i \neq p_j,\ p_i \in \mathcal{P})$

$$\lambda(n) = LCM\left[\lambda(p_1^{\alpha_1}), \dots, \lambda(p_r^{\alpha_r})\right].$$

Here LCM = least common multiple.

Let $\omega(n)$ = number of distinct prime factors of n, $\Omega(n)$ = number of prime power divisors of n.

$$\varphi(n) = \prod_{j=1}^{r} p_j^{\alpha_j - 1}(p_j - 1) \text{ the Euler's totient function.}$$

$P(n)$ = largest prime divisor of n; $p(n)$ = smallest prime divisor of n.

Let $x_1 = \log x,\ x_2 = \log x_1 \dots$.

Let $\lambda^{(k)}(n),\ \varphi^{(k)}(n)$ be the kth iterate of $\lambda(n)$ and of $\varphi(n)$, respectively, i.e. $\lambda^{(0)}(n) = n,\ \varphi^{(0)}(n) = n$, and $\lambda^{(k+1)}(n) = \lambda(\lambda^{(k)}(n)),\ \varphi^{(k+1)}(n) = \varphi(\varphi^{(k)}(n))$.

1.2. In this paper we shall formulate some theorems on λ, φ and on their iterates. Some of these theorems can be proved by known methods which were applied earlier, and we omit their complete proof.

1.3. Let $q \geq 2$ be a fixed prime, $\gamma(n)$ be that exponent, for which $q^{\gamma(n)} \,\|\, \varphi(n)$. M. Wijsmuller [3] investigated the completely additive function β defined on $p \in \mathcal{P}$ by $q^{\beta(p)} \,\|\, p+1$, and proved a global central limit theorem for $\beta(n)$. Her method can be used to prove central limit theorem for $\gamma(n)$ as well. In [1], [2] we developed a method by which we can prove local central limit theorem for $\gamma(n)$ and $\beta(n)$. We are unable to give the asymptotic of $\#\{p \leq x,\ p \in \mathcal{P},\ \gamma(p+1) = k\}$, and that of $\{n \leq x,\ \gamma(n^2+1) = k\}$. Global central limit theorem can be proved for $\gamma(p+1)$, and $\gamma(n^2+1)$.

1.4. Let $\nu(n)$ be defined by $q^{\nu(n)} \,\|\, \lambda(n)$. Let $\mathcal{P}_k := \{p \mid p \in \mathcal{P},\ p \equiv 1 \pmod{q^k}\}$; $\mathcal{P}_k^* = \mathcal{P}_k \backslash \mathcal{P}_{k+1}$. Let furthermore

$$\omega_k(n) = \sum_{\substack{p \mid n \\ p \in \mathcal{P}_k}} 1, \tag{1.1}$$

$$t_k(x) := \prod_{\substack{p \equiv 1 \pmod{q^k} \\ p \leq x}} \left(1 - \frac{1}{p}\right). \tag{1.2}$$

From the Siegel–Walfisz theorem (Lemma 7) one can obtain that

$$\log t_k(x) = -\sum_{\substack{p\le x\\ p\equiv 1(q^k)}} \frac{1}{p} + O\left(\frac{1}{q^k}\right) = -\frac{x_2}{\varphi(q^k)} + O\left(\frac{1}{q^k}\right) \tag{1.3}$$

valid if $1 \le q^k \le c\,x_2$.

The following assertion can be proved by routine application of the asymptotic sieve.

Theorem 1. *Let $q \ge 2$ be a fixed prime,*

$$\alpha_k(x) := \frac{x_2}{\varphi(q^k)}. \tag{1.4}$$

Assume that $k = k(x) \to \infty$ and that $x_2 \cdot q^{-k} \to \infty$. Then

$$\begin{aligned} &\frac{1}{\left(1-\frac{1}{q}\right)x}\#\{n \le x,\ (n,q)=1,\ \nu(n)=k,\ \omega_k(n)=r\} = \\ &\qquad = (1+o_x(1))t_k(x)\sum \frac{1}{\varphi(p_1\cdots p_r)} \end{aligned} \tag{1.5}$$

valid for $0 \le r \le \frac{x}{x_3^2}$. The last sum is extended over those $p_1 < \ldots < p_r$ for which $p_i \in \mathcal{P}_k^*$, $p_1 < \ldots < p_r \le x$. In this range of r we have

$$\sum \frac{1}{\varphi(p_1\cdots p_r)} = (1+o_x(1))\left(\frac{x_2}{q^k}\right)^r \cdot \frac{1}{r!}. \tag{1.6}$$

Assume that $q^k/x_2 \to \infty$, $q^k < x^{1/3}$. Then

$$\sum_{n\le x}\omega_k(n) = x\sum_{\substack{p\le x\\ p\in\mathcal{P}_k}}\frac{1}{p} + O(\pi(x,q^k,1)), \tag{1.7}$$

and

$$\sum_{n\le x}\omega_k(n)(\omega_k(n)-1) = \sum_{\substack{p_1\ne p_2\\ p_1p_2\le x\\ p_1,p_2\in\mathcal{P}_k}}\frac{x}{p_1p_2} + O\left(\sum_{\substack{p_1<\sqrt{x}\\ p_1\in\mathcal{P}_k}}\pi\left(\frac{x}{p_1},q^k,1\right)\right). \tag{1.8}$$

By using the Brun–Titchmarsh theorem (Lemma 8), we obtain that the error term on the right hand side of (1.8) is less than $(\operatorname{li} x)q^{-2k}x_2$. From (1.7), (1.8) we can deduce a Turán–Kubilius type inequality and from that

Theorem 2. *Let $q \in \mathcal{P}$ be fixed, $k = k(x)$ be such that $q^k/x_2 \to \infty$ and that $q^k < c\,x_1^A$ hold with arbitrary constants c, A. Then*

$$\frac{1}{x}\#\{n \le x \mid \nu(n) \ge k\} = (1 + o_x(1)) \sum_{\substack{p \le x \\ p \in \mathcal{P}_k}} \frac{1}{p}, \tag{1.9}$$

furthermore

$$\sum_{\substack{p \le x \\ p \in \mathcal{P}_k}} \frac{1}{p} = \alpha_k(x) + O\left(\frac{1}{q^k}\right). \tag{1.10}$$

Remark. By using the Barban–Linnik–Tshudakov theorem (Lemma 9) (1.9) remains valid up to $q^k < x^\delta$, where δ is a suitable positive constant.

We can prove also the following Theorem 3, 4, 5.

Theorem 3. *Assume that $k = k(x)$ is such a sequence for which $q^k/x_2 \to \to \infty$ and that $q^k < c\,x_1^A$ with arbitrary constants c, A. Then*

$$\frac{1}{\operatorname{li} x}\#\{p \le x \mid \nu(p+1) \ge k\} = (1 + o_x(1))\alpha_k(x). \tag{1.11}$$

Furthermore

$$\frac{1}{\operatorname{li} x}\#\{p \le x \mid \nu(p+1) \ge k,\ \nu(p-1) \ge l\} = (1 + o_x(1))\alpha_k(x) \cdot \alpha_l(x) \tag{1.12}$$

holds, if additionally $q^l/x_2 \to \infty,\ q^l < c\,x_1^A$.

