Annales Univ. Sci. Budapest., Sect. Comp. 35 (2011) 95–105

BIORTHOGONAL SYSTEMS TO RATIONAL FUNCTIONS

S. Fridli (Budapest, Hungary)
F. Schipp (Budapest, Hungary)

Dedicated to Professor Antal Járai on his 60th birthday

Abstract. In this paper we start from a given rational function system and take the linear space spanned by it. Then in this linear space we construct a rational function system that is biorthogonal to the original one. By means of biorthogonality expansions in terms of the original rational functions can be easily given. For the discrete version we need to choose the points of discretization and the weight function in the discrete scalar product in a proper way. Then we obtain that the biorthogonality relation holds true for the discretized systems as well.

1. Introduction

There is a wide range of applications of rational function systems. For instance in system, control theories they are effectively used for representing the transfer function, see e.g. [1], [4], [5]. Another area where they have been found to be very efficient is signal processing [8]. Recently we have been using them for

2010 Mathematics Subject Classification: Primary 26C15, Secondary 42C10.

Key words and phrases: rational orthogonal and biorthogonal systems, Malmquist–Takenaka systems, discretization.

The Project is supported by the European Union and co-financed by the European Social Fund (grant agreement no. TAMOP 4.2.1./B-09/1/KMR-2010-0003).

representing and decomposing ECG signals [3]. In several cases the so called Malmquist–Takenaka orthogonal systems are generated and used in applications. There are, however, applications when the result should be expressed by the original rational functions rather than by the terms of the orthonormed system generated by them. Then it makes sense to use the corresponding biorthogonal system. This is the basic motivation behind our construction.

Let us take basic rational functions of the form

$$r_{a,n}(z) := \frac{1}{(1-\overline{a}z)^n} \qquad (|a|<1,\ |z|\le 1,\ n\in\mathbb{P}). \tag{1}$$

($\mathbb{P}$ stands for the set of positive integers.) They form a generating system for the linear space of rational functions that are analytic on the closed unit disc $\overline{\mathbb{D}} = \{z\in\mathbb{C} : |z|\le 1\}$, where $\mathbb{D} := \{z\in\mathbb{C} : |z|<1\}$ stands for the open unit disc. Indeed, by partial fraction decomposition any analytic function can be written as a finite linear combination of such functions. $a^* := 1/\overline{a} = a/|a|^2$ is the pole of $r_{a,n}$ the order of which is n. On the basis of the relation $a^*\overline{a} = 1$ the parameter a will be called inverse pole.

In our construction we will use the following modified basic functions

$$\phi_{a,n}(z) := \frac{z^{n-1}}{(1-\overline{a}z)^n} \qquad (z\in\overline{\mathbb{D}},\ a\in\mathbb{D},\ n\in\mathbb{P})\,. \tag{2}$$

If $a\ne 0$ then this modification makes no difference in the generated subspaces, i.e.

$$\operatorname{span}\{r_{a,k} : 1\le k\le n\} = \operatorname{span}\{\phi_{a,k} : 1\le k\le n\} \qquad (n\in\mathbb{P},\ a\ne 0)\,.$$

It is easy to see that the transition between the system of basic and the system of modified basic functions is very simple. We note that, however, if $a=0$ then the two subspaces are different. Indeed, in this special special case we receive the set of polynomials of order $(n-1)$ on the right side.

Let the set of rational functions that are analytic on $\overline{\mathbb{D}}$ be denoted by $\mathfrak{R}$. It is actually the set of linear combinations of modified basic functions given in (2). $\mathfrak{R}$ will be considered as the normed subspace of the Hardy space $H^2(\mathbb{D})$. Recall that $H^2(\mathbb{D})$ is the collection of functions $F:\mathbb{D}\to\mathbb{C}$ which are analytic on $\mathbb{D}$, and for which

$$\|F\|_{H^2} := \sup_{0\le r<1}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |F(re^{it})|^2\,dt\right)^{1/2} < \infty$$

holds. It is known that for any $F\in H^2(\mathbb{D})$ the limit

$$F(e^{it}) := \lim_{r\to 1-0} F(re^{it})$$

exists for a.e. $t \in \mathbb{I} := [-\pi, \pi)$. The radial limit function defined on the torus $\mathbb{T}$ belongs to $L^2(\mathbb{T})$. This way a scalar product can be defined on $H^2(\mathbb{D})$ as follows

$$\langle F, G \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{it})\overline{G}(e^{it})\, dt \qquad (F, G \in H^2(\mathbb{T}))\,.$$

