

MEAN VALUES OF MULTIPLICATIVE FUNCTIONS ON THE SET OF $\mathcal{P}_k + 1$, WHERE \mathcal{P}_k RUNS OVER THE INTEGERS HAVING k DISTINCT PRIME FACTORS

L. Germán (Paderborn, Germany)

Dedicated to the 60th anniversary of Professor Antal Jári

Abstract. We investigate the limit behaviour of

$$\sum_{\substack{n \leq x \\ n \in \mathcal{P}_k}} g(n+1)$$

as x tends to infinity where g is multiplicative with values in the unit disc and \mathcal{P}_k runs over the integers having k distinct prime factors. We let k vary in the range $2 \leq k \leq \epsilon(x) \log \log x$ where $\epsilon(x)$ is an arbitrary function tending to zero as x tends to infinity.

Throughout this work n denotes a positive integer and $P(n)$, $p(n)$ denote the largest and the smallest prime factors of n , respectively. p, q with or without suffixes will always denote prime numbers. As usual, the number of primes up to x will be denoted by $\pi(x)$, and $\log_k x := \log(\log_{k-1} x)$ for all positive integers k where $\log_1 x = \log x$ means the natural logarithm of x . If

$$(1) \quad n = p_1^{r_1} \cdot p_2^{r_2} \cdots p_k^{r_k}, \quad p_1 < p_2 < \dots < p_k, \quad r_i, i = 1, \dots, k$$

are positive integers, $p_i, i = 1, \dots, k$ are distinct primes then let $\omega(n) := k$. A typical integer n for which $\omega(n) = k$ will be denoted by π_k . We denote the set of integers having k distinct prime factors with \mathcal{P}_k , that is

$$\mathcal{P}_k := \{\pi_k \in \mathbb{N}\}.$$

The set of integers in \mathcal{P}_k up to x is denoted by $\mathcal{P}_k(x)$. We introduce the counting function for the set \mathcal{P}_k in arithmetic progressions. If $(d, l) = 1$ then let

$$\pi_k(x, d, l) = \sum_{\substack{\pi_k \leq x \\ \pi_k \equiv l \pmod{d}}} 1.$$

In the special case $d = l = 1$ we use $\pi_k(x)$ instead of $\pi_k(x, 1, 1)$.

An arithmetical function $g : \mathbb{N} \rightarrow \mathbb{C}$ is said to be *multiplicative* if $g(nm) = g(n)g(m)$ holds for all integers n, m with $(n, m) = 1$. It is called *additive* if $g(nm) = g(n) + g(m)$ for $(n, m) = 1$ and is called *strongly additive* if additionally $g(p^\alpha) = g(p)$ holds for all p and $\alpha \in \mathbb{N}$.

In the middle of the twentieth century Delange did some pioneering work concerning mean value estimations for multiplicative functions on the set \mathbb{N} . One of his results was the following (See [2])

Theorem (Delange). *Let g be a multiplicative function with $|g(n)| \leq 1$, satisfying*

$$\sum_p \frac{1 - \operatorname{Re} g(p)}{p} < \infty.$$

Then

$$\frac{1}{x} \sum_{n \leq x} g(n) = \prod_{p \leq x} \left(1 - \frac{1}{p}\right) \left(1 + \sum_{m \geq 1} \frac{g(p^m)}{p^m}\right) + o(1)$$

as x tends to infinity.

Although this result provides sufficient condition for multiplicative functions to have zero mean value, the full description of such multiplicative functions was given by Wirsing [12] for real and by Halász [4] for complex multiplicative functions of modulus ≤ 1 . The result of Halász extends Delange's theorem in the following way:

Theorem (Delange, Wirsing, Halász). *Let g be a multiplicative function with $|g(n)| \leq 1$, satisfying*

$$\sum_p \frac{1 - \operatorname{Re} g(p)p^{-i\tau}}{p} < \infty$$

for some real τ . Then

$$\frac{1}{x} \sum_{n \leq x} g(n) = \frac{x^{i\tau}}{1 + i\tau} \prod_{p \leq x} \left(1 - \frac{1}{p}\right) \left(1 + \sum_{m \geq 1} \frac{g(p^m)}{p^{m(1+i\tau)}}\right) + o(1)$$

as x tends to infinity. On the other hand, if there is no such τ then

$$\frac{1}{x} \sum_{n \leq x} g(n) = o(1) \quad (x \rightarrow \infty).$$

Kátai in [7, 8] began to investigate the mean behaviour of multiplicative functions on the set of shifted primes. Through the contribution of Hildebrand [6] and Timofeev [11] it turned out that the situation is basically different from the case of the whole set of natural numbers. Their result is

Theorem (Kátai, Hildebrand, Timofeev). *Let g be a multiplicative function with $|g(n)| \leq 1$ and suppose that there are a real τ and a primitive character χ_d modulo d for some modulus d such that*

$$\sum_p \frac{1 - \operatorname{Re} \chi_d(p) f(p) p^{-i\tau}}{p}$$

converges. Then

$$\begin{aligned} \frac{1}{\pi(x)} \sum_{n \leq x} f(p+1) &= \frac{\mu(d)}{\varphi(d)} \frac{x^{i\tau}}{1+i\tau} \times \\ &\times \prod_{\substack{p \leq x \\ p \nmid d}} \left(1 + \sum_{r \geq 1} \frac{\chi_d(p^r) f(p^r) p^{-ri\tau} - \chi_d(p^{r-1}) f(p^{r-1}) p^{-(r-1)i\tau}}{\varphi(p^r)} \right) + o(1) \end{aligned}$$

as $x \rightarrow \infty$, which is not necessarily $o(1)$ as x tends to infinity, if χ_d is a real character.

The main result of this paper is

Theorem 1. *Let $g(n)$ be a multiplicative function of modulus one, such that there are a primitive character $\chi \pmod{d}$ for some fixed d and a real τ such that*

$$\sum_p \frac{1 - \operatorname{Re} \chi(p) g(p) p^{-i\tau}}{p}$$

converges. Let furthermore $\epsilon(x)$ be an arbitrary function tending to zero as x tends to infinity. Then

$$\begin{aligned} \pi_k(x)^{-1} \sum_{\substack{n \leq x \\ \omega(n)=k}} g(n+1) &= \\ &= \frac{x^{i\tau}}{1+i\tau} \frac{\mu(d)}{\varphi(d)} \prod_{\substack{p \leq x \\ p \nmid d}} \left(1 - \frac{1}{p-1} + \sum_{\alpha \geq 1} \frac{g(p^\alpha) p^{-i\alpha\tau} \chi(p^\alpha)}{p^\alpha} \right) + o(1) \quad (x \rightarrow \infty) \end{aligned}$$

uniformly for all k , if $1 \leq k \leq \epsilon(x) \log \log x$.

We will use the method of [3] since as we deduce the results from the analogue for $D\mathcal{P} + 1$ where \mathcal{P} denotes the set of primes.

Let

$$M(x, f, D) := \sum_{Dp+1 \leq x} f(Dp+1).$$

Theorem 2. *Let $f(n)$ be a multiplicative function of modulus 1. Let furthermore d be a positive integer. Suppose that there is a real τ such that the series*

$$(2) \quad \sum_p \frac{|\chi(p)f(p)p^{i\tau} - 1|^2}{p}$$

converges for some primitive character $\chi \pmod{d}$. Let $0 < \epsilon < 1/2$. Then

$$\begin{aligned} & \left(\pi \left(\frac{x-1}{D} \right) \right)^{-1} M(x, f, D) = \\ & = \frac{x^{i\tau}}{1+i\tau} \frac{\mu(d)}{\varphi(d)} \prod_{\substack{p \leq x \\ p \nmid dD}} \left(1 - \frac{1}{p-1} + \sum_{\alpha \geq 1} \frac{f(p^\alpha)p^{-i\alpha\tau}\chi(p^\alpha)}{p^\alpha} \right) + o(1) \quad (x \rightarrow \infty) \end{aligned}$$

holds uniformly for all $x > 2$ and $D \leq x^{1/2-\epsilon}$ with $(d, D) = 1$.