Remark. One could prove in general that

$$\frac{1}{\operatorname{li} x}\#\{p \le x \mid \nu(p+t_j) \ge k_j,\ j = 1, \dots, h\} = (1 + o_x(1))\alpha_{k_1}(x) \dots \alpha_{k_h}(x)$$

if $t_1, \dots, t_h$ are distinct nonzero integers and $q^{k_j}/x_2 \to \infty,\ q^{k_j} \le cx_1^A$ $(j = = 1, \dots, h)$.

Theorem 4. *Let q be an odd prime. Assume that $k = k(x) \to \infty,\ x_2 q^{-k} \to \to \infty$. Then*

$$\begin{aligned} &\frac{1}{\operatorname{li} x}\#\{p \le x,\ (p+1, q) = 1,\ \nu(p+1) = k,\ \omega_k(p+1) = r\} = \\ &= (1 + o_x(1))(\operatorname{li} x) t_k^*(x) \frac{1}{r!}\left(\frac{x_2^k}{q^k}\right)^r \end{aligned} \tag{1.13}$$

if $0 \le r \le \frac{x_2}{x_3}$*. Here*

$$t_k^*(x) = \prod_{\substack{p<x \\ p\in\mathcal{P}_k}} \left(1 - \frac{1}{p-1}\right). \tag{1.14}$$

Remark. Since

$$\log \frac{t_k^*(x)}{t_k(x)} = O\left(\sum_{p\in\mathcal{P}_k} \frac{1}{p^2}\right) = O\left(\frac{1}{q^k}\right),$$

(1.13) remains valid with $t_k(x)$ instead of $t_k^*(x)$.

Theorem 5. *Let q be an odd prime, $k = k(x)$ be such a sequence for which $x_2 q^{-k} \to \infty$. Let $\rho(m)$ be the number of residue classes $n \pmod m$, for which $n^2+1 \equiv 0 \pmod m$.*

Let

$$s_k(x) = \prod_{\substack{p<x \\ p\in\mathcal{P}_k}} \left(1 - \frac{\rho(p)}{p-1}\right). \tag{1.15}$$

Then

$$\begin{aligned} &\frac{1}{x}\#\left\{n \le x,\ (n^2+1, q) = 1,\ \nu(n^2+1) = k, \omega_k(n^2+1) = r\right\} = \\ &= (1+o_x(1))\left(1 - \frac{\rho(q)}{q}\right) s_k(x) \frac{1}{r!}\left(\sum_{\substack{\pi<x \\ \pi\in\mathcal{P}_k}} \frac{\rho(\pi)}{\pi - 1}\right)^r \end{aligned} \tag{1.16}$$

if $0 \le r \le \frac{x_2}{x_3}$.

1.5. In their paper [6] W.D. Banks, F. Luca, I.E. Shparlinski investigated some arithmetic properties of $\varphi(n), \lambda(n)$, and that of $\xi(n) = \frac{\varphi(n)}{\lambda(n)}$. Among others they investigated the distribution of $P(\xi(n))$. Namely they proved that

$$1 + o(1) \le \frac{1}{x\cdot x_3} \sum_{n\le x} \log P(\xi(n)) \le 2 + o(1), \tag{1.17}$$

and that

$$(0<)c_1 \le \frac{1}{x x_2^3} \sum_{n\le x} P(\xi(n)) \le c_2 \quad (x \ge 1) \tag{1.18}$$

holds with suitable positive constants.

We can prove that $P(\xi(n))$ is distributed in limit according to the Poisson law.

Let

$$\kappa_q(n) := \sum_{\substack{p\,|n \\ p\equiv 1 \pmod{q^2}}} 1; \quad f_Y(n) := \sum_{q>Y} \kappa_q(n).$$

Since

$$\sum_{n\le x} \kappa_q(n) = \sum_{p\equiv 1 \pmod{q^2}} \left[\frac{x}{p}\right] \le x \sum_{\substack{p\le x \\ p\equiv 1 \pmod{q^2}}} \frac{1}{p} \le \frac{cxx_2}{q^2}$$

holds with a suitable constant c, and

$$\sum_{q\ge Y} \frac{1}{q^2} = \frac{1}{Y\log Y} + O\left(\frac{1}{Y(\log Y)^2}\right),$$

we obtain that

$$\sum_{n\le x} f_Y(n) \le \frac{c\,x\,x_2}{Y\log Y}.$$

If q is an odd prime, $q^2 \mid \lambda(n)$, then either $q^3 \mid n$, or there exists some $p \mid n$ for which $q^2 \mid p-1$. We obtain

$$\#\left\{n \le x \;\mid\; q^2 \mid \lambda(n) \text{ for some } q > x_2^2\right\} \le \frac{c\,x}{x_2 x_3}. \tag{1.19}$$

Let

$$f_Y^*(n) = \sum_{Y\le q\le x_2^2} \kappa_q(n),$$

$$\sum\nolimits_1 := \sum_{n\le x} f_Y^*(n), \qquad \sum\nolimits_2 := \sum_{n\le x} f_Y^{*2}(n).$$

From the Siegel–Walfisz theorem one can prove that

$$\sum_{\substack{p\le x \\ p\equiv 1 \pmod{k}}} \frac{1}{p} = \frac{1}{\varphi(k)} x_2 + O\left(\frac{x_3}{\varphi(k)}\right) \text{ if } 1\le k\le x_2^A,$$

where A is an arbitrary constant, whence we deduce that

$$\sum\nolimits_1 = xx_2\ A_{Y,x} + O\left(\frac{xx_3}{Y\log Y}\right),$$

$$A_{Y,x} := \sum_{Y\le q\le x_2^2} \frac{1}{\varphi(q^2)} = \frac{1}{Y\log Y} + O\left(\frac{1}{Y(\log Y)^2}\right).$$

Furthermore $\sum_2 = \sum_{2,1} + \sum_{2,2}$, where

$$\sum\nolimits_{2,1} = \sum_{Y \le q \le x_2^2} \sum_{n \le x} \kappa_q^2(n), \qquad \sum\nolimits_{2,2} = \sum_{\substack{q_1 \ne q_2 \\ Y \le q_1, q_2 \le x_2^2}} \sum_{n \le x} \kappa_{q_1}(n) \kappa_{q_2}(n).$$

In this section q, q_1, q_2 run over the set of primes.

