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ON THE THEOREM OF H. DABOUSSI OVER THE GAUSSIAN INTEGERS

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Dedicated to Professor Antal Járai on his 60th birthday

Abstract. Some analogues of the theorem of Daboussi over the set of Gaussian integers are investigated.

1. Introduction

Let $c, c_1, c_2, \ldots, K, K_1, K_2, \ldots$ be positive constants, not necessarily the same at every occurrence. Let $\mathcal{M}$ be the set of complex valued multiplicative functions and $\mathcal{M}_1$ be the set of those $g \in \mathcal{M}$ for which additionally $|g(n)| \leq 1$ $(n \in \mathbb{N})$ holds as well. Let $e(\alpha) := e^{2\pi i\alpha}$.

A famous theorem of H. Daboussi published in the paper written jointly with H. Delange in [2] asserts that

$$\sup_{f\in\mathcal{M}_1} \left| \frac{1}{x} \sum_{n\leq x} f(n)e(n\alpha) \right| = \varrho_x \to 0 \qquad (x \to \infty), \tag{1.1}$$

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whenever α is an irrational number. This famous theorem has been generalized in different aspects in [1], [3]–[20]. In [2] the following assertion was proved:

Let S be an arithmetical function satisfying the following conditions:

(i) *S is almost-periodic B^1,*

(ii) *the Fourier series of S is $\lambda + \sum \lambda_\nu e(\alpha_\nu n)$, where all the α_ν are irrational.*

Then, as x tends to infinity, we have

$$\sup_{f\in\mathcal{M}_1}\left|\frac{1}{x}\sum_{n\le x} f(n)S(n) - \frac{1}{\lambda}\sum f(n)\right| \le \varrho_x(S),$$

$\varrho_x(S) \to 0$ *as* $(x\to\infty)$.

In [20] the following theorem is proved.

Let $k\ge 1$ be fixed, $J_1,\dots,J_k \subseteq [0,1)$ be such sets which are the union of finitely many intervals. Let $P_1(x),\dots,P_k(x)$ be non-constant real valued polynomials,

$$Q_{m_1,\dots,m_k}(x) = m_1P_1(x)+\dots+m_kP_k(x)$$

for $m_1,\dots,m_k\in\mathbb{Z}$.

Assume that $Q_{m_1,\dots,m_k}(x) - Q_{m_1,\dots,m_k}(0)$ has at least one irrational coefficient for every $m_1,\dots,m_k\in\mathbb{Z}$, except when $m_1=\ldots=m_k=0$.

Let

$$S := \{n \mid n\in\mathbb{N},\quad \{P_l(n)\}\in J_l,\quad l=1,\dots,k\}.$$

Let λ be the Lebesgue measure.

Theorem A. *Under the conditions stated for $P_1,\dots,P_k,J_1,\dots,J_k$ we have*

$$\sup_{g\in\mathcal{M}_1}\left|\frac{1}{x}\sum_{\substack{n\le x\\ n\in S}} g(n) - \frac{\lambda(J_1)\dots\lambda(J_k)}{x}\sum_{n\le x} g(n)\right| = \tau_x, \tag{1.2}$$

$\tau_x \to 0$ *as* $x\to\infty$.

By using the same method and Theorem B we can prove

Theorem 1. *Let $J_1,\dots,J_k,P_1,\dots,P_k,S$ be as above. Let P be a non-constant real valued polynomial.*

Let $R_{m_0,m_1,\dots,m_k}(x) = m_0P(x)+Q_{m_1,\dots,m_k}(x)$. Assume that

$$R_{m_0,m_1,\dots,m_k}(x) - R_{m_0,m_1,\dots,m_k}(0)$$

has at least one irrational coefficient for every $m_0, m_1, \dots, m_k$ *except the case when* $m_0 = m_1 = \dots = m_k = 0$.

Then

$$\sup_{g\in\mathcal{M}_1} \frac{1}{x}\left|\sum_{\substack{n\le x\\ n\in S}} g(n)e(P(n))\right| = \varrho_x \to 0, \qquad \text{as} \quad x\to\infty. \tag{1.3}$$

ϱ_x *may depend on* S *and on* P.

Theorem B. (See [7].) (1.3) *is true, if* $S = \mathbb{N}$.

Applying Theorem A for $g(n) = 1$ we obtain that

$$\frac{1}{x}\#\{n\le x \mid n\in S\} \to \lambda(J_1)\dots\lambda(J_k).$$

From Theorem 1, by using Weyl's criterion for uniformly distributed sequences we get

Theorem 2. *Let* $J_1, \dots, J_k, P, P_1, \dots, P_k, S$ *as in Theorem* 1. *Let* $\mathcal{A}$ *be the set of additive arithmetical functions,* $S = \{t_1, t_2, \dots\}$, $t_j < t_{j+1}$ $(j = = 1, 2, \dots)$, $\xi_n(f) := f(t_n) + P(t_n)$ $(n = 1, \dots)$,

$$\Delta_N(f\mid S) :=$$
$$:= \sup_{[\alpha,\beta)\subseteq[0,1)}\left|\frac{1}{N}\#\{\xi_n(f) \mod 1 \in [\alpha,\beta], n\in N\} - (\beta-\alpha)\right|. \tag{1.4}$$

Then

$$\sup_{f\in\mathcal{A}} \Delta_N(f|S) = \varrho_N \to 0 \qquad \text{as} \quad N\to\infty. \tag{1.5}$$

ϱ_N *may depend on* S.