Then $H^2(\mathbb{D})$ becomes a Hilbert space since the norm induced by this scalar product is equivalent to the original $\|\cdot\|_{H^2}$ norm.

Let $\mathfrak{b} := (b_n \in \mathbb{D},\ n \in \mathbb{N})$ be a sequence of inverse poles. Taking the segment $b_0, b_1, \cdots, b_n$ we count how many times the value of b_n occurs in that. That number will be called the multiplicity of b_n and denoted by ν_n. In other words ν_n is the number of indices $j \le n$ for which $b_j = b_n$. Then we introduce the following subspaces of $\mathfrak{R}$ and of $H^2(\mathbb{D})$ generated by $\mathfrak{b}$

$$\mathfrak{R}_n^{\mathfrak{b}} := \operatorname{span}\{\phi_{b_k,\nu_k} : 0 \le k < n\} \qquad (n \in \mathbb{P}), \qquad \mathfrak{R}^{\mathfrak{b}} := \bigcup_{n=0}^{\infty} \mathfrak{R}_n^{\mathfrak{b}} \subset \mathfrak{R}.$$

We note that $\mathfrak{R}^{\mathfrak{b}}$ is everywhere dense in the Hilbert space $H^2(\mathbb{D})$, i.e. the system $\{\phi_{b_n,m_n} :\ n \in \mathbb{N}\}$ is closed in $H^2(\mathbb{D})$, if and only if ([7], [11])

$$\sum_{n=0}^{\infty}(1 - |b_n|) = \infty\,.$$

By means of the Cauchy integral formula the scalar product of a function $F \in H^2(\mathbb{D})$ and a modified basic function $\phi_{a,k}$ in (2) can be written in an explicit form. Indeed, by definition

$$\begin{aligned} \langle F, \phi_{a,k} \rangle &= \frac{1}{2\pi} \int_{\mathbb{I}} \frac{F(e^{it})e^{-i(k-1)t}}{(1 - ae^{-it})^k}\, dt = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{F(\zeta)}{(\zeta - a)^k}\, d\zeta = \\ &= \frac{F^{(k-1)}(a)}{(k-1)!} \qquad (a \in \mathbb{D},\ k \in \mathbb{P})\,. \end{aligned} \tag{3}$$

Using this formula one can give an explicit form for the members of the so called Malmquist–Takenaka (MT) system. The Malmquist–Takenaka system $(\Phi_n,\ n \in \mathbb{N})$ is generated from $(\phi_{b_k,m_k},\ k \in \mathbb{N})$ by Gram-Schmidt orthogonalization is of the form [12]:

$$\Phi_n(z) := \frac{\sqrt{1 - |b_n|^2}}{1 - \bar{b}_n z} \prod_{k=0}^{n-1} B_{b_k}(z) \qquad (z \in \overline{\mathbb{D}},\ n \in \mathbb{N}), \tag{4}$$

where

$$B_b(z) := \frac{z - b}{1 - \bar{b}z} \qquad (z \in \overline{\mathbb{D}},\ b \in \mathbb{D}) \tag{5}$$

is the Blaschke function of parameter b. The Blaschke functions enjoy several nice propeties. For instance they are bijections on the disc $\mathbb{D}$ and on the torus $\mathbb{T}$, they define a metric on $\mathbb{D}$ as follows

$$\rho(z_1, z_2) := |B_{z_1}(z_2)| = \frac{|z_1 - z_2|}{|1 - \overline{z}_1 z_2|} \qquad (z_1,\, z_2 \in \mathbb{D})\,.$$