We have

$$\sum\nolimits_{2,1} = \sum\nolimits_1 + \sum_{Y \le q \le x_2^2} \sum_{\substack{p_1 \ne p_2 \\ q^2/p_j - 1}} \left[\frac{x}{p_1 p_2} \right] = \sum\nolimits_1 + x \sum_{Y \le q \le x_2^2} \frac{x_2^2}{\varphi(q^2)^2} +$$

$$+ O\left(x x_2 x_3 \sum_{q > Y} 1/q^4 \right) = \sum\nolimits_1 + O\left(\frac{x x_2^2}{Y^3 \log Y} \right)$$

and

$$\sum\nolimits_{2,2} = x \sum_{\substack{q_1 \ne q_2 \\ q_j \in [Y, x_2^2]}} \sum_{\substack{p_j \equiv 1 \pmod{q_j^2} \\ p_1 p_2 \le x}} \frac{1}{p_1 p_2} + x \sum_{\substack{q_1 \ne q_2 \\ q_j \in [Y, x_2^2]}} \sum_{\substack{p \le x \\ p \equiv 1 \pmod{q_1^2 q_2^2}}} \frac{1}{p} + O(x)$$

whence we obtain that

$$\sum\nolimits_{2,2} = \left(1 + O\left(\frac{x_3}{x_2} \right) \right) x x_2^2 A_{y,x}^2 + O\left(x x_2 \left(\sum_{q > Y} \frac{1}{q^2} \right)^2 \right) =$$

$$= x x_2^2 A_{y,x}^2 + O\left(x x_3 x_2 \cdot \frac{1}{Y^2 \log^2 Y} \right).$$

After some easy computation we obtain that

$$\frac{1}{x} \sum_{n \le x} (f_Y^*(n) - x_2 A_{Y,x})^2 \ll \frac{x_2}{Y \log Y} + \frac{x_2 x_3}{(Y \log Y)^2} + \frac{x_2^2}{Y^3 \log Y}. \tag{1.20}$$

From (1.20) we can deduce

Theorem 6. *Let $\varepsilon_x \to 0$. Then*

$$x^{-1} \# \left\{ n \le x \mid P(\lambda(n)) \in \left[\varepsilon_x \cdot \frac{x_2}{x_3}, \frac{1}{\varepsilon_x} \cdot \frac{x_2}{x_3} \right] \right\} \to 1 \quad (x \to \infty).$$

Proof. Indeed, choose first $Y = \varepsilon_x \cdot \dfrac{x_2}{x_3}$, then $Y = \dfrac{1}{\varepsilon_x} \cdot \dfrac{x_2}{x_3}$ and apply (1.20). ∎

We can prove also

Theorem 7. *Let* $\varepsilon_x \to 0$. *Then*

$$\frac{1}{\operatorname{li} x}\#\left\{p \le x \;\middle|\; P(\lambda(p-1)) \in \left[\varepsilon_x \cdot \frac{x_2}{x_3}, \frac{1}{\varepsilon_x}\cdot\frac{x_2}{x_3}\right]\right\} \to 1 \quad (x\to\infty).$$

1.6. Assume that $Y = O(x_2^2)$, $Y \ge x_2^{3/2}$, $u(n) := e^{i\tau f_Y^*(n)}$, where $\tau \in \mathbb{R}$. Then u is a strongly multiplicative function, for $p \in \mathcal{P}$

$$u(p) := \begin{cases} e^{i\tau} & \text{if } p \equiv 1 \pmod{q^2} \text{ for some } \ q \in [Y, x_2^2],\\ 1 & \text{otherwise.}\end{cases}$$

Let h be the Moebius transform of u, i.e.

$$h(p) = \begin{cases} e^{i\tau} - 1 & \text{if } q^2 \mid p-1 \text{ for some } q \in [Y, x_2^2],\\ 0 & \text{otherwise,}\end{cases}$$

$h(p^\alpha) = 0$ if $p \in \mathcal{P}$, $\alpha \ge 2$.

Let

$$S_1(x,\tau) := \sum_{n\le x} e^{i\tau f_Y^*(n)};\ S_2(x,\tau) = \sum_{n \le x} u(n).$$

If $f_Y^*(n) \ne u(n)$ for some n, then there exists a prime divisor p of n, and $q_1, q_2 \in \mathcal{P}$, $q_1, q_2 > Y$, $q_1 \ne q_2$ such that $p \equiv 1 (\operatorname{mod} q_1^2 q_2^2)$.

Then

$$|S_1(x,\tau) - S_2(x,\tau)| \le x \sum_{\substack{q_1,q_2\in[Y,x_2^2]\\ q_1\ne q_2}} \sum_{\substack{p\equiv 1 \pmod{q_1^2, q_2^2}\\ p\le x}} \frac{1}{p} \ll$$

$$\ll x x_2 \left(\sum_{q>Y} \frac{1}{q^2}\right)^2 \ll x x_2 \left(\frac{1}{Y\log Y}\right)^2 = O\left(\frac{x}{x_2^2}\right).$$

There are several ways to prove that

$$\frac{S_2(x,\tau)}{x} = (1+o_x(1)) \prod_{\substack{p<x\\ p\equiv 1\ (q^2)\\ q>Y\\ q\in\mathcal{P}}} \left(1 + \frac{e^{i\tau}-1}{p}\right) = \tag{1.21}$$

$$= (1+o_x(1)) \exp\left(\left(e^{i\tau}-1\right)\frac{x_2}{Y\log Y}\right).$$

One way to prove (1.21) is to copy the argument of the theorem of H. Delange
for the arithmetical mean of multiplicative functions of moduli 1. (See [7], or [4] pp. 331–336.) Another method is to compute the asymptotic of $\sum_{n\le x} f_Y^{*h}(n)$ for $h = 1, 2, \dots$ and use the Frechlet–Shohat theorem (see J. Galambos [11]). A relevant paper is written by J. Šiaulys [8]. We can prove

Theorem 8. *Let* $\alpha_Y = x_2 \sum_{q>Y} \dfrac{1}{\varphi(q^2)}$. *Assume that* $\alpha_Y \in [c_1, c_2]$, *where* $c_1 < c_2$ *are arbitrary positive constants. Then*

$$\lim_{x\to\infty} \sup_{\alpha_Y\in[c_1,c_2]} \sup_{k\ge 0} \left| \frac{1}{x}\#\left\{n \le x \mid f_Y^*(n) = k\right\} - \frac{\alpha_Y^k}{k!}\exp(-\alpha_Y)\right| = 0. \tag{1.22}$$

Similarly, we have

$$\lim_{x\to\infty} \sup_{\alpha_Y\in[c_1,c_2]} \sup_{k\ge 0} \left| \frac{1}{\operatorname{li} x}\#\left\{p \le x \mid f_Y^*(p-1) = k\right\} - \frac{\alpha_Y^k}{k!}\exp(-\alpha_Y)\right| = 0. \tag{1.23}$$

Assume that Q is such a prime for which $(Q\log Q)/x_2 \in [c_1, c_2]$, where c_1, c_2 are positive constants. We would like to estimate the number of those integers $n \le x$ for which $P(\xi(n)) = Q$. By using the asymptotic sieve one can obtain quite immediately that

$$\frac{1}{x}\#\{n \le x \mid P(\xi(n)) < Q\} = (1 + o_x(1)) \prod_{\substack{p\le x\\ q^2/p-1\\ q\ge Q}} \left(1 - \frac{1}{p}\right).$$

Let

$$\tau(Q, x) = x_2 \cdot \sum_{q\ge Q} \frac{1}{\varphi(q^2)}.$$

Then

$$\frac{1}{x}\#\{n \le x \mid P(\xi(n)) < Q\} = (1 + o_x(1))\exp(-\tau(Q, x)).$$

Let $\mathcal{B}_{Q,r}$ be the set of those n for which $P(\xi(n)) = Q$, and there exists exactly r distinct prime divisors $p_1, p_2, \dots, p_r$ of n for which $Q^2 \mid p_j - 1$. Then

$$\frac{1}{x}\#\{n \le x \mid n \in \mathcal{B}_{Q,r}\} = (1 + o_x(1))\exp(-\tau(Q, x)) \cdot \frac{1}{r!}\left\{\sum_{\substack{p\equiv 1(Q^2)\\ p\le x}} \frac{1}{p}\right\}^r$$

valid for every fixed $r = 0, 1, 2, \dots$.