As an application of Theorem 2 we are able to analyze the mean behavior of multiplicative functions on the set $\mathcal{P}_k + 1$ in some cases. We need the following

Lemma 1. *Let $\epsilon(x) \rightarrow 0$ as $x \rightarrow \infty$. Then there exist sequences $y_x \rightarrow \infty$, $\delta_x \rightarrow 0$ as $x \rightarrow \infty$ such that*

$$(3) \quad P(n) > x^{1-\delta_x}, \quad y_x < p(n), \quad n \text{ is square-free}$$

hold for all but $o(\pi_k(x))$ elements of $\mathcal{P}_k(x)$, uniformly for all

$$2 \leq k \leq \epsilon(x) \log \log x \quad \text{as } x \rightarrow \infty.$$

Proof. The following sets have zero relative density in \mathcal{P}_k .

1. If $A_1 = \{n \in \mathcal{P}_k, n \leq x : \exists p^2 | n\}$, then we have

$$\#A_1 \leq \sum_{\substack{p^\alpha \leq x^{1/2} \\ \alpha \geq 2}} \pi_{k-1} \left(\frac{x}{p^\alpha} \right) + \sum_{\substack{p^\alpha > x^{1/2} \\ \alpha \geq 2}} \frac{x}{p^\alpha} \ll \pi_k(x) \frac{k}{\log \log x} \sum_{\substack{p^\alpha \leq x^{1/2} \\ \alpha \geq 2}} \frac{1}{p^\alpha} + \mathcal{O}(x^{3/4}).$$

Here we used that

$$\frac{\pi_{k-1}(x)}{\pi_k(x)} \sim \frac{k}{\log \log x} (\rightarrow 0) \quad (x \rightarrow \infty)$$

holds uniformly for $2 \leq k \leq \epsilon(x) \log \log x$. This is a direct consequence of the asymptotic estimation

$$(4) \quad \pi_k(x) = \frac{x}{\log x} \frac{\log \log^{k-1} x}{(k-1)!} \left(1 + \mathcal{O} \left(\frac{1}{\log \log x} \right) \right),$$

which is uniform for $1 \leq k \leq \epsilon(x) \log \log x$ (see for example in [9]).

2. If $A_2 = \{n \in \mathcal{P}_k, n \leq x : p(n) < y_x\}$, then we have

$$\#A_2 \leq \sum_{\substack{p^\alpha \leq x^{1/2} \\ p < y_x}} \pi_{k-1} \left(\frac{x}{p^\alpha} \right) + \sum_{\substack{p^\alpha > x^{1/2} \\ \alpha \geq 2}} \frac{x}{p^\alpha} \ll \pi_k(x) \frac{k}{\log \log x} \sum_{p < y_x} \frac{1}{p} + \mathcal{O}(x^{3/4}).$$

By means of these last two steps we can assume that $p(n) > y_x$, and n is square-free. Finally we have

$$\begin{aligned} \sum_{\substack{\pi_k \leq x \\ P(\pi_k) \leq x^{1-\delta_x}}} 1 &\ll \sum_{\pi_k \leq x^{1/2}} 1 + \sum_{\substack{x^{1/2} \leq \pi_k \leq x \\ P(\pi_k) \leq x^{1-\delta_x}}} 1 \ll \\ &\ll x^{1/2} + \frac{1}{\log x} \sum_{\substack{x^{1/2} \leq \pi_k \leq x \\ P(\pi_k) \leq x^{1-\delta_x}}} \log \pi_k \ll \\ &\ll \frac{1}{\log x} \sum_{p \leq x^{1-\delta_x}} \pi_{k-1} \left(\frac{x}{p} \right) \log p + x^{1/2} \ll \\ &\ll \frac{x}{\log x} \frac{\log^{k-2} \log x}{(k-2)!} \sum_{p \leq x^{1-\delta_x}} \frac{\log p}{p \log(x/p)} + x^{1/2} \ll \\ &\ll \frac{1}{\delta_x} \pi_k(x) \frac{k}{\log \log x} \end{aligned}$$

and the proof is finished. ■

Proof of Theorem 1. The case $k = 1$ was proved by Kátai, Hildebrand and Timofeev, and is included in Theorem 2. Therefore we can suppose that $k \geq 2$. Let $U_k(x)$ be the set of those elements of $\mathcal{P}_k(x)$, for which (3) holds true. Let S_x be the set of those π_{k-1} , for which there exists at least one prime $p > P(\pi_{k-1})$ such that $\pi_{k-1}p \in U_k(x)$. Let $p^* = p_{\pi_{k-1}}$ be the smallest p with this property. Then $\pi_{k-1}p \in U_k(x)$ for all $p^* \leq p \leq \frac{x}{\pi_{k-1}}$. Using Lemma 1 we have that $\pi_{k-1} < x^{\lambda_x}$, with an appropriate $\lambda_x \rightarrow 0$, as x tends to infinity. Further,

$$P(\pi_{k-1}) < p, \quad \text{and} \quad p(\pi_{k-1}) > y_x,$$

where $y_x \rightarrow \infty$ as $x \rightarrow \infty$, slowly. We obtain

$$\begin{aligned}
 \sum_{\substack{n \leq x \\ \omega(n)=k}} g(n+1) &= \sum_{\pi_{k-1} \in S_x} \sum_{p_{\pi_{k-1}}^* \leq p \leq \frac{x}{\pi_{k-1}}} g(\pi_{k-1}p+1) + o(\pi_k(x)) = \\
 (5) \quad &= \sum_{\pi_{k-1} \in S_x} M(g, x, \pi_{k-1}) - \sum_{\pi_{k-1} \in S_x} \sum_{p \leq p_{\pi_{k-1}}^*} g(\pi_{k-1}p+1) + o(\pi_k(x))
 \end{aligned}$$

as $x \rightarrow \infty$.

Let

$$\psi(x, D) := \frac{x^{i\tau}}{1+i\tau} \frac{\mu(d)}{\varphi(d)} \prod_{\substack{p \leq x \\ p \nmid dD}} \left(1 - \frac{1}{p-1} + \sum_{\alpha \geq 1} \frac{f(p^\alpha) p^{-i\alpha\tau} \chi(p^\alpha)}{p^\alpha} \right).$$

Note that using Lemma 1 we have $y_x \leq p(\pi_{k-1})$, therefore in our case π_{k-1} and d are coprimes for large x . Furthermore,

$$(6) \quad \sum_{\pi_{k-1} \in S_x} \pi(p_{\pi_{k-1}}^*) \ll x^{1/2} + \sum_{\pi_{k-1} \in S_x} \sum_{P(\pi_{k-1}) < p < p_{\pi_{k-1}}^*} 1$$

which, by the definition of S_x , equals $o(\pi_k(x))$ as x tends to infinity. Thus, the second sum on the most right hand side of (5) is $o(\pi_k(x))$. For the estimation of the first sum here we apply Theorem 2 and we deduce

$$\sum_{\substack{n \leq x \\ \omega(n)=k}} g(n+1) = \sum_{\pi_{k-1} \in S_x} \psi(x, \pi_{k-1}) \pi\left(\frac{x}{\pi_{k-1}}\right) + o(\pi_k(x)) \quad (x \rightarrow \infty).$$

Defining $K(x, D)$ by the identity

$$\psi(x, 1) = \psi(x, D)K(x, D),$$

such that

$$K(x, D) = \prod_{\substack{p \leq x \\ p \nmid D}} \left(1 - \frac{1}{p-1} + \sum_{\alpha \geq 1} \frac{f(p^\alpha) p^{-i\alpha\tau} \chi(p^\alpha)}{p^\alpha} \right)$$

holds, we have that the left hand side of (5) equals

$$\begin{aligned}
 &\psi(x, 1) \sum_{\pi_{k-1} \in S_x} \pi\left(\frac{x}{\pi_{k-1}}\right) + \\
 &+ \sum_{\pi_{k-1} \in S_x} \pi\left(\frac{x}{\pi_{k-1}}\right) \psi(x, \pi_{k-1}) [1 - K(x, \pi_{k-1})] + o(\pi_k(x)) \quad (x \rightarrow \infty).
 \end{aligned}$$

Since $y_x \leq p(\pi_{k-1})$, and since

$$K(x, \pi_{k-1}) = \exp \left[\sum_{\substack{p \leq x \\ p | \pi_{k-1}}} \frac{f(p^\alpha) \chi(p^\alpha) p^{i\tau} - 1}{p} + \mathcal{O} \left(\sum_{\substack{p \leq x \\ p | \pi_{k-1}}} \frac{1}{p^2} \right) \right],$$

the right hand side of (5) equals

$$\psi(x, 1) \sum_{\pi_{k-1} \in S_x} \pi \left(\frac{x}{\pi_{k-1}} \right) + o(1) \sum_{\pi_{k-1} \in S_x} \pi \left(\frac{x}{\pi_{k-1}} \right) + o(\pi_k(x)) \quad (x \rightarrow \infty).$$

By the same argument as in the estimation of (5) and then using (6) again we obtain

$$\pi_k^{-1}(x) \sum_{\pi_{k-1} \in S_x} \pi \left(\frac{x}{\pi_{k-1}} \right) \rightarrow 1 \quad (x \rightarrow \infty)$$

and the assertion follows. ■

In order to show Theorem 2 we need an analogue of the Turán–Kubilius inequality.