Moreover the maps ϵB_b $(b \in \mathbb{D}, \epsilon \in \mathbb{T})$ can be identified with the congruences in the Poincaré model of the hyperbolic plane.
The orthogonal expansions with respect to Malmquist–Takenaka systems generated by a sequence of inverse poles turned to be very useful in several applications. On the other hand there are problems when the expansion in terms of the generating basic or modified basic functions would be more useful. This is the case for example in system identification when a partial fraction representation of the transfer function is taken, and the poles should be determined [10]. In such cases a biorthogonal system is needed to deduce such an expansion. In the next section we construct a biorthogonal system to a finite system of modified basic functions. The elements of the biorthogonal system are in the subspace generated by the basic functions. In Section 3 we define a set of points of discretization. By means of that and a proper weight function we prove a discrete type biorthogonality as well. We note that a similar problem was addressed in [9] except that equidistant subdivision was taken there and the members of the biorthogonal system were polynomials.

2. Rational biorthogonal systems

Let $\mathfrak{b}$ be a sequence of inverse poles in $\mathbb{D}$ and fix $N \in \mathbb{P}$. Let $a_0, a_1, \cdots, a_n$ denote the distinct elements in $\{b_0, \ldots, b_{N-1}\}$. Then m_j will stand for the number of occurrences of a_j in $\{b_0, \ldots, b_{N-1}\}$. We will use the simplified notations $\phi_{\ell j} := \phi_{a_\ell, j}$, and $\mathfrak{R}_N := \mathfrak{R}_N^{\mathfrak{b}}$. Then the following equations hold

$$\begin{aligned}
&\mathfrak{R}_N = \operatorname{span}\{\phi_{\ell j} \,:\, 1 \le j \le m_\ell, 0 \le \ell \le n\}\\
&\{b_k \,:\, 0 \le k < N\} = \{a_j \,:\, 0 \le j \le n\}\,,\\
&m_0 + m_1 + \cdots + m_n = N.
\end{aligned}$$

In this section we will construct a system $\{\Psi_{\ell j} \,:\, 1 \le j \le m_\ell, k, \ell = 0, 1, \ldots, n\}$ within $\mathfrak{R}_N$ which is biorthogonal to the generating system $\{\phi_{\ell j} \,:\, 1 \le j \le m_\ell, 0 \le \ell \le n\}$. In notation

i) $\operatorname{span}\{\Psi_{\ell j} : 1 \le j \le m_\ell,\ \ell = 0, 1, \cdots, n\} = \mathfrak{R}_N$,

ii) $\langle \Psi_{\ell j}, \phi_{ki} \rangle = \delta_{ij}\delta_{k\ell} \qquad (1 \le i \le m_k,\ 1 \le j \le m_\ell, k,\ \ell = 0, 1, \cdots, n)\,.$

Then the operator P_N of projection onto $\mathfrak{R}_N$ can be expressed as a biorthogonal expansion

$$P_N f = \sum_{k=0}^{n} \sum_{i=1}^{m_k} \langle f, \Psi_{ki} \rangle \phi_{ki} \, .$$

In the construction of the explicit form of the biorthogonal system the formula in (3), that relates biorthogonality with Hermite interpolation, will play a key role. Using the Blaschke functions defined in (5) we introduce the function Ω_n as follows

$$\Omega_{\ell n}(z) := \frac{1}{(1-\overline{a}_\ell z)^{m_\ell}} \prod_{i=0, i\neq \ell}^{n} B_{a_i}^{m_i}(z) \qquad (0 \le \ell \le n) \, . \tag{6}$$

We will show that the members of the biorthogonal system can be written in the form

$$\Psi_{\ell j}(z) = P_{\ell j}(z) \frac{\Omega_{\ell n}(z)}{\Omega_{\ell n}(a_\ell)} \, , \tag{7}$$

where

$$P_{\ell j}(z) = \sum_{s=0}^{m_\ell - 1} \frac{P_{\ell j}^{(s)}(a_\ell)}{s!} (z - a_\ell)^s \tag{8}$$

is a polynomial of order $(m_\ell - 1)$.
Indeed, by (3) we have

$$\langle \Psi_{\ell j}, \phi_{ki} \rangle = \frac{\Psi_{\ell j}^{(i-1)}(a_k)}{(i-1)!} \qquad (1 \le i, j \le m_k) \, . \tag{9}$$