We can prove furthermore

Theorem 9. *We have*

$$\frac{1}{\operatorname{li} x}\#\{p \le x \mid p-1 \in \mathcal{B}_{Q,r}\} = (1+o_x(1))\exp(-\tau(Q,x))\cdot\frac{1}{r!}\left\{\sum_{\substack{p\equiv 1 \pmod{Q^2}\\ p\le x}} 1/p\right\}^r$$

for every fixed $r = 0, 1, 2, \dots$.

1.7. For $p_1, p_2,\ q \in \mathcal{P}$ let

$$f_q(p_1,p_2) = \begin{cases} 1 & \text{if } p_1 \equiv p_2 \equiv 1 \pmod{q}, \quad p_1 < p_2, \\ 0 & \text{otherwise.} \end{cases} \tag{1.24}$$

Let

$$\Delta_Y(n) := \sum_{q>Y}\sum_{p_1p_2\mid n} f_1(p_1,p_2). \tag{1.25}$$

We observe that $\Delta_Y(n) \ne 0$ implies that $q^2 \mid \varphi(n)$ for some $q > Y$. On the other hand, if $q^2 \mid \varphi(n)$, then either $q^3 \mid n$; or $q^2 \mid n$ and $p \mid n$ with some $p \equiv 1 \pmod{q}$, or $p \mid n$ with some $p \equiv 1 \pmod{q^2}$; or there exist $p_1 \ne p_2$, $p_1 \equiv p_2 \equiv 1 \pmod{q}$, $q > Y$, and $p_1p_2 \mid n$.

Thus

$$\frac{1}{x}\#\{n \le x \mid \Delta_Y(n) \ne 0\} - \frac{1}{x}\#\{n \le x \mid q^2|\varphi(n) \text{ for some } q > Y\} \ll \frac{x}{Y\log Y}. \tag{1.26}$$

By using our method developed by De Koninck and myself [1], [2] we can compute the asymptotic of $\sum_{n\le x}\Delta_Y^h(n)$ and from the Frechet–Shohat theorem deduce

Theorem 10. *Let* $0 < c_1 < c_2 < \infty$ *be fixed constants,* $\alpha = \alpha_x \in [c_1, c_2]$, $Y = Y_x = \frac{1}{2\alpha}\cdot x_2^2/2x_3$. *Then*

$$x^{-1}\#\{n \le x \mid \Delta_{Y_x}(n) = k\} = (1+o_x(1))\frac{\alpha^k}{k!}e^{-\alpha} \quad (x \to \infty) \tag{1.27}$$

for every fixed $k = 0, 1, 2, \dots$ *uniformly as* $\alpha_x \in [c_1, c_2]$.

Furthermore we obtain that

$$\frac{1}{\operatorname{li} x}\#\{p \le x \mid \Delta_{Y_x}(p-1) = k\} = (1+o_x(1))\frac{\alpha^k}{k!}e^{-\alpha} \quad (x \to \infty) \tag{1.28}$$

for every fixed $k = 0, 1, \dots$ *uniformly as* $\alpha_x \in [c_1, c_2]$.

We shall prove this theorem in Section 3.

The following theorem can be deduced easily from Theorem 10.

Let $\kappa_Y(n)$ be the number of those $q > Y$ for which $q^2 \mid \varphi(n)$.

Theorem 11. *Let* Y_x *be the same as in Theorem* 10. *Then*

$$x^{-1}\#\{n \le x \mid \kappa_{Y_x}(n) = k\} = (1 + o_x(1))\frac{\alpha^k}{k!}e^{-\alpha} \quad (x \to \infty), \tag{1.29}$$

and

$$\frac{1}{\operatorname{li} x}\#\{p \le x \mid \kappa_{Y_x}(p-1) = k\} = (1 + o_x(n))\frac{\alpha^k}{k!}e^{-\alpha} \quad (x \to \infty). \tag{1.30}$$

Remark. By using our method we can determine the distribution of

$$\delta_Y^{(k,r)}(n) = \delta_Y(n) = \#\{q > Y,\ q \in \mathcal{P},\ q^r \mid \varphi_k(n)\}$$

and that of $\delta_Y^{(k,r)}(p-1)$, where $Y_x = \alpha\ (x_2^{kr}/x_3)^{1/(r-1)}$. We shall prove it in another paper.

1.8. In a paper of F. Luca and C. Pomerance [17] the conjecture of Erdős, namely that $\varphi(n - \varphi(n)) < \varphi(n)$ holds on a set of asymptotic density 1 is proved.

They deduce that

$$\left|\frac{\varphi(n - \varphi(n))}{n - \varphi(n)} - \frac{\varphi(n)}{n}\right| < \varepsilon_n \tag{1.31}$$

holds for almost all n, with a sequence $\varepsilon_n \to 0$, which implies the conjecture of Erdős. Namely they prove (1.31) with $\varepsilon_n = 2\dfrac{\log\log\log n}{\log\log n}$ but this is not necessary for obtaining Erdős conjecture.

By their method one can prove that

$$\left|\frac{f_i(n \pm f_j(n))}{n \pm f_j(n)} - \frac{f_i(n)}{n}\right| < \varepsilon_n \tag{1.32}$$

holds on a set of asymptotic density 1, where $\varepsilon_n \to 0$, and $f_1(n), f_2(n)$ can take the values $\varphi(n),\ \sigma(n) : (f_1, f_2) = (\varphi, \varphi);\ (\varphi, \sigma), (\sigma, \varphi), (\sigma, \sigma)$.

We can prove (1.32) also, if n runs over the set of shifted primes. We shall give a complete proof only in the case $f_1 = f_2 = \varphi$, $\pm = -$, and over the set of prime $+1'$s.

Theorem 12. *There exists a suitable sequence $\varepsilon_p \to 0$ $(p \in \mathcal{P}, p \to \infty)$ such that*

$$\left| \frac{\varphi(p-1-\varphi(p-1))}{p-1-\varphi(p-1)} - \frac{\varphi(p-1)}{p-1} \right| < \varepsilon_p$$

holds for $p \in \mathcal{P}$ with the possible exception of $o_x(1)\pi(x)$ of $p \in \mathcal{P}$ up to x.

1.9. J.-M. De Koninck and F. Luca [17] investigated

$$H(n) := \frac{\sigma(\varphi(n))}{\varphi(\sigma(n))}.$$

In particular, they obtain the maximal and minimal orders of $H(n)$, its average order, and also proved some density theorems.