Lemma 2. *Let $0 \leq \epsilon < 1$ and let $0 < \theta_x$ be an arbitrary sequence tending to zero as x tends to infinity. Let D be a positive integer, and let $x \geq 2D$. Let h be a real strongly additive function and*

$$h_x(n) = \sum_{\substack{p^\alpha | n \\ p \leq (\frac{x-1}{D})^{1-\theta_x}}} h(p).$$

Then

$$(7) \quad \frac{1}{\pi(\frac{x-1}{D})} \sum_{p \leq (x-1)/D} \left| h_x(Dp+1) - \sum_{\substack{q \leq x \\ q \nmid D}} \frac{h(q)}{\varphi(q)} \right|^2 \ll \frac{1}{\theta_x} \sum_{q \leq x} \frac{|h(q)|^2}{q}$$

uniformly for all x and all $D \leq x^\epsilon$.

Proof. With $x_D := (x-1)/D$ let

$$h_{1,x}(n) := \sum_{\substack{p^\alpha | n \\ p \leq x_D^{1/8}}} h(p) \quad \text{and} \quad h_{2,x}(n) := \sum_{\substack{p^\alpha | n \\ x_D^{1/8} < p \leq x_D^{1-\theta_x}}} h(p).$$

Further, define

$$A(y) := \sum_{\substack{p \leq y \\ q \nmid D}} \frac{h(p)}{\varphi(p)} \quad \text{and} \quad B^2(y) := \sum_{p \leq y} \frac{|h(p)|^2}{p}.$$

The left hand side of (7) is $\ll \Sigma_1 + \Sigma_2 + \Sigma_3$, where

$$\begin{aligned}\Sigma_1 &= \frac{1}{\pi(x_D)} \sum_{p \leq x_D} |h_{1,x}(Dp+1) - A(x_D^{1/8})|^2, \\ \Sigma_2 &= \frac{1}{\pi(x_D)} \sum_{p \leq x_D} |h_{2,x}(Dp+1)|^2, \\ \Sigma_3 &= \frac{1}{\pi(x_D)} \sum_{p \leq x_D} |A(x) - A(x_D^{1/8})|^2.\end{aligned}$$

Using the Cauchy–Schwarz inequality we have

$$\Sigma_3 \ll \left(\sum_{x_D^{1/8} \leq p \leq x} \frac{1}{p} \right) \left(\sum_{x_D^{1/8} \leq p \leq x} \frac{|h(p)|^2}{p} \right) \ll \sum_{p \leq x} \frac{|h(p)|^2}{p}.$$

In order to estimate Σ_2 note that a positive integer, $n \leq x$, can have at most a bounded number of distinct prime divisors $q > x_D^{1/8}$. Thus, using the Brun–Titchmarsh inequality (Theorem I.4.9 in [10]) we deduce

$$\begin{aligned}\Sigma_2 &= \frac{1}{\pi(x_D)} \sum_{p \leq x_D} \left| \sum_{q|Dp+1} h_{2,x}(q) \right|^2 \ll \frac{1}{\pi(x_D)} \sum_{\substack{q \leq x_D^{1-\theta_x} \\ q \nmid D}} |h(q)|^2 \pi(x_D, q, l_{D,q}) \ll \\ &\ll \frac{x_D}{\pi(x_D)} \sum_{q \leq x_D^{1-\theta_x}} \frac{|h(q)|^2}{q \log(\frac{x_D}{q})} \ll \\ &\ll \frac{1}{\theta_x} \sum_{q \leq x_D^{1-\theta_x}} \frac{|h(q)|^2}{q}.\end{aligned}$$

Here we used that if $Dp+1 = aq$ then there exists a unique residue class $l_{D,q} \pmod{q}$ such that $p \equiv l_{D,q} \pmod{q}$ holds.

It remains to estimate Σ_1 . Performing the multiplications we obtain

$$\sum_{p \leq x_D} \left| h_{1,x}(Dp+1) - A(x_D^{1/8}) \right|^2 = S_1 - 2S_2 + S_3,$$

where

$$\begin{aligned}S_1 &= \sum_{p \leq x_D} |h_{1,x}(Dp+1)|^2, \\ S_2 &= A(x_D^{1/8}) \sum_{p \leq x_D} h_{1,x}(Dp+1), \\ S_3 &= A(x_D^{1/8})^2 \pi(x_D).\end{aligned}$$

Further,

$$(8) \quad S_1 = \sum_{p \leq x_D} \left(\sum_{q | Dp+1} h_{1,x}(q) \right)^2 = \sum_{\substack{q \leq x_D \\ q_1 \uparrow D}} h_{1,x}^2(q) \pi(x_D, q, l_{D,q}) + \\ + \sum_{\substack{q_1, q_2 \leq x_D \\ q_1 \neq q_2, q_1 \uparrow D, q_2 \uparrow D}} h_{1,x}(q_1) h_{1,x}(q_2) \pi(x_D, q_1 q_2, l_{D, q_1 q_2}).$$

Since $h_{1,x}(q) = 0$ for $q > x_D^{1/8}$, the Brun–Titchmarsh theorem is applicable and we deduce that the first term on the right hand side of (8) does not exceed $c\pi(x_D)B^2(x)$.

The second term on the right hand side of (8) equals

$$(9) \quad \sum_{\substack{q_1, q_2 \leq x_D^{1/8} \\ q_1 \neq q_2, q_1 \uparrow D, q_2 \uparrow D}} h_{1,x}(q_1) h_{1,x}(q_2) \frac{\pi(x_D)}{\varphi(q_1 q_2)} + \\ + \sum_{\substack{q_1, q_2 \leq x_D^{1/8} \\ q_1 \neq q_2, q_1 \uparrow D, q_2 \uparrow D}} h_{1,x}(q_1) h_{1,x}(q_2) \left\{ \pi(x_D, q_1 q_2, l_{D, q_1 q_2}) - \frac{\pi(x_D)}{\varphi(q_1 q_2)} \right\}.$$

Let T_1, T_2 be the sums in (9). We have

$$\frac{T_1}{\pi(x_D)} = A^2(x_D^{1/8}) - \sum_{\substack{q_1 \leq x_D^{1/8} \\ q_1 \uparrow D}} \frac{h_{1,x}^2(q_1)}{\varphi^2(q_1)} = A^2(x_D^{1/8}) + \mathcal{O}(B^2(x)).$$

For T_2 we use the Cauchy–Schwarz inequality to obtain

$$T_2^2 \ll \sum_{\substack{q_1, q_2 \leq x_D^{1/8} \\ q_1 \neq q_2, q_1 \uparrow D, q_2 \uparrow D}} \frac{h_{1,x}^2(q_1)}{\varphi(q_1)} \frac{h_{1,x}^2(q_2)}{\varphi(q_2)} \times \\ \times \sum_{\substack{q_1, q_2 \leq x_D^{1/8} \\ q_1 \neq q_2, q_1 \uparrow D, q_2 \uparrow D}} \varphi(q_1 q_2) \left\{ \pi(x_D, q_1 q_2, l_{D, q_1 q_2}) - \frac{\pi(x_D)}{\varphi(q_1 q_2)} \right\}^2 \ll \\ \ll B^4(x) \sum_{\substack{q_1, q_2 \leq x_D^{1/8} \\ q_1 \neq q_2, q_1 \uparrow D, q_2 \uparrow D}} \varphi(q_1 q_2) \left\{ \pi(x_D, q_1 q_2, l_{D, q_1 q_2}) - \frac{\pi(x_D)}{\varphi(q_1 q_2)} \right\}^2.$$