It follows from the definition of $\Omega_{\ell n}$ in (6) that if $k \neq \ell$ then a_k is a root of the nominator of $\Psi_{\ell j}$ of order exactly m_k. Therefore the scalar product product is 0, and orthogonality holds in (9) for $k \neq \ell$. In case $k = \ell$ biorthogonality is equivalent to

$$\langle \Psi_{\ell j}, \phi_{\ell i} \rangle = \frac{\Psi_{\ell j}^{(i-1)}(a_\ell)}{(i-1)!} = \delta_{ij} \qquad (1 \le i, j \le m_\ell) \, . \tag{10}$$

Set

$$\omega_{\ell n}(z) = \frac{\Omega_{\ell n}(a_\ell)}{\Omega_{\ell n}(z)} \, . \tag{11}$$

We note that $\omega_{\ell n}$ is analytic in a proper neighborhood of a_ℓ since $\Omega_{\ell n}(a_\ell) \neq 0$. By definition, see (7), we have

$$P_{\ell j}(z) = \Psi_{\ell j}(z) \omega_{\ell n}(z) \, .$$

Using the product rule of differentiation and the condition (10) we obtain

$$P_{\ell j}^{(s)}(a_\ell) = \sum_{r=0}^{s} \binom{s}{r} \Psi_{\ell j}^{(r)}(a_\ell)\omega_{\ell n}^{(s-r)}(a_\ell) = \binom{s}{j-1}(j-1)!\,\omega_{\ell n}^{(s-j+1)}(a_\ell)$$

for the coefficients of the polynomial $P_{\ell j}$ in (8). Hence

$$\frac{P_{\ell j}^{(s)}(a_\ell)}{s!} = \begin{cases} 0, & (0 \le s < j-1)\,; \\ \dfrac{\omega_{\ell n}^{(s-j+1)}(a_\ell)}{(s-j+1)!}, & (j-1 \le s < m_\ell)\,. \end{cases} \tag{12}$$

For the calculation of the derivatives of $\omega_{\ell n}$ we will use the following logarithmic formula for the Blaschke functions, for definition see (5),

$$\begin{aligned} \frac{d}{dz}\log(B_a(z)) &= \frac{d}{dz}[\log(z-a) - \log(1-\overline{a}z)] \\ &= \frac{1}{z-a} + \frac{\overline{a}}{1-\overline{a}z} = \frac{1}{z-a} - \frac{1}{z-a^*} \qquad (a^* := 1/\overline{a}). \end{aligned} \tag{13}$$

Thus

$$\begin{aligned} \frac{d}{dz}\log(\Omega_{\ell n}(z)) &= \frac{d}{dz}[-m_\ell \log(1-\overline{a}_\ell z) + \sum_{i=1, i\neq \ell}^{n} m_i \log(B_{a_i}(z))] = \\ &= -\frac{m_\ell}{z-a_\ell^*} + \sum_{i=1, i\neq\ell}^{n} \left(\frac{m_i}{z-a_i} - \frac{m_i}{z-a_i^*}\right). \end{aligned} \tag{14}$$

Since

$$\frac{\omega_{\ell n}'(z)}{\omega_{\ell n}(z)} = \frac{d}{dz}\log(\omega_{\ell n}(z)) = -\frac{d}{dz}\log(\Omega_{\ell n}(z))$$

we can conclude by (14) that

$$\omega_{\ell n}'(z) = \omega_{\ell n}(z)\rho_{\ell n}(z) \tag{15}$$

with

$$\rho_{\ell n}(z) := \frac{m_\ell}{z-a_\ell^*} - \sum_{i=1, i\neq\ell}^{n} m_i \left(\frac{1}{z-a_i} - \frac{1}{z-a_i^*}\right).$$