Since

$$H(n) = \frac{\sigma(\varphi(n))}{\varphi(n)} \cdot \frac{\sigma(n)}{\varphi(\sigma(n))} \cdot \frac{\varphi(n)}{\sigma(n)},$$

therefore

$$\log H(n) = \kappa_1(n) + \kappa_2(n) + \kappa_3(n),$$

where

$$\kappa_1(n) = \sum_{p^\alpha \,\|\, \varphi(n)} \log\left(1 + \frac{1}{p} + \cdots + \frac{1}{p^\alpha}\right),$$

$$\kappa_2(n) = \sum_{p|\sigma(n)} \log \frac{1}{1-\frac{1}{p}},$$

$$\kappa_3(n) = \sum_{p^\alpha \,\|\, n} \log \frac{1-\frac{1}{p}}{1+\frac{1}{p}+\cdots+\frac{1}{p^\alpha}}.$$

By using a known theorem of P. Erdős one can prove that

$$\left| \kappa_j(n) - \sum_{p<x_2/x_3^2} \log \frac{1}{1-\frac{1}{p}} \right| < \varepsilon_x \quad (j = 1, 2)$$

holds for all but at most $o(x)$ integers $n \le x$, where $\varepsilon_x \to 0$ $(x \to \infty)$. Since $\kappa_3(n)$ is an additive function satisfying the conditions of the Erdős–Wintner theorem, we obtain immediately that

$$\frac{1}{x}\#\left\{ n \le x \;\middle|\; \log H(n) - \sum_{p<x_2/x_3^2} \log \frac{1}{1-\frac{1}{p}} < y \right\} = F_x(y)$$

tends to $F(y)$, where F is the distribution function defined as

$$F(y) = \lim_{x\to\infty} \frac{1}{x}\#\{n \le x \mid \kappa_3(n) < y\}.$$

Erdős proved that F is a continuous singular function.

Distribution of H on the set of shifted primes, on polynomial values, and on prime places of polynomial values can be proved similarly. Let

$$s(x) = \prod_{p<x}\left(1-\frac{1}{p}\right)^{-1}.$$

Then $s(x) = e^{\gamma}x_1\ (1+o_x(1))$.

Theorem 13. *Let $k,l \ge 0$, $f_{k,l}^{(1)}(n) := \sigma_k(\varphi_l(n))$, $f_{k,l}^{(2)}(n) = \varphi_k(\sigma_l(n))$. Then for every $n \le x$ dropping at most $o(x)$ integers*

$$\frac{\sigma_k(n)}{\sigma_{k-1}(n)} = s(x_2^{k-1})(1+o_x(1)) \quad (k \ge 2), \tag{1.33}$$

$$\frac{\varphi_k(n)}{\varphi_{k-1}(n)} = \frac{1}{s(x_2^{k-1})}(1+o_x(1)) \quad (k \ge 2), \tag{1.34}$$

and for $k,l \ge 1$

$$\frac{f_{k,l}^{(1)}(n)}{f_{k-1,l}^{(1)}(n)} = \frac{1}{s(x_2^{k+l-1})}(1+o_x(1)) \quad (k \ge 1), \tag{1.35}$$

$$\frac{f_{k,l}^{(2)}(n)}{f_{k-1,l}^{(2)}(n)} = s(x_2^{k+l-1})(1+o_x(1)) \quad (k \ge 1). \tag{1.36}$$

Furthermore the relations (1.33), (1.34), (1.35), (1.36) *are valid on the set of shifted primes $p+a$ $(a \ne 0)$, with the exception of no more than $o(\operatorname{li} x)$ integers $p+a$ up to x.*

This theorem is an immediate consequence of the following

Theorem 14. *Let $k,l \ge 1$. Then, with the exception of at most $\delta_x x$ integers $n \le x$, for the others*

α) $p^\alpha \mid \varphi_k(n)$, $p^\alpha \mid \sigma_k(n)$ if $p^\alpha \le x_2^{k-\varepsilon_x}$, *and*

$$\sum_{\substack{p\mid\varphi_k(n)\\ p>x_2^{k+\varepsilon_x}}} \frac{1}{p} < \varepsilon_x; \quad \sum_{\substack{p\mid\varphi_k(n)\\ p>x_2^{k+\varepsilon_x}}} \frac{1}{p} < \varepsilon_x,$$

β) $p^{\alpha} \mid f_{k+l}^{(1)}(n),\ p^{\alpha} \mid f_{k+l}^{(2)}(n)$ if $p^{\alpha} \le x_2^{k+l-\varepsilon_x}$,
and

$$\sum_{\substack{p \mid f_{k+l}^{(1)}(n) \\ p > x_x^{k+l+\varepsilon_x}}} \frac{1}{p} < \varepsilon_x; \quad \sum_{\substack{p \mid f_{k+l}^{(2)}(n) \\ p > x_2^{k+l+\varepsilon_x}}} \frac{1}{p} < \varepsilon_x,$$

where $\varepsilon_x \to 0$. *Here* $\delta_x \to 0$.

The same assertions hold if n *runs over the set of shifted primes, i.e. dropping no more than* $\delta_x \mathrm{li}\, x$ *integers* $p + a \le x$ (a fix, $a \ne 0$), *for the other* $p + a$ *the relations* α), β) *hold true.*

Remark. Theorem 14. α) for $k = 1$ is due to Erdős [11], for arbitrary k is given in [12]. The proof of β), can be proved similarly. One can use the method using in the papers [13], [15], [16].

From Theorem 13, 14 and from Erdős–Wintner theorem (see in [5]) we can deduce several generalizations of the theorem of De Koninck and Luca [16].

Examples.

1. The function

$$\nu_k(n) = \frac{\varphi_k(n)}{n} \cdot (k-1)!(\log\log\log n)^{k-1} \cdot e^{(k-1)\gamma}$$

has a limit distribution, which is the same as the limit distribution of $\dfrac{\varphi(n)}{n}$.

2. The function

$$\mu_k(n) = \frac{\sigma_k(n)}{n} \frac{(\log\log\log n)^{-(k-1)}}{(k-1)!} e^{-(k-1)\gamma}$$

is distributed in limit as $\dfrac{\sigma(n)}{n}$.

3. The function

$\nu_k(p+a)$ is distributed in limit as $\dfrac{\varphi(p+a)}{p+a}$;

$\mu_k(p+a)$ is distributed in limit as $\dfrac{\sigma(p+a)}{p+a}$.

Here $a \ne 0$, p runs over the set of primes.

4. The function

$$\rho_{k,l}^{(1)}(n) := \frac{f_{k,l}^{(1)}(n)}{n}(\log\log\log n)^{l-1-k}\frac{(l-1)!}{l(l+1)\dots(l+k-1)}e^{l-1-k}\gamma$$

is distributed in limit as $\frac{\varphi(n)}{n}$;

the function

$$\rho_{k,l}^{(2)}(n) = \frac{f_{k,l}^{(2)}(n)}{n}\cdot\frac{l(l+1)\dots(l+k-1)}{(l-1)!}e^{(k-l+1)\gamma}\cdot(\log\log\log n)^{k-l+1}$$

is distributed in limit as $\frac{\sigma(n)}{n}$.

5. Let $a \neq 0$, fixed integer. The functions

$$\rho_{k,l}^{(1)}(p+a); \qquad \rho_{k,l}^{(2)}(p+a)$$

are distributed in limit as $\frac{\varphi(p+a)}{p+a}$, $\frac{\sigma(p+a)}{p+a}$ respectively. Here p runs over the set of primes

2. Lemmata

We shall use Selberg's sieve theorem as it is formulated in Elliott ([4], Chapter 2, Lemma 2.1).