Using the Brun–Titchmarsh inequality

$$T_2^2 \ll B^4(x) \pi(x_D) \sum_{\substack{q_1, q_2 \leq x_D^{1/8} \\ q_1 \neq q_2, q_1 \uparrow D, q_2 \uparrow D}} \left| \pi(x_D, q_1 q_2, l_{D, q_1 q_2}) - \frac{\pi(x_D)}{\varphi(q_1 q_2)} \right|,$$

and an application of the Bombieri–Vinogradov theorem (Chapter 28. in [1]) shows

$$T_2 \ll B^2(x) \frac{\pi(x_D)}{\log^A x_D},$$

where $A > 0$ is an arbitrary large constant. Since by the Cauchy–Schwarz inequality we have

$$A(y) = \sum_{\substack{q \leq y \\ q \nmid D}} \frac{h(q)}{\varphi(q)} \ll \left(\sum_{q \leq y} \frac{h^2(q)}{q} \right)^{1/2} \log \log^{1/2} y \ll B(y) \log \log^{1/2} y,$$

for $y \geq e^2$, in a similar way as in the estimation of T_2 we deduce

$$\begin{aligned} S_2 - A^2(x_D^{1/8})\pi(x_D) &\ll A(x_D^{1/8})B(x) \frac{\pi(x_D)}{\log^A x_D} \ll \\ &\ll B^2(x) \log \log x_D \frac{\pi(x_D)}{\log^A x_D} \ll \\ &\ll B^2(x)\pi(x_D), \end{aligned}$$

and the proof is finished. ■

Lemma 3. *Let D, q be two coprime positive integers and let $(l_{D,q} =)l_D$ be the unique residue class satisfying $l_D \equiv 1 \pmod{q}$. Let further $0 < \epsilon < 1/2$ and $x_D := (x - 1)/D$ whenever $x > 2$ and let $a > \frac{1-2\epsilon}{1+2\epsilon}$. Then*

$$(10) \quad \sum_{\substack{q > x_D^a \\ q \text{ prime}, q \nmid D}} q\pi^2(x_D, q, l_D) \ll \pi^2(x_D)$$

holds uniformly for all $x > 2$ and $D \leq x^{1/2-\epsilon}$. The constant implied by \ll depends on a .

Proof. The sum on the left hand side of (10) equals

$$(11) \quad \sum_{q > x_D^a} q \sum_{\substack{a_1 q = Dp_1 + 1 \\ a_1 \leq x/q}} \sum_{\substack{a_2 q = Dp_2 + 1 \\ a_2 \leq x/q}} 1 \leq 2x \sum_{\substack{a_1 \leq x x_D^{-a} \\ (a_1, D) = 1}} \frac{1}{a_1} \sum_{\substack{a_2 < a_1 \\ (a_2, D) = 1}} \sum_{\substack{a_1 q = Dp_1 + 1 \leq x \\ a_2 q = Dp_2 + 1 \leq x}} 1.$$

Denote the inner sum by $(\Sigma(a_1, a_2) =) \Sigma$. It is nonempty only if $a_1 \equiv a_2 \pmod{D}$. Suppose, a_1, a_2 is fixed and

$$q = Dn + l_{a_1 D}.$$

Then

$$Dp_1 + 1 = a_1Dn + a_1l_{a_1D}, \quad Dp_2 = a_2Dn + a_2l_{a_1D}.$$

Thus, the primes we want to count in Σ satisfy

$$q = Dn + l_{a_1D}, \\ p_1 = a_1n + t_{Da_1}, \quad p_2 = a_2n + t_{Da_2},$$

where

$$a_1l_{a_1D} - Dt_{Da_1} = 1 \quad \text{and} \quad a_2l_{a_1D} - Dt_{Da_2} = 1.$$

It follows,

$$\Sigma \ll \#\left\{n \leq \frac{x_D}{a_1} : q = Dn + l_{a_1D}, p_1 = a_1n + t_{Da_1}, p_2 = a_2n + t_{Da_2} \text{ primes}\right\}.$$

Let

$$E = Da_1a_2(a_1 - a_2),$$

and let $\varrho(p)$ be the number of solutions of

$$(Dn + l_{a_1D})(a_1n + t_{Da_1})(a_2n + t_{Da_2}) \equiv 0 \pmod{p}.$$

Since $E \leq x_D^A$ for some appropriate $A > 0$, by Theorem 5.7 of [5]

$$\Sigma \ll \frac{x_D}{a_1 \log^3 \frac{x_D}{a_1}} \prod_p \left(1 - \frac{\varrho(p) - 1}{p - 1}\right) \left(1 - \frac{1}{p}\right)^{-2}.$$

Noting that $(D, a_1a_2) = 1$ we have

$$\varrho(p) = \begin{cases} 1 & \text{if } p|D, p|\frac{a_1-a_2}{D} \text{ or } p|a_1, p|a_2 \\ 2 & \text{if } p|D, p \nmid \frac{a_1-a_2}{D} \text{ or } p|a_1a_2, p \nmid (a_1, a_2) \\ 3 & \text{otherwise.} \end{cases}$$

Now, making use of the inequality $\log(1 - z) = 1 + z + \mathcal{O}(z^2)$ which holds uniformly for all real numbers $|z| \leq 1/2$ we obtain

$$\prod_p \left(1 - \frac{\varrho(p) - 1}{p - 1}\right) \left(1 - \frac{1}{p}\right)^{-2} \ll \\ \ll \prod_{p|D} \left(1 + \frac{1}{p}\right) \prod_{p|\frac{a_1-a_2}{D}} \left(1 + \frac{1}{p}\right) \prod_{p|a_1} \left(1 + \frac{2}{p}\right) \prod_{p|a_2} \left(1 + \frac{2}{p}\right).$$

Thus, the right hand side of (11) is at most

$$c \frac{x^2}{D} \prod_{p|D} \left(1 + \frac{1}{p}\right) \sum_{\substack{a_1 \leq xx_D^{-a} \\ (a_1, D)=1}} \frac{1}{a_1^2 \log^3 \frac{x_D}{a_1}} \prod_{p|a_1} \left(1 + \frac{2}{p}\right) \times \\ \times \sum_{\substack{a_2 \leq a_1 \\ a_1 \equiv a_2 \pmod{D}}} \prod_{p|\frac{a_1-a_2}{D}} \left(1 + \frac{1}{p}\right) \prod_{p|a_2} \left(1 + \frac{2}{p}\right).$$

Since $|ab| \leq a^2 + b^2$ holds for all real a, b we deduce

$$\sum_{\substack{a_2 \leq a_1 \\ a_1 \equiv a_2 \pmod{D}}} \prod_{p|\frac{a_1-a_2}{D}} \left(1 + \frac{1}{p}\right) \prod_{p|a_2} \left(1 + \frac{2}{p}\right) \ll \\ \ll \sum_{\substack{a_2 \leq a_1 \\ a_1 \equiv a_2 \pmod{D}}} \left\{ \sum_{d|\frac{a_1-a_2}{D}} \frac{2^{\omega(d)} \mu^2(d)}{d} + \sum_{d|a_2} \frac{4^{\omega(d)} \mu^2(d)}{d} \right\} \ll \\ \ll \sum_{d \leq \frac{a_1}{D}} \frac{2^{\omega(d)} \mu^2(d)}{d} \sum_{\substack{a_2 \leq a_1 \\ a_2 \equiv a_1 \pmod{D} \\ \frac{a_1-a_2}{D} \equiv 0 \pmod{d}}} 1 + \sum_{\substack{d \leq a_1 \\ (d, D)=1}} \frac{4^{\omega(d)} \mu^2(d)}{d} \sum_{\substack{a_2 \leq a_1 \\ a_2 \equiv 0 \pmod{d} \\ a_1 \equiv a_2 \pmod{D}}} 1 \ll \\ \ll \frac{a_1}{D}.$$

Since $a > \frac{1-2\epsilon}{1+2\epsilon}$ and $a_1 \leq xx_D^{-a}$ we have $\log \frac{x_D}{a_1} \gg_a \log x \gg \log x_D$. Further,

$$\sum_{\substack{a_1 \leq xx_D^{-a} \\ (a_1, D)=1}} \frac{1}{a_1} \prod_{p|a_1} \left(1 + \frac{2}{p}\right) = \prod_{\substack{p \leq xx_D^{-a} \\ p \nmid D}} \left(1 + \frac{1}{p} \left(1 + \frac{2}{p}\right)\right) \ll \\ \ll \prod_{p \leq xx_D^{-a}} \left(1 + \frac{1}{p}\right) \prod_{p|D} \left(1 + \frac{1}{p}\right)^{-1} \ll \\ \ll \log x_D \prod_{p|D} \left(1 + \frac{1}{p}\right)^{-1}.$$