This provides a recursion process for the calculation of the derivatives of $\omega_{\ell n}$. As an example, the second and third derivatives are shown below:

$$\begin{aligned} \omega_{\ell n}^{(2)} &= \omega_{\ell n}'\ \rho_{\ell n} + \omega_{\ell n}\ \rho_{\ell n}' = \omega_{\ell n}(\rho_{\ell n}^2 + \rho_{\ell n}'), \\ \omega_{\ell n}^{(3)} &= \omega_{\ell n}'(\rho_{\ell n}^2 + \rho_{\ell n}') + \omega_{\ell n}(2\rho_{\ell n}\ \rho_{\ell n}' + \rho_{\ell n}^{(2)}) = \omega_{\ell n}(\rho_{\ell n}^3 + 3\rho_{\ell n}\ \rho_{\ell n}' + \rho_{\ell n}^{(2)}). \end{aligned}$$

where the terms $\rho_{\ell n}^{(j)}(z)$ are

$$\rho_{\ell n}^{(j)}(z) = (-1)^j j! \left(\frac{m_\ell}{(z-a_\ell^*)^{j+1}} - \sum_{i=1, i\neq \ell}^{n} m_i \left(\frac{1}{(z-a_i)^{j+1}} - \frac{1}{(z-a_i^*)^{j+1}} \right) \right).$$

In summary, we have proved the following theorem.

Theorem 1. *Let $\Omega_{\ell n}$, and $\omega_{\ell n}$ be defined as in* (6)*, and* (11)*. Then the systems*

$$\begin{aligned} \phi_{ki}(z) &:= \frac{z^{i-1}}{(1-\overline{a}_k z)^i}, \\ \Psi_{\ell j}(z) &:= \frac{\Omega_{\ell n}(z)(z-a_\ell)^{j-1}}{\Omega_{\ell n}(a_\ell)} \sum_{s=0}^{m_\ell - j} \frac{\omega_{\ell n}^{(s)}(a_\ell)}{s!}(z-a_\ell)^s \end{aligned}$$

$(z \in \overline{\mathbb{D}},\, 1 \le i \le m_k,\, 1 \le j \le m_\ell, 0 \le k, \ell \le n)$ *are biorthogonal to each other with respect to the scalar product in $H^2(\mathbb{D})$.*
The two systems span the same linear space.
The derivatives of $\omega_{\ell n}$ can be calculated by recursion based on the relation in (15)*.*

3. Discrete rational biorthogonal systems

In this section we introduce a discrete scalar product in $\mathfrak{R}_N$ as follows

$$[F, G]_N := \sum_{z \in \mathbb{T}_N} F(z)\overline{G}(z)\rho_N(z) \qquad (F, G \in \mathfrak{R}_N)\,, \tag{16}$$

where the discrete set $\mathbb{T}_N \subset \mathbb{T}$ with number of elements equals to N, and the positive weight function ρ_N on it will be defined later.
The Blaschke function B_a admits a representation on the unit circle of the form

$$B_a(e^{it}) = e^{i\beta_a(t)} \qquad (t \in \mathbb{R})\,, \tag{17}$$

where $\beta_a : \mathbb{R} \to \mathbb{R}$ is strictly increasing for which $\beta_a(t + 2\pi) = \beta_a(t) + 2\pi$ holds. Moreover,

$$\beta_a'(t) = \frac{1-r^2}{1-2r\cos(t-\alpha)+r^2} \qquad (t \in \mathbb{R},\, a = re^{i\alpha} \in \mathbb{D})\,. \tag{18}$$

Indeed, let us continue (13) to obtain

$$\frac{d}{dz}\log(B_a(e^{it})) = ie^{it}\left(\frac{1}{e^{it}-re^{i\alpha}} - \frac{1}{e^{it}-\frac{1}{r}e^{i\alpha}}\right)$$
$$= i\frac{1-r^2}{1-2r\cos(t-\alpha)+r^2}\,.$$