Lemma 1. *Let a_n $(n = 1, \dots, N)$ be integers, $f(n) \geq 0$. Let $r > 0$, and $p_1 < p_2 < \dots < p_s \leq r$ be rational primes. Set $Q = p_1 \dots p_s$. If $d \mid Q$ then let*

$$\sum_{\substack{n=1 \\ a_n \equiv 0 \pmod d}}^{N} f(n) = \eta(d)X + R(N,d),$$

where X, R are real numbers, $X \geq 0$, and $\eta(d_1 d_2) = \eta(d_1)\cdot\eta(d_2)$ whenever d_1 and d_2 are coprime divisors of Q.
Assume that for each prime p, $0 \leq \eta(p) < 1$. Let

$$I(N,Q) := \sum_{\substack{n=1 \\ (a_n,Q)=1}}^{N} f(n).$$

Then the estimate

$$I(N,Q)=\{1+2\Theta_1 H\}\times\prod_{p|Q}(1-\eta(p))+2\Theta_2\sum_{\substack{d|Q\\ d\le z^3}}3^{\omega(d)}|R(N,d)|$$

holds uniformly for $r\ge 2$, $\max(\log r,S)\le\frac{1}{8}\log z$, *where* $|\Theta_1|\le 1$, $|\Theta_2|\le 1$ *and*

$$H=\exp\left(-\frac{\log z}{\log r}\left\{\log\left(\frac{\log z}{S}\right)-\log\log\left(\frac{\log z}{S}\right)-\frac{2S}{\log z}\right\}\right),$$

$$S=\sum_{p|Q}\frac{\eta(p)}{1-\eta(p)}\log p.$$

The next lemma can be found in Halberstam and Richert [5], Corollary 2.4.1.

Lemma 2. *Let* k *be a positive integer,* l,a,b *be nonzero integers,* $k\le x$. *Then*

$$\#\{p\le x\mid p\equiv l\pmod k,\ \ ap+b\in\mathcal{P},\ p\in\mathcal{P}\}\le$$
$$\le c\prod_{p|kab}\left(1-\frac{1}{p}\right)^{-1}\cdot\frac{x}{\varphi(k)\log^2\frac{x}{k}},$$

where c *is an absolute constant.*

Lemma 3 (E. Bombieri and A.I. Vinogradov). *For fixed* $A>0$, *there exists* $B=B(A)>0$ *such that*

$$\sum_{k\le\frac{\sqrt{x}}{x_1^B}}\max_{(l,k)=1}\max_{2\le y\le x}\left|\pi(y,k,l)-\frac{li\,y}{\varphi(k)}\right|\ll\frac{x}{x_1^A}.$$

For a proof see [4].

Lemma 4. *Let* f *be a multiplicative non-negative function which for suitable* A *and* B *satisfies*

(i) $$\sum_{p\le y}f(p)\log p\le Ay\quad(y\ge 0),$$

(ii) $$\sup_p\sum_{\nu\ge 2}\frac{f(p^\nu)}{p^\nu}\log p^\nu\le B.$$

Then, for $x>1$,

$$\sum_{n\le x} f(n) \le (A+B+1)\frac{x}{x_1}\sum_{n\le x}\frac{f(n)}{n}.$$

This assertion is Theorem 5 in Tenenbaum [4], Part III. Chapter 5.

Lemma 5. *We have for $l=1,2,\ 1\le k\le x$*

$$\sum_{\substack{p\le x\\ p\equiv l (mod\, k)}} \frac{1}{p} \le c\frac{x_2}{\varphi(k)}.$$

(See [5].)

Lemma 6 (Frechet and Shohat [9]). *Let $F_n(u)$ $(n=1,2,\dots)$ be a sequence of distribution functions. For each non-negative integer l let*

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} u^l \,\mathrm{d}F_n(u)$$

exist. Then there exists a subsequence $F_{n_k}(u)$, $n_1<n_2<\dots$ which converges weakly to the limiting distribution $F(u)$ satisfying

$$\alpha_l = \int_{-\infty}^{\infty} u^l\,\mathrm{d}F(u), \quad (l=0,1,\dots).$$

Moreover, if the sequence of moments α_l determines $F(u)$ uniquely, then the sequence $F_n(u)$ converges to $F(u)$ weakly.

Lemma 7 (Siegel and Walfisz). *We have*

$$\pi(x,k,l)=\frac{li\,x}{\varphi(k)}\left(1+O\left(e^{-c\sqrt{x_1}}\right)\right)$$

uniformly as $(k,l)=1,\ 1\le k\le x_1^A$. A is an arbitrary constant.

(See in [4].)

Lemma 8 (Brun–Titchmarsh). *We have*

$$\pi(x,k,l)\le\frac{cx}{\varphi(k)\log x/k},$$

if $1\le k<x,\ (k,l)=1$. c is an absolute constant.

(See in [18].)

Lemma 9 (Barban, Linnik and Tshudakov [10]). *Let q be an odd prime. Then*

$$\pi(x, q^r, l) = \frac{li\, x}{\varphi(q^r)}\left(1 + O\left(e^{-c\sqrt{x_1}}\right)\right)$$

uniformly as $(l, q) = 1,\ q^r \le x^{1/3}$.

3. Proof of Theorem 10 and Theorem 11

First we prove the relation (1.27). Let

(3.1) $$\delta_q(n) = \sum_{p_1 p_2 | n} f_q(p_1, p_2),$$

(3.2) $$\Delta_Y^*(n) = \sum_{Y < q \le x_2^2} \sum_{p_1 p_2 | n} f_q(p_1, p_2).$$

We observe that

(3.3) $$\#\{n \le x \mid \Delta_{x_2}^2(n) \ne 0\} \le \sum_{n \le x} \Delta_{x_2^2}(n) \le \\ \le \sum_{q \ge x_2^2} \sum_{\substack{p_1 p_2 \le x \\ p_j \equiv 1(q)}} \left[\frac{x}{p_1 p_2}\right] \le c x x_2^2 \sum_{q \ge x_2^2} \frac{1}{q^2} = O\left(\frac{x}{x_3}\right).$$

Let $r \ge 1$, and

(3.4) $$\tau_r(n) = \Delta_{Y_x}^*(n)\left(\Delta_{Y_x}^*(n) - 1\right) \dots \left(\Delta_{Y_x}^*(n) - (r-1)\right).$$

If $z_1, z_2, \dots, z_M \in \{0, 1\}$, then

(3.5) $$\sum_{i_1 < i_2 < \dots < i_r} z_{i_1} z_{i_2} \dots z_{i_r} = \frac{T(T-1)\dots(T-(r-1))}{r!},$$

(3.6) $$T = z_1 + z_2 + \dots + z_m.$$

The relation (3.5) can be proved by using induction on r.