Thus, the right hand side of (11) does not exceed

$$c \frac{x_D^2}{\log^3 x_D} \prod_{p|D} \left(1 + \frac{1}{p}\right) \sum_{\substack{a_1 \leq xx_D^{-a} \\ (a_1, D)=1}} \frac{1}{a_1} \prod_{p|a_1} \left(1 + \frac{2}{p}\right) \ll \pi^2(x_D),$$

which proves the assertion. ■

Proof of Theorem 2. First suppose that $\tau = 0$. We set $r = \log \log x$, and $x_D = \frac{x-1}{D}$. Let

$$K_D(x) := \{Dp + 1 \leq x : p \text{ prime}\}.$$

We have

$$(12) \quad \begin{aligned} & \#\{n \in K_D(x) \mid \exists q^2 \mid n, q > y\} \leq \\ & \leq \sum_{y < q < (\frac{x-1}{D})^a} \pi\left(\frac{x-1}{D}, q^2, l_q\right) + \frac{x-1}{D} \sum_{q \geq (\frac{x-1}{D})^a} \frac{1}{q^2} = \delta(y)\pi\left(\frac{x-1}{D}\right), \end{aligned}$$

where $\delta(y) \rightarrow 0$ ($y \rightarrow \infty$). Let f^* be a multiplicative function defined by

$$f^*(p^\alpha) = \begin{cases} f(p^\alpha), & \text{if } p \leq r \\ f(p), & \text{if } r < p \leq x_D^{1-\vartheta} \\ \bar{\chi}(p), & \text{otherwise.} \end{cases}$$

Since $\chi(q) \neq 0$ for $q > d$, there exists a function $g(q) \in [-\pi, \pi)$ such that $f(q) = \chi(q)e^{ig(q)}$. By (12)

$$\begin{aligned} & \left| \sum_{Dp+1 \leq x} \{f(Dp+1) - f^*(Dp+1)\} \right| \leq \\ & \sum_{\substack{Dp+1 \leq x \\ \exists q^2 \mid Dp+1, q > r}} 1 + \sum_{\substack{Dp+1 \leq x \\ \exists q \mid Dp+1, q > x_D^{1-\vartheta}}} |e^{i\tilde{g}(Dp+1)} - 1| \leq \\ & \leq \sum_{\substack{Dp+1 \leq x \\ \exists q \mid Dp+1, q > x_D^{1-\vartheta}}} |e^{i\tilde{g}(Dp+1)} - 1| + o(\pi(x_D)) \quad (x \rightarrow \infty), \end{aligned}$$

where

$$\tilde{g}(p^\alpha) = \begin{cases} g(p), & \text{if } x_D^{1-\vartheta} < q, \quad \alpha = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \sum_{\substack{Dp+1 \leq x \\ \exists q \mid Dp+1, q > x_D^{1-\vartheta}}} |e^{i\tilde{g}(Dp+1)} - 1| & \leq \sum_{\substack{Dp+1 \leq x \\ \exists q \mid Dp+1, q > x_D^{1-\vartheta}}} |\tilde{g}(Dp+1)| \\ & \leq \sum_{x_D^{1-\vartheta} < q \leq x} |g(q)|\pi(x_D, q, t_D), \end{aligned}$$

where $(t_{D,q} =)t_D$ is the unique residue class satisfying

$$Dt_D \equiv -1 \pmod{q}.$$

Applying the Cauchy–Scwarz inequality then using Lemma 3 we obtain

$$\begin{aligned} & \sum_{x_D^{1-\vartheta_x} < q \leq x} |g(q)|\pi(x_D, q, t_D) \ll \\ & \ll \left(\sum_{x_D^{1-\vartheta_x} < q \leq x} \frac{g(q)^2}{q} \right)^{1/2} \left(\sum_{x_D^{1-\vartheta_x} < q \leq x} q\pi^2(x_D, q, t_D) \right)^{1/2} \ll \\ & \ll \pi(x_D) \left(\sum_{x_D^{1-\vartheta_x} < q \leq x} \frac{g(q)^2}{q} \right)^{1/2}. \end{aligned}$$

Noting that

$$|g(q)|^2 \ll |f(q)\bar{\chi}(q) - 1|^2,$$

by (2) we obtain

$$(13) \quad \sum_{Dp+1 \leq x} \{f(Dp+1) - f^*(Dp+1)\} = o(\pi(x_D)) \quad (x \rightarrow \infty).$$

Let f_r be a further multiplicative function defined by

$$f_r(p^\alpha) = \begin{cases} f(p^\alpha), & \text{if } p \leq r \\ \bar{\chi}(p), & \text{if } r < p. \end{cases}$$

Next we give an alternative representation of $M(x, f_r, D)$. It can be written as follows

$$(14) \quad \sum_{Dp+1 \leq x} f_r(Dp+1) = \sum_{\substack{m \leq x+1 \\ P(m) \leq r \\ (D,m)=1}} f(m) \sum_{\substack{p \leq x_D \\ p \equiv 1_D \pmod{m} \\ (\frac{Dp+1}{m}, \mathcal{P}(r))=1}} \bar{\chi}\left(\frac{Dp+1}{m}\right) + Err(x, r),$$

where

$$\mathcal{P}(r) := \prod_{p \leq r} p,$$

and $(l_{D,m} =)l_D$ is the unique residue class satisfying

$$Dl_D \equiv -1 \pmod{m},$$

and by (12)

$$Err(x, r) \ll \sum_{\substack{Dp+1 \leq x \\ \exists q^2 | Dp+1, r < q}} 1 = o(\pi(x_D)) \quad (x \rightarrow \infty).$$

Furthermore, $(\frac{Dp+1}{m}, \mathcal{P}(r)) = 1$. Hence, $\frac{Dp+1}{m}$ is always odd and there is at most one prime p satisfying $Dp+1 = m\frac{Dp+1}{m}$ if D and m have the same parity. The contribution of these integers to the sum on the right hand side of (14) is at most

$$\sum_{\substack{m \leq x \\ P(m) \leq r}} 1 \ll x \exp\left(-\frac{1}{2} \frac{\log x}{\log r}\right),$$

which inequality is well known in number theory (Theorem III.5.1 in [10]). The sum over the integers $m > e^r$ on the right hand side of (14) is at most

$$\sum_{\substack{e^r \leq m \leq \sqrt{x} \\ P(m) \leq r}} \pi(x_D, m, l_D) + \sum_{\substack{\sqrt{x} \leq m \leq x \\ P(m) \leq r}} \frac{x_D}{m} = \Sigma_1 + \Sigma_2.$$

Using the Brun–Titchmarsh theorem we obtain

$$\begin{aligned} \Sigma_1 &\ll \pi(x_D) \sum_{\substack{e^r \leq m \leq \sqrt{x} \\ P(m) \leq r}} \frac{1}{\varphi(m)} \ll \frac{\pi(x_D)}{r} \sum_{\substack{m \leq x \\ P(m) \leq r}} \frac{\log m}{\varphi(m)} \ll \\ &\ll \frac{\pi(x_D)}{r} \sum_{p \leq r} \sum_{\alpha} \log p^\alpha \sum_{\substack{mp^\alpha \leq x \\ P(m) \leq r, (m,p)=1}} \frac{1}{\varphi(p^\alpha m)} \ll \\ &\ll \frac{\pi(x_D) \log r}{r} \sum_{p \leq r} \frac{\log p}{p} \ll \\ &\ll \pi(x_D) \frac{\log^2 r}{r}. \end{aligned}$$

Further, using the inequality $|\log(1 - y) - y| \leq 2y^2$, which is valid for all real

y with $|1 - y| \leq 1/2$ we have

$$\begin{aligned} \Sigma_2 &\ll x_D x^{-1/8} \sum_{\substack{m \leq x \\ P(m) \leq r}} \frac{1}{m^{3/4}} \ll x_D x^{-1/4} \prod_{p \leq r} \left(1 - \frac{1}{p^{3/4}}\right)^{-1} \ll \\ &\ll x_D x^{-1/4} \exp\left(\sum_{p \leq r} \frac{1}{p^{3/4}}\right) \ll \\ &\ll x_D x^{-1/4} e^r. \end{aligned}$$