Hence (17) and (18) follow. Then by the definition of $\{a_0,\cdots,a_n\}$ at the beginning of Section 2 we have that the Blaschke products can be written as

$$\prod_{k=0}^{N-1} B_{b_k}(e^{it}) = \prod_{j=0}^{n} e^{im_j\beta_{a_j}(t)} = e^{i\theta_N(t)} \qquad (t\in\mathbb{R})\,,$$

where

$$\theta_N(t) := \sum_{j=0}^{n} m_j\beta_{a_j}(t) \quad (t\in\mathbb{R})\,.$$

θ_N is strictly increasing and $\theta_N(t+2\pi) = \theta_N(t) + 2N\pi$. Therefore, for any $t_0\in\mathbb{I}$ and $k=1,2,\cdots,N-1$ there exists exactly one $t_k\in(t_0,t_0+2\pi)$ for which

$$\theta_N(t_k) = 2\pi k + \theta_N(t_0) \qquad (k=0,1,\cdots,N-1) \tag{19}$$

holds.
Then the set of discretization $\mathbb{T}_N$ and the weight function ρ_N in (16) are defined as follows

$$\mathbb{T}_N := \{e^{it_k} : k=0,1,\cdots,N-1\}\,, \qquad \rho_N(e^{it}) = \frac{1}{\theta'_N(t)}\,.$$

Then the following theorem holds for this discrete model and the rational functions.

Theorem 2. *The MT-system Φ_n $(n=0,1,\cdots,N-1)$ is orthonormed system with respect to the scalar product in* (16)*, i.e.*

$$[\Phi_k,\Phi_\ell]_N = \delta_{k\ell} \qquad (0\le k,\ell<N)\,.$$

The $\Psi_{\ell j}$, and $\phi_{\ell j}$ $(1\le j\le m_\ell, 0\le\ell\le n)$ systems are biorthogonal to each other with respect to the scalar product in (16)*, i.e.*

$$[\Psi_{\ell r},\phi_{ks}]_N = \delta_{k\ell}\delta_{rs} \qquad (1\le r\le m_\ell, 1\le s\le m_k, 0\le k,\ell\le n)\,.$$

Proof. For the proof we will use the following closed form the Dirichlet kernels of the MT-systems [2] (or see e.g. [6], pp. 320, [4], pp. 82):

$$D_N(t,\tau) := \sum_{j=0}^{N-1} \Phi_j(e^{it})\overline{\Phi}_j(e^{i\tau}) = \frac{e^{i(\theta_N(t)-\theta_N(\tau))}-1}{e^{i(t-\tau)}-1} \qquad (t,\tau\in\mathbb{R}, t\neq\tau)\,. \tag{20}$$

By the definition of t_k, see (19), we have

$$D_N(t_k,t_\ell)=0 \qquad (k\neq\ell,\ 0\le k,\ell<N)\,.$$

In the special case $t=\tau$ one can deduce from the continuity of the kernel and from (20) that

$$D_N(t,t)=\lim_{\tau\to t} D_n(t,\tau)=\lim_{\tau\to t}\left(\frac{e^{i\tau}}{e^{i\theta_N(\tau)}}\cdot\frac{e^{i\theta_N(t)}-e^{i\theta_N(\tau)}}{t-\tau}\cdot\left(\frac{e^{it}-e^{i\tau}}{t-\tau}\right)^{-1}\right)= \\ =\theta_N'(t)\,.$$

This along with (20) imply

$$\sum_{j=0}^{N-1} u_{jk}\overline{u}_{j\ell}=\frac{D_N(t_k,t_\ell)}{D_N(t_k,t_k)}=\delta_{k\ell}\quad (0\le k,\ell<N),$$

for the matrix

$$u_{jk}:=\frac{\Phi_j(t_k)}{\sqrt{D_N(t_k,t_k)}}\quad (0\le k,\ell<N)\,.$$

This means that the matrix is unitarian. Taking the adjoint matrix we have

$$\sum_{j=0}^{N-1} u_{kj}\overline{u}_{\ell j}=\sum_{j=0}^{N-1}\frac{\Phi_k(t_j)\overline{\Phi}_\ell(t_j)}{D_N(t_j,t_j)}=[\Phi_k,\Phi_\ell]_N=\delta_{k\ell} \qquad (0\le k,\ell<N)\,.$$

The first part of our theorem on the discrete orthogonality of the MT-sytems is proved.