We can write

$$\tau_r(n) = \sum_{\substack{\pi_j,\pi_j',q_j\\ \pi_j\pi_j'|n}} \prod_{j=1}^{r} f_{q_j}(\pi_j,\pi_j'), \tag{3.7}$$

where $\pi_j,\pi_j',q_j \in \mathcal{P},\ q_j \in [Y_x, x_2^2]$.

Let $\tau_r(n) = \tau_r^{(1)}(n)+\tau_r^{(2)}(n)$, where in $\tau_r^{(1)}(n)$ we sum over those π_j,π_j' $(j = 1,\dots,r)$ for which $\{\pi_u,\pi_u'\} \cap \{\pi_v,\pi_v'\} = \emptyset$ if $u \neq v$, and in $\tau_r^{(2)}(n)$ we sum over the others.

We have

$$\sum\nolimits_2 := \sum_{n\le x} \tau_r^{(2)}(n) \le \sum_{q_j,\pi_j,\pi_j'}^{*} \left[\frac{x}{LCM(\pi_1,\pi_1',\dots,\pi_r,\pi_r')}\right]$$

where $*$ indicates that no more than $(2r-1)$ distinct primes occur among $\pi_1,\pi_1',\dots,\pi_r,\pi_r'$.

By using Lemma 3 we obtain that

$$\frac{1}{x}\sum\nolimits_2 \ll x_2^{2r-1}\left\{\sum_{q>Y_x}\frac{1}{q^2}\right\}^r \ll x_2^{2r-1}\cdot\frac{1}{(Y_x\log Y_x)^r} = o_x(1).$$

Let

$$\sum\nolimits_1 := \sum_{n\le x}\tau_r^{(1)}(n).$$

Then

$$\sum\nolimits_1 = \sum_{\substack{\pi_j,\pi_j',q_j\\ \pi_j<\pi_j'}} \left[\frac{x}{\pi_1\pi_1'\cdots\pi_r\pi_r'}\right], \tag{3.8}$$

where in the right hand side $\pi_1,\pi_1'\dots,\pi_r,\pi_r'$ are distinct primes $q_j|\pi_j-1$, $q_j|\pi_j'-1$ and $q_j\in[Y_x,x_2^2]$.

By using our method in [1] one can obtain that

$$\frac{1}{x}\sum\nolimits_1 = (1+o_x(1))\frac{1}{2^r}\sum_{p_1p_2\cdots p_{2r}\le x}\frac{1}{p_1p_2\cdots p_{2r}}\left\{\sum_{Y_x\le q\le x_2^2}\frac{1}{(q-1)^2}\right\}^r. \tag{3.9}$$

Since

$$\sum_{Y_x\le q\le x_2^2}\frac{1}{(q-1)^2} = (1+o_x(1))\frac{1}{Y_x\log Y_x} = (1+o_x(1))\frac{2\alpha}{x_2^2},$$

and

$$\sum_{p_1\cdots p_{2r}\le x}\frac{1}{p_1\cdots p_{2r}}=(1+o_x(1))x_2^{2r},$$

we obtain that

$$\frac{1}{x}\sum\nolimits_1=(1+o_x(1))\alpha^r, \tag{3.10}$$

and so

$$\frac{1}{x}\sum_{n\le x}\tau_r(n)=(1+o_x(1))\alpha^r\quad(x\to\infty)$$

uniformly as $\alpha=\alpha_x\in[c_1,c_2],\ 0<c_1<c_2<\infty$.

By the Frechet–Shohat theorem and that $\frac{\alpha^r}{r!}$ are the factorial moments of the Poisson-distribution, furthermore taking into consideration (1.26), we obtain (1.27).

The proof of (1.28) is similar, somewhat more complicated.

Let $r\ge 1$ be fixed. Count those primes $p\le x$ for which there exists such a couple of primes $\pi<\pi'$ for which $\pi\pi'\mid p-1$ and $\pi\equiv 1 \pmod q$, $\pi'\equiv 1 \pmod q$, $q>Y_x$, furthermore $\pi'>x^{1/4r}$. We shall apply Lemma 2. We write $p-1$ as $a\pi\pi'$. Let a,π,q be fixed, $\pi\equiv 1\pmod q$. Since $\pi'>x^{1/4r}$, therefore $a\pi<x^{1-1/4r}$. We have

$$\#\{p\le x\mid p-1=a\pi\pi';\ p,\pi'\in\mathcal{P},\ p'\equiv 1\pmod q\}\le c\frac{x}{a\pi q\log^2\frac{x}{a\pi q}}.$$

Let us sum over $q<x^{1/8r}$, a, $\pi\equiv 1\pmod q$. Since $a\pi q\le x^{1-1/8r}$, therefore this sum is

$$\le\sum_{q\ge Y_x}\frac{c(\operatorname{li}x)x_2}{q^2}=o_x(1)\operatorname{li}x.$$

The contribution of those π,π' for which $q\ge x^{1/8r}$ is

$$\le\sum_{q\ge x^{1/8r}}\ \sum_{\pi\pi'\le x}\left[\frac{x}{\pi\pi'}\right]\le xx_2^2\sum_{q\ge x^{1/8r}}\frac{1}{q^2}=o(\operatorname{li}x).$$

Let

$$\tilde\Delta_Y(n)=\sum_{Y<q}\ \sum_{\substack{p_1p_2\mid n\\ p_1<p_2<x^{1/4r}}}f_q(p_1,p_2).$$

By using the Brun–Titchmarsh inequality (Lemma 8), we obtain that

$$\frac{1}{\operatorname{li}x}\#\{p\le x\mid\tilde\Delta_{x_2^2}(p-1)\ne 0\}=o(\operatorname{li}x).$$

Let $\tilde{\Delta}_Y^*(n) = \tilde{\Delta}_Y(n) - \tilde{\Delta}_{x_2^2}(n)$, and

$$\tilde{\tau}_r(n) = \tilde{\Delta}_Y^*(n)(\tilde{\Delta}_Y^*(n) - 1)\cdots(\tilde{\Delta}_Y^*(n) - (r-1)). \tag{3.11}$$

Let $\tilde{\tau}_r(n) = \tilde{\tau}_r^{(1)}(n) + \tilde{\tau}_r^{(2)}(n)$. Arguing as earlier, we deduce that

$$\sum_{p\le x} \tau_r^{(2)}(p-1) = o(\operatorname{li} x),$$

and that

$$\sum_{p\le x} \tau_r^{(1)}(p-1) = \sum_{\substack{\pi_j,\pi_j',q_j\\ \pi_j<\pi_j'<x^{1/8r}}} \pi(x, \pi_1\pi_1'\dots\pi_r\pi_r', 1).$$

By using the Bombieri–Vinogradov theorem (Lemma 3) we can continue the proof as we did in the proof of (1.27).

Now we prove Theorem 11.