The inner sum on the right hand side of (14) equals

$$\begin{aligned} &\sum_{\substack{Dp \leq x \\ Dp \equiv -1 \pmod{m}}} \bar{\chi}_d \left(\frac{Dp+1}{m}\right) \sum_{\delta | (\frac{Dp+1}{m}, \mathcal{P}(r))} \mu(\delta) = \\ &= \sum_{\substack{\delta | \mathcal{P}(r) \\ (\delta, Dd)=1}} \mu(\delta) \sum_{\substack{Dp \leq x \\ Dp+1 \equiv 0 \pmod{\delta m}}} \bar{\chi}_d \left(\frac{Dp+1}{m}\right) = \\ &= \sum_{\substack{\delta | \mathcal{P}(r) \\ (\delta, Dd)=1}} \mu(\delta) \sum_{\substack{b=1 \\ (b,d)=1}}^d \bar{\chi}_d(b) J(x, m, \delta, b), \end{aligned}$$

where

$$J_m(x, m, \delta, b) := \#\left\{p \leq x_D : Dp + 1 \equiv 0 \pmod{\delta m}, \frac{Dp+1}{m} \equiv b \pmod{d}\right\}.$$

Note that $J_m(x, m, \delta, b) \ll 1$ for all b with $(bm - 1, d) \neq 1$. There is a unique $l_\delta \pmod{d}$ such that $\delta l_\delta \equiv b \pmod{d}$, therefore

$$Dp + 1 = c\delta m \quad \text{and} \quad Dp + 1 = mb + tdm,$$

implies

$$Dp + 1 \equiv ml_\delta \delta \pmod{m\delta d}.$$

Thus,

$$J_m(x, m, \delta, b) = \#\{p \leq x_D : Dp + 1 \equiv m\delta l_\delta \pmod{\delta dm}\}.$$

We arrive at

$$\begin{aligned} M(x, f_r, D) &= \sum'_{\substack{m \leq e^r \\ P(m) \leq r}} f(m) \sum_{\substack{b=1 \\ (b,d)=1 \\ (bm-1,d)=1}}^d \bar{\chi}_d(b) \sum_{\substack{\delta | \mathcal{P}(r) \\ (\delta, Dd)=1}} \mu(\delta) \pi(x_D, \delta dm, m\delta l_\delta) + \\ (15) \quad &+ o(\pi(x_D)) \quad (x \rightarrow \infty), \end{aligned}$$

where Σ' indicates that m and D are of opposite parity. The right hand side of (15) equals

$$\sum'_{\substack{m \leq e^r \\ P(m) \leq r}} f(m) \sum_{\substack{b=1 \\ (b,d)=1 \\ (bm-1,d)=1}}^d \bar{\chi}_d(b) \sum_{\substack{\delta | \mathcal{P}(r) \\ (\delta, Dd)=1}} \mu(\delta) \frac{\pi(x_D)}{\varphi(\delta dm)} + \\ + \mathcal{O} \left(\sum_{\substack{\delta | \mathcal{P}(r) \\ (\delta, Dd)=1}} \sum_{\substack{m \leq e^r \\ P(m) \leq r}} \left| \pi(x_D, \delta dm, m\delta l_\delta) - \frac{\pi(x_D)}{\varphi(\delta dm)} \right| \right) = M + Err_2(x, r).$$

Applying the Cauchy–Schwarz inequality and then the Brun–Titchmarsh theorem we obtain that $Err_2^2(x, r)$ is at most

$$c \left(\sum_{\delta \leq \frac{\sqrt{x}}{\sqrt{D} \log^A x}} 4^{\omega(\delta)} \max_{(l,\delta)=1} \left| \pi(x_D, \delta, l) - \frac{\pi(x_D)}{\varphi(\delta)} \right| \right)^2 \ll \\ \ll \sum_{\delta \leq \frac{\sqrt{x}}{\sqrt{D} \log^A x}} \frac{16^{\omega(\delta)}}{\varphi(\delta)} \sum_{\delta \leq \frac{\sqrt{x}}{\sqrt{D} \log^A x}} \varphi(\delta) \max_{(l,\delta)=1} \left| \pi(x_D, \delta, l) - \frac{\pi(x_D)}{\varphi(\delta)} \right|^2 \ll \\ \ll \prod_{p \leq x} \left(1 + \frac{16}{p} \right) \pi(x_D) \sum_{\delta \leq \frac{\sqrt{x}}{\sqrt{D} \log^A x}} \max_{(l,\delta)=1} \left| \pi(x_D, \delta, l) - \frac{\pi(x_D)}{\varphi(\delta)} \right|,$$

which by the Bombieri–Vinogradov theorem does not exceed $\frac{\pi^2(x_D)}{\log^A x}$, where $A > 0$ is an arbitrary large fixed constant.

Since

$$\varphi(\delta dm) = \delta dm \prod_{p|dm} \left(1 - \frac{1}{p} \right) \prod_{\substack{p|\delta \\ p \nmid dm}} \left(1 - \frac{1}{p} \right) = \varphi(dm) \delta \prod_{\substack{p|\delta \\ p \nmid dm}} \left(1 - \frac{1}{p} \right),$$

we have

$$\sum_{\substack{\delta | \mathcal{P}(r) \\ (\delta, Dd)=1}} \frac{\mu(\delta)}{\varphi(\delta md)} = \frac{1}{\varphi(dm)} \prod_{\substack{p \leq r \\ p \nmid Dd \\ p|dm}} \left(1 - \frac{1}{p} \right) \prod_{\substack{p \leq r \\ p \nmid Dd \\ p \nmid dm}} \left(1 - \frac{1}{p-1} \right) = \\ = \frac{1}{\varphi(dm)} \prod_{\substack{p \leq r \\ p \nmid Dd \\ p|m}} \left(1 - \frac{1}{p} \right) \prod_{\substack{p \leq r \\ p \nmid Dd \\ p \nmid m}} \left(1 - \frac{1}{p-1} \right).$$

Further, by the inclusion-exclusion principle and by the orthogonality relation of the Dirichlet characters we have

$$\sum_{\substack{b=1 \\ (b, dm)=1}}^d \bar{\chi}(b) = \sum_{(b, d)=1} \bar{\chi}(b) \sum_{k|d} \frac{\mu(k)}{\varphi(k)} \sum_{\chi \pmod{k}} \chi_k(bm).$$

Thus,

$$(16) \quad \frac{1}{\pi(x_D)} M = \sum_{(b, d)=1} \bar{\chi}(b) \sum_{k|d} \frac{\mu(k)}{\varphi(k)} \sum_{\chi \pmod{k}} \chi_k(b) \times \\ \times \sum'_{\substack{m \\ P(m) \leq r}} \frac{f(m) \chi_k(m)}{\varphi(dm)} \prod_{\substack{p \leq r \\ p \nmid Dd \\ p|m}} \left(1 - \frac{1}{p}\right) \prod_{\substack{p \leq r \\ p \nmid Dd \\ p \nmid m}} \left(1 - \frac{1}{p-1}\right) + Err_3(r),$$

where

$$Err_3(r) \ll \sum_{\substack{m > e^r \\ P(m) \leq r}} \frac{1}{\varphi(dm)} \prod_{\substack{p \leq r \\ p \nmid Dd \\ p|m}} \left(1 - \frac{1}{p}\right) \prod_{\substack{p \leq r \\ p \nmid Dd \\ p \nmid m}} \left(1 - \frac{1}{p-1}\right) \ll \\ \ll \prod_{\substack{p \leq r \\ p \nmid Dd}} \left(1 - \frac{1}{p}\right) \sum_{\substack{m > e^r \\ P(m) \leq r}} \frac{1}{\varphi(m)} \ll \\ \ll \frac{\log^2 r}{r}.$$

Keeping in mind that m and D has opposite parity

$$(17) \quad \sum'_{\substack{m \\ P(m) \leq r}} \frac{f(m) \chi_k(m)}{\varphi(dm)} \prod_{\substack{p \leq r \\ p \nmid Dd \\ p|m}} \left(1 - \frac{1}{p}\right) \prod_{\substack{p \leq r \\ p \nmid Dd \\ p \nmid m}} \left(1 - \frac{1}{p-1}\right)$$

can be written as

$$\prod_{\substack{p \leq r \\ p \nmid Dd}} \left(1 - \frac{1}{p-1} + \sum_{\alpha \geq 1} \frac{f(p^\alpha) \chi_k(p^\alpha)}{p^\alpha}\right) \prod_{\substack{p \leq r \\ p \nmid D \\ p|d}} \left(1 + \sum_{\alpha \geq 1} \frac{f(p^\alpha) \chi_k(p^\alpha)}{p^\alpha}\right).$$