The proof of the second part of our theorem follows from the equivalence of the scalar products $\langle\cdot,\cdot\rangle$ and $[\cdot,\cdot]_N$ in the subspace $\mathfrak{R}_N$:

$$\langle F,G\rangle=[F,G]_N \qquad (F,G\in\mathfrak{R}_N)\,.$$

Indeed, if $F,G\in\mathfrak{R}_N$ then they can be expressed as linear combinations of the Φ_k $(k=0,1,\cdots,N-1)$ MT-functions:

$$F=\sum_{k=0}^{N-1}\lambda_k\Phi_k,\quad G=\sum_{k=0}^{N-1}\mu_k\Phi_k.$$

Since, as it has already been shown, the MT-functions are orthonormed with respect to both scalar products we have

$$\begin{aligned}\langle F, G\rangle &= \sum_{k=0}^{N-1}\sum_{\ell=0}^{N-1}\lambda_k\overline{\mu}_\ell\langle\Phi_k,\Phi_\ell\rangle = \sum_{k=0}^{N-1}\lambda_k\overline{\mu}_k = \\ &= \sum_{k=0}^{N-1}\sum_{\ell=0}^{N-1}\lambda_k\overline{\mu}_\ell[\Phi_k,\Phi_\ell]_N = [F,G]_N.\end{aligned}$$

Hence our statement on discrete biorthogonality follows by Theorem 1. ∎

References

[1] **Bokor, J. and F. Schipp,** Approximate linear H^∞ identification in Laguerre and Kautz basis, *IFAC AUTOMATICA J.,* **34** (1998), 463–468.

[2] **Dzhrbashyan, M.M.,** Expansions for systems of rational functions with fixed poles, *Izv. Akad. Nauk. Armyan. SSSR., ser. mat.,* **2** (1967), 3–51.

[3] **Fridli, S. and F. Schipp,** Rational function systems in ECG processing, *Computer Aided Systems Theory - EUROCAST 2011,* Lecture Notes in Computer Science (to appear).

[4] **Heuberger, S.C., P.M.J. Van den Hof and B. Wahlberg,** *Modelling and Identification with Rational Orthogonal Basis Functions,* New York, Springer-Verlag, 2005.

[5] **Hangos K.M. and Nádai L.** (eds.) *Proceedings of the Workshop on System and Control Theory,* Budapest University of Technology and Economics, Computer and Automation Research Institute of HAS, Budapest, 2009.

[6] **Lorentz, G.G., M. Golitschek and Y. Makozov,** *Constructive Aproximation. Advanced Problems,* Grundlehren der Mathematischen Wissenschaften, 304, Springer-Verlag, Berlin, 1996.

[7] **Malmquist, F.,** Sur la détermination d'une classe de fonctions analytiques par leurs valeurs dans ensemble donné de points, *Comptes Rendus du Sixiéme Congrés des Mathématiciens Scandinaves,* (Copenhague, 1925) (1926), 253–259.

[8] **Marple, S.,** *Digital Spectral Analysis with Application,* Englewood Cliffs: Prentice Hall, 1987.

[9] **Pap, M. and F. Schipp,** Interpolation by rational functions, *Annales Univ. Sci. Budapest., Sect. Comp.,* **23** (2004), 223–237.

[10] **Soumelidis, A., M. Pap, F. Schipp and J. Bokor,** Frequency domain identification of partial fraction models, In: *Proceedings of the 15 th IFAC World Congress,* Barcelona, Spain, Lune 2002, p. on CD.

[11] **Takenaka, S.,** On the orthogonal functions and a new formula of interpolation, *Jap. J. Math.,* **2** (1925, 129–145.

[12] **Walsh, J.L.,** *Interpolation and Approximation by Rational Functions in the Complex Domain,* Amer. Math. Soc. , Rhode Island, 2nd ed., 1956.

S. Fridli and F. Schipp
Department of Numerical Analysis
Eötvös Loránd University
Pázmány P. sétány 1/C.
H-1117 Budapest, Hungary
fridli@inf.elte.hu
schipp@ludens.elte.hu