It is clear that $\Delta_{Y_x}(n) \ge \kappa_{Y_x}(n)$. It is enough to prove that

$$x^{-1}\#\{n\le x \mid \kappa_{Y_x}(n) \ne \Delta_{Y_x}(n)\} \to 0 \quad (x\to\infty), \tag{3.12}$$

and that

$$\frac{1}{\operatorname{li} x}\#\{p\le x \mid \kappa_{Y_x}(p-1) \ne \Delta_{Y_x}(p-1)\} \to 0 \quad (x\to\infty). \tag{3.13}$$

If $\kappa_{Y_x}(n) \ne \Delta_{Y_x}(n)$, then there exists $q > Y_x$ and $\pi_1 < \pi_2 < \pi_3$, $\pi_j \in \mathcal{P}$, $q \mid \pi_j - 1$ $(j = 1,2,3)$ such that $\pi_1\pi_2\pi_3 \mid n$. Thus (3.12) is less than

$$\sum_{q>Y} \sum_{\substack{\pi_1\pi_2\pi_3\\ q\mid\pi_j-1}} \frac{x}{\pi_1\pi_2\pi_3} \ll x\cdot x_2^3 \sum_{q>Y_x}\frac{1}{q^3} \ll \frac{xx_2^3}{Y_x^2\log Y_x} = o(x).$$

(3.14) can be proved similarly. We have to overestimate the size of those $p\le x$ for which there exists $q > Y_x$ and primes $\pi_1 < \pi_2 < \pi_3$ such that $\pi_1\pi_2\pi_3 \mid p-1$, and $q \mid \pi_j - 1$ $(j=1,2,3)$.

We can drop the contribution of those primes $p \le x$ for which $q > x_2^2$, say. Now we may assume that $q \le x_2^2$. By using the Brun–Titchmarsh inequality, we can drop also the contribution of those primes p for which $\pi_1\pi_2\pi_3 < x^{1-\delta}$, where δ is a fixed positive constant. It remains the case when $p-1 = a\pi_1\pi_2\pi_3$, $\pi_1\pi_2\pi_3 \ge x^{1-\delta}$, $\pi_j \equiv 1 \pmod q$, $\pi_j \in [Y_x, x_2^2]$. From Lemma 5 we obtain that the number of these primes is $o(\operatorname{li} x)$.

4. Proof of Theorem 12

Let $e(n)=\dfrac{\varphi(n)}{n}$, $\log\dfrac{1}{e(n)}=t(n)=\sum_{q\mid n}\log\dfrac{1}{1-\frac{1}{q}}$.

Let $\delta_x\to 0$ slowly, $t(n)=t_1(n)+t_2(n)+t_3(n)+t_4(n)$ where

$$t_1(n)=\sum_{\substack{q\mid n\\ q<x_2^{1-\delta_x}}} t(q);\quad t_2(n)=\sum_{\substack{x_2^{1-\delta_x}<q<x_2^{1+\delta_x}\\ q\mid n}} t(q),$$

$$t_3(n)=\sum_{\substack{q\mid n\\ x_2^{1+\delta}<q<x_1}} t(q);\quad t_4(n)=\sum_{\substack{q>x_1\\ q\mid n}} t(q).$$

It is clear that $\max\limits_{n\le x} t_2(n)=o_x(1)$, $\max\limits_{n\le x} t_4(n)=o_x(1)$.

By using sieve theorems one can prove that for all but $o(\operatorname{li} x)$ of primes $p\le x$, $q\mid\varphi(p-1)$ holds for all $q<x_2^{1-\delta_x}$, if $\delta_x\to 0$ sufficiently slowly. This implies that $t_1(p-1)=t_1(p-1-\varphi(p-1))$ for all but $o(\pi(x))$ of primes $p\le x$.

Since

$$(4.1)\quad \sum_{p\le x} t_3(p-1)\ll \sum_{x_1\ge q>x_2^{1+\delta_x}} (1/q)\ \pi(x,q,1)\ll \operatorname{li} x\cdot \sum_{q>x_2^{1+\delta_x}} 1/q^2=o(\operatorname{li} x)$$

we obtain that $t_3(p-1)=o_x(1)$ holds for all but $o(\pi(x))$ primes $p\le x$.

Now we shall prove that $t_3(p-1-\varphi(p-1))=o_x(1)$ holds for all but $o(\pi(x))$ of primes $p\le x$.

Let us write each $p-1$ as Qm, where Q is the largest prime factor of $p-1$. The size of those $p\le x$ for which $P(p-1)<x^{\delta_x}$, or $P(p-1)>x^{1-\delta_x}$ is $o(\operatorname{li} x)$. This is wellknown, easy consequence of sieve theorems. We shall drop all these primes. Starting from (4.1) it is enough to prove that

$$\sum_{\substack{p\le x\\ P(p-1)\in\left[x^{\delta_x},x^{1-\delta_x}\right]}} t_3^*(p-1-\varphi(p-1))=o(\operatorname{li} x),$$

where

$$t_3^*(p-1-\varphi(p-1))=\sum_{\substack{p-1-\varphi(p-1)\equiv 0(q)\\ q\nmid p-1}} 1/q.$$

Let $Q\in\mathcal{P}$, $S_Q=\{p\le x,\ p-1=Qm,\ P(p-1)=Q\}$. Observe that if $p-1=Qm$, $q\mid p-1-\varphi(p-1)$, then $Q(m-\varphi(m))+\varphi(m)\equiv 0 \pmod q$. If $q\mid m-\varphi(m)$, then the above equation has a solution Q only if $q\mid\varphi(m)$, and so if $q\mid m$. Such kind of q's are excluded in t_3^*.

Hence

$$\begin{aligned}\sum := &\sum_{P(p-1)\in[x^{\delta_x},x^{1-\delta_x}]} t_3^*(p-1-\varphi(p-1)) \ll \\ &\ll \sum_{x_1\ge q>x_1^{1+\delta_x}} \frac{1}{q} \sum_{\substack{m\le x^{1-\delta_x}\\ q\nmid m}} \#\{Q\in\mathcal{P}, Qm\le \\ &\qquad \le x, Q(m-\varphi(m))+\varphi(m)\equiv O(q)\} \ll \\ &\ll \sum_{x_2^{1+\delta_x}\le q<x_1} \frac{1}{q} \sum_{\substack{m\le x_2^{1-\delta_x}\\ q\nmid m}} \#\{p,Q\in\mathcal{P},\ p=Qm+1,\ Q(m-\varphi(m))+ \\ &\quad +\varphi(m)\equiv 0 \pmod q\}.\end{aligned}$$

Let us apply Lemma 1 with substituting in it $x\to\frac{x}{n}$, $p\to Q$, $k\to q$. We have

$$\sum \ll \sum_{x_2^{1+\delta_x}<q<x_1} \frac{1}{q^2} \sum_{m<x^{1-\delta_x}} \frac{x}{m\log^2\frac{x}{mq}}.$$

The right hand side is clearly $o(\operatorname{li} x)$.

We are almost ready. Let $e_j(n):=e^{t_j(n)}$. Then $e(n)=e_1(n)e_2(n)e_3(n)e_4(n)$. We have to consider

$$u_{p-1}:=e(p-1-\varphi(p-1))-e(p-1).$$

We proved that $e_j(p-1-\varphi(p-1))=1+o_x(1)$, $e_j(p-1)=1+o_x(1)$ hold for all but $o(\operatorname{li} x)$ primes $p\le x$, for $j=2,3,4$, and claimed that $e_1(p-1)= = e_1(p-1-\varphi(p-1))$ is satisfied for all but $o(\operatorname{li} x)$ primes $p\le x$.

The proof of the theorem is completed. ■

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