Thus, the first term on the right hand side of (16) equals

$$\begin{aligned}
 & \sum_{\substack{b=1 \\ (b,d)=1}}^d \frac{\bar{\chi}(b)}{\varphi(d)} \sum_{k|d} \frac{\mu(k)}{\varphi(k)} \times \\
 (18) \quad & \times \sum_{\chi \pmod{k}} \chi_k(b) \prod_{\substack{p \leq r \\ p \nmid dD}} \left(1 - \frac{1}{p-1} + \sum_{\alpha \geq 1} \frac{f(p^\alpha) \chi_k(p^\alpha)}{p^\alpha} \right) \times \\
 & \times \prod_{\substack{p \leq r \\ p|d, p \nmid 2D}} \left(1 + \sum_{\alpha \geq 1} \frac{f(p^\alpha) \chi_k(p^\alpha)}{p^\alpha} \right).
 \end{aligned}$$

Since the character induced by $\chi_k \cdot \bar{\chi}$ is not the principal character if $\chi_k \neq \chi$ we obtain using Dirichlet's theorem in arithmetic progressions that

$$\sum_{z \leq p \leq r} \frac{|1 - \chi_k \cdot \bar{\chi}(p)|^2}{p} \gg \log \left(\frac{\log r}{\log z} \right) \gg \log \left(\frac{\log_3 x}{\log_4 x} \right),$$

if $z = \log_3 x$. Here we used that $\chi_k \cdot \bar{\chi}(p)$ is at most a $\varphi(d)$ -th root of unity. Further,

$$|\chi_k(p)f(p) - 1|^2 \gg |1 - \bar{\chi}(p)\chi_k(p)|^2 - |1 - \bar{\chi}(p)f(p)|^2,$$

therefore

$$\begin{aligned}
 & \sum_{z \leq p \leq r} \frac{|1 - \chi_k(p)f(p)|^2}{p} \gg \\
 \gg & \sum_{z \leq p \leq r} \frac{|1 - \chi_k(p)\bar{\chi}(p)|^2}{p} + \mathcal{O} \left(\sum_{z \leq p \leq r} \frac{|1 - \bar{\chi}(p)f(p)|^2}{p} \right) \gg \\
 & \gg \log \left(\frac{\log_3 x}{\log_4 x} \right) + o(1) \quad (x \rightarrow \infty).
 \end{aligned}$$

Thus,

$$\begin{aligned}
 \left| \prod_{\substack{p \leq r \\ p \nmid dD}} \left(1 - \frac{1}{p-1} + \sum_{\alpha \geq 1} \frac{f(p^\alpha) \chi_k(p^\alpha)}{p^\alpha} \right) \right| & \ll \left| \exp \left(\sum_{p \leq r} \frac{f(p) \chi_k(p) - 1}{p} \right) \right| \ll \\
 & \ll \exp \left(- \sum_{z \leq p \leq r} \frac{1 - \operatorname{Re} f(p) \chi_k(p)}{p} \right) = \\
 & = o(1) \quad (x \rightarrow \infty).
 \end{aligned}$$

Putting it back into (18) we deduce

$$(19) \quad \begin{aligned} \frac{1}{\pi(x_D)} M(x, f_r, D) &= \frac{\mu(d)}{\varphi(d)} \prod_{\substack{p \leq r \\ p \nmid dD}} \left(1 - \frac{1}{p-1} + \sum_{\alpha \geq 1} \frac{f(p^\alpha) \chi(p^\alpha)}{p^\alpha} \right) \times \\ &\times \prod_{\substack{p \leq r \\ p \mid d, p \nmid 2D}} \left(1 + \sum_{\alpha \geq 1} \frac{f(p^\alpha) \chi(p^\alpha)}{p^\alpha} \right) + o(1) \quad (x \rightarrow \infty). \end{aligned}$$

Since $\chi(p^\alpha) = 0$ for all $p \mid d$, introducing the notation

$$P(y) := \prod_{\substack{p \leq y \\ p \nmid dD}} \left(1 - \frac{1}{p-1} + \sum_{\alpha \geq 1} \frac{f(p^\alpha) \chi(p^\alpha)}{p^\alpha} \right),$$

we proved that

$$(20) \quad \pi(x_D)^{-1} M(x, f_r, D) = \frac{\mu(d)}{\varphi(d)} P(r) + o(1) \quad (x \rightarrow \infty).$$

Here we note that if (2) converges for $\tau = 0$ then $1 \ll |P(r)| \leq 1$. Now we can prove that

$$\frac{\mu(d)}{\varphi(d)} P(x_D)$$

is a good approximation of the sum $M(x, f, D)$. Now

$$\begin{aligned} &\left| \pi^{-1}(x_D) M(x, f, D) - \frac{\mu(d)}{\varphi(d)} P(x_D) \right| \leq \\ &\leq \left| \pi^{-1}(x_D) M(x, f^*, D) - \pi^{-1}(x_D) M(x, f_r, D) \frac{P(x_D)}{P(r)} \right| + \\ &\quad + \pi(x_D)^{-1} |M(x, f^*, D) - M(x, f, D)| + \\ &\quad + \left| \frac{\mu(d)}{\varphi(d)} P(x_D) - \pi^{-1}(x_D) M(x, f_r, D) \frac{P(x_D)}{P(r)} \right|, \end{aligned}$$

therefore by (13) and by (20) we have to show that

$$(21) \quad \pi^{-1}(x_D) \left| M(x, f^*, D) - M(x, f_r, D) \frac{P(x_D)}{P(r)} \right| = o(1) \quad (x \rightarrow \infty).$$

We note that, if $d < r$, then

$$|f^*(p^\alpha)| = |f_r(p^\alpha)| = 1.$$

Hence there is a strongly additive function $g_r^*(p) \in (-\pi, \pi]$ with

$$f_r^*(n) = f^* \cdot \bar{f}_r(n) = e^{ig_r^*(n)}.$$

We note that if

$$p \leq r, \quad \text{or} \quad p > x_D^{1-\vartheta_x}, \quad \text{then} \quad g_r^*(p) = 0.$$

By Lemma 2 we have

$$(22) \quad \sum_{Dp+1 \leq x} \left| g_r^*(Dp+1) - \sum_{\substack{q \leq x_D \\ q \nmid D}} \frac{g_r^*(q)}{q} \right|^2 \ll \frac{1}{\vartheta_x} \pi(x_D) \sum_{p \leq x_D} \frac{|g_r^*(p)|^2}{p}.$$

Let

$$A(x) := \sum_{\substack{p \leq x_D \\ p \nmid D}} \frac{g_r^*(p)}{p}.$$

We obtain that the left hand side of (21) is at most

$$\begin{aligned} & \frac{c}{\pi(x_D)} \left| \sum_{Dp+1 \leq x} f^*(Dp+1) - f_r(Dp+1) \frac{P(x_D)}{P(r)} \right| \ll \\ & \ll \frac{1}{\pi(x_D)} \sum_{Dp+1 \leq x} \left| f_r^*(Dp+1) - \frac{P(x_D)}{P(r)} \right| \ll \\ & \ll \frac{1}{\pi(x_D)} \sum_{Dp+1 \leq x} |f_r^*(Dp+1) - \exp[iA(x)]| + \left| \exp[iA(x)] - \frac{P(x_D)}{P(r)} \right| = \\ & = \Sigma'_1 + \Sigma'_2. \end{aligned}$$

Using the Cauchy–Schwarz inequality again we obtain

$$\begin{aligned} \Sigma'_1 &= \pi(x_D)^{-1} \sum_{Dp+1 \leq x} \left| \exp[i(g_r^*(Dp+1) - A(x))] - 1 \right| \leq \\ & \leq \pi(x_D)^{-1/2} \left(\sum_{Dp+1 \leq x} |g_r^*(Dp+1) - A(x)|^2 \right)^{1/2}. \end{aligned}$$

Thus, by (22) we deduce that Σ_1 is at most

$$\left(\frac{c}{\vartheta_x} \sum_{\substack{p \leq x_D \\ p \nmid D}} \frac{|g_r^*(p)|^2}{p} \right)^{1/2}.$$

Further,

$$|g_r^*(p)|^2 \ll |f_r^*(p) - 1|^2 = |f(p) - f_r(p)|^2,$$

therefore

$$\Sigma'_1 \ll \left(\frac{1}{\vartheta_x} \sum_{r \leq p \leq x} \frac{|\chi(p)f(p) - 1|^2}{p} \right)^{1/2},$$

which according to condition (2) tends to zero as $r \rightarrow \infty$ with a suitable choice of ϑ_x .

We have to estimate Σ'_2 . It can be written as

$$\left| 1 - \prod_{\substack{r < p \leq x_D \\ p \nmid D}} \left(1 - \frac{1}{p-1} + \sum_{m \geq 1} \frac{f(p^m)\chi(p^m)}{p^m} \right) \exp\left(-i \sum_{\substack{r < p \leq x_D \\ p \nmid D}} \frac{g_r^*(p)}{p}\right) \right|,$$

which equals

$$\left| 1 - \exp\left[\mathcal{O}\left(\sum_{r < p \leq x_D} \frac{|f(p)\chi(p) - 1|^2}{p} + \sum_{r < p} \frac{1}{p^2}\right)\right] \right|,$$

which again tends to zero as $x \rightarrow \infty$, such that (21) follows. Finally we note that $x^{1-\varepsilon} < x_D$, therefore we have

$$\begin{aligned} |P(x_D) - P(x)| &\ll \left| \prod_{x_D < p \leq x} \left(1 + \frac{f(p)\chi(p) - 1}{p} + \mathcal{O}\left(\frac{1}{p^2}\right) \right) - 1 \right| = \\ &= \left| \exp\left(\sum_{x_D < p \leq x} \frac{f(p)\chi(p) - 1}{p} + \mathcal{O}\left(\frac{1}{p^2}\right)\right) - 1 \right|, \end{aligned}$$

which tends to zero as $x \rightarrow \infty$ inasmuch as

$$\begin{aligned} &\left| \sum_{x_D < p \leq x} \frac{f(p)\chi(p) - 1}{p} \right| \ll \\ (23) \quad &\ll \left(\sum_{x_D < p \leq x} \frac{1}{p} \right)^{1/2} \left(\sum_{x_D < p \leq x} \frac{|f(p)\chi(p) - 1|^2}{p} \right)^{1/2} = o(1) \quad (x \rightarrow \infty). \end{aligned}$$

We proved Theorem 2 in the case $\tau = 0$.

Now consider the case of an arbitrary τ . We proved that

$$\begin{aligned} \pi(x_D)^{-1}M(x, f(n)n^{-i\tau}, D) &= \frac{\mu(d)}{\varphi(d)} \prod_{\substack{p \leq x \\ p \nmid dD}} \left(1 - \frac{1}{p-1} + \sum_{\alpha \geq 1} \frac{f(p^\alpha)p^{-i\alpha\tau}\chi(p^\alpha)}{p^\alpha} \right) + \\ &+ o(1) =: \psi(x) + o(1) \end{aligned}$$

as $x \rightarrow \infty$. Using a summation by parts we obtain that

$$(24) \quad \begin{aligned} \sum_{Dp+1 \leq x} f(Dp+1) &= x^{i\tau} \sum_{Dp+1 \leq x} f(Dp+1)(Dp+1)^{-i\tau} \\ &- i\tau \int_2^x \sum_{Dp+1 \leq u} f(Dp+1)(Dp+1)^{-i\tau} u^{i\tau-1} du. \end{aligned}$$

If $D < x^\varepsilon$, then $D < x^{\gamma\varepsilon'}$ with some other $\varepsilon < \varepsilon' < 1$ and an appropriate $0 \leq \gamma < 1$. Therefore the estimation

$$\begin{aligned} &\pi\left(\frac{u-1}{D}\right)^{-1} M(u, f(n)n^{-i\tau}, D) = \\ &= \frac{\mu(d)}{\varphi(d)} \prod_{\substack{p \leq u \\ p \nmid dD}} \left(1 - \frac{1}{p-1} + \sum_{\alpha \geq 1} \frac{f(p^\alpha)p^{-i\alpha\tau}\chi(p^\alpha)}{p^\alpha} \right) + o(1) \quad (x \rightarrow \infty) \end{aligned}$$

remains valid in the range $x^\gamma < u < x$. Thus, we can estimate the integral on the right hand side of (24) in this range as

$$(25) \quad \begin{aligned} &\int_{x^\gamma}^x \sum_{Dp+1 \leq u} f(Dp+1)(Dp+1)^{-i\tau} u^{i\tau-1} du = \\ &= \frac{\mu(d)}{\varphi(d)} \int_{x^\gamma}^x \pi(u_D)\psi(u)u^{i\tau-1} du + o(1) \int_{x^\gamma}^x \frac{1}{D \log u} du \quad (x \rightarrow \infty). \end{aligned}$$

Now if $x^\gamma \leq u \leq x$, then as in (23) we have

$$|\psi(x) - \psi(u)| = o(1)$$

as $x \rightarrow \infty$. Therefore the right hand side of (25) equals

$$\pi(x_D) \frac{x^{i\tau}}{1+i\tau} \frac{\mu(d)}{\varphi(d)} \psi(x) + o(\pi(x_D)) \quad (x \rightarrow \infty).$$

Using the trivial bound

$$|M(u, f(n)n^{i\tau}, D)| \leq \pi(u_D),$$

we have that the integral on the right hand side of (24) in the range $2 \leq u \leq x^\gamma$ is not more than

$$\mathcal{O}\left(\frac{1}{D} \int_{2D+1}^{x^\gamma} \frac{1}{\log(u/D)} du\right) \ll \int_2^{x^\gamma/D} \frac{1}{\log(u)} du = o(\pi(x_D)) \quad (x \rightarrow \infty).$$

In summary we have

$$\sum_{Dp+1 \leq x} f(Dp+1) = \pi(x_D) \frac{x^{i\tau}}{1+i\tau} \frac{\mu(d)}{\varphi(d)} \psi(x) + o(\pi(x_D)) \quad (x \rightarrow \infty),$$

as asserted. ■

References

- [1] **Davenport, H.**, *Multiplicative Number Theory*, 3 ed., Springer-Verlag, New York, 2000.
- [2] **Delange, H.**, Sur les fonctions arithmétiques multiplicatives, *Ann. Scient. EC. Norm. Sup.*, **78** (1961), 273–304.
- [3] **Germán, L.**, The distribution of an additive arithmetical function on the set of shifted integers having k distinct prime factors, *Annales Univ. Sci. Budapest., Sect. Comp.*, **27** (2007), 187–215.
- [4] **Halász, G.**, Über die Mittelwerte multiplikativer zahlentheoretischer Funktionen, *Acta Math. Hungar.*, **19** (1968), 365–403.
- [5] **Halberstam H. and H.-E. Richert**, *Sieve Methods*, Acad. Press, London, 1974.
- [6] **Hildebrand, A.**, Additive and multiplicative functions on shifted primes, *Proc. London Math. Soc.*, **53** (1989), 209–232.
- [7] **Kátai, I.**, On the distribution of arithmetical functions on the set of primes plus one, *Composito Math.*, **19** (1968), 278–289.
- [8] **Kátai, I.**, On the distribution of arithmetical functions, *Acta Math. Hungar.*, **20(1-2)** (1969), 69–87.
- [9] **Kubilius, J.**, *Probabilistic Methods in the Theory of Numbers*, American Mathematical Society, Providence, Rhode Island, 1964.

- [10] **Tenenbaum, G.**, *Introduction to Analytic and Probabilistic Number Theory*, Cambridge University Press, 1995.
- [11] **Timofeev, N.M.**, Multiplicative functions on the set of shifted prime numbers, *Math. USSR Izvestiya*, **39(3)** (1992), 1189–1207.
- [12] **Wirsing, E.**, Das asymptotische Verhalten von Summen ber multiplikative Funktionen II, *Acta Math. Hungar.*, **18** (1967), 411–467.

L. Germán

Institute of Mathematics
University of Paderborn
Warburger Str. 100
D-33098 Paderborn
Germany
laszlo@math.upb.de