Let $\mathcal{N}_k$ be the set of the integers the number of the prime power factors of which is k. Let $N_k(x)$ be the size of $n\le x$, $n\in\mathcal{N}_k$. In our paper [10] we proved

Theorem C. *Let* $0 < \delta(< 1)$ *be an arbitrary constant, and* α *be an irrational number. Then*

$$\lim_{x\to\infty} \sup_{\delta\le\frac{k}{\log\log x}\le 2-\delta} \sup_{f\in\mathcal{M}_1} \frac{1}{N_k(x)}\left|\sum_{\substack{m\le x\\ m\in\mathcal{N}_k}} f(m)e(m\alpha)\right| = 0. \tag{1.6}$$

The proof depends on an important assertion due to Dupain, Hall, Tenenbaum [4], namely that

$$(1.7)\qquad \sup_{\frac{k}{\log\log x}\le 2-\delta} \frac{1}{N_k(x)}\left|\sum_{\substack{m\le x\\ m\in\mathcal{N}_k}} e(m\alpha)\right| \to 0 \qquad \text{as} \quad x\to\infty.$$

Theorem 3.

1.) *Let* $P(n)=\alpha n$, $P_j(n)=\alpha_j n$, $(j=1,\dots,k)$, $J_1,\dots,J_k$ *and* S *as earlier. Assume that* $m\alpha+m_1\alpha_1+\dots+m_k\alpha_k$ *is irrational for every nontrivial choice of* $m,m_1,\dots,m_k$. *Let* $S_k(x)=\#\{n\le x \mid n\in\mathcal{N}_k,\ n\in S\}$.

Then

$$(1.8)\qquad \lim_{x\to\infty}\ \sup_{\delta\le\frac{k}{\log\log x}\le 2-\delta}\ \sup_{f\in\mathcal{M}_1} \frac{1}{S_k(x)}\left|\sum_{\substack{n\le x\\ n\in\mathcal{N}_k\cap S}} f(n)e(n\alpha)\right| = 0.$$

2.) *Let* $P_1,\dots,P_k,J_1,\dots,J_k$ *and* S *as earlier. Assume that* $m_1\alpha_1+\dots+m_k\alpha_k$ *is irrational for every nontrivial choice of* $m_1,\dots,m_k$. *Then*

$$(1.9)\qquad \lim_{x\to\infty}\ \sup_{\delta\le\frac{k}{\log\log x}\le 2-\delta}\ \sup_{f\in\mathcal{M}_1}\left|\frac{1}{S_k(x)}\sum_{\substack{n\le x\\ n\in\mathcal{N}_k\cap S}} f(n) - \frac{1}{N_k(x)}\sum_{\substack{n\le x\\ n\in\mathcal{N}_k}} f(n)\right| = 0.$$

Since the Theorems 1, 2, 3 can be deduced from already published papers by the method used in [20], we omit the proofs of them. In the next section we formulate and prove Theorem 4.

2.

Let $\mathbb{Z}[i]$ be the ring of Gaussian integers, $\mathbb{Z}^*[i]=\mathbb{Z}[i]\setminus\{0\}$ be the multiplicative group of nonzero Gaussian integers.

Let χ be such an additive character on $\mathbb{Z}[i]$, for which $\chi(1)=e(A)$, $\chi(i)=e(B)$. Let $\mathcal{K}_1$ be the set of multiplicative functions $g:\mathbb{Z}^*[i]\to\mathbb{C}$ satisfying $|g(\alpha)|\le 1\quad(\alpha\in\mathbb{Z}^*[i])$. Let W be the union of finitely many convex bounded domain in $\mathbb{C}$. In our paper [11] written jointly with N.L. Bassily and J.-M. De Koninck we proved

Theorem D. *Assume that at least one of A or B is irrational. Then*

$$\lim_{x\to\infty} \sup_{g\in\mathcal{K}_1} \frac{1}{|xW|} \left| \sum_{\beta\in xW} g(\beta)\chi(\beta) \right| = 0. \tag{2.1}$$

Let $I = [0,1)\times[0,1)$, $S = S_1\cup\ldots\cup S_r \subseteq I$, where S_j are domains the boundary of which is a rectifiable continuous curve for every j. For some small $\Delta > 0$ let

$$S^{(-\Delta)} = \{(u,v) \mid [u-\Delta, u+\Delta]\times[v-\Delta, v+\Delta] \subseteq S\},$$
$$S^{(+\Delta)} = \{(u,v) \mid [u-\Delta, u+\Delta]\times[v-\Delta, v+\Delta] \cap S \neq 0\}.$$

Let

$$f(x,y) = \begin{cases} 1, & \text{if } (x,y)\in S \\ 0, & \text{if } (x,y)\in I\setminus S, \end{cases} \tag{2.2}$$

and let us extend the definition of f over $\mathbb{R}^2$ by

$$f(x+k, y+l) = f(x,y) \qquad (k,l\in\mathbb{Z}).$$

Let $\sum\limits_{m,n\in\mathbb{Z}} a_{m,n} e(mx+ny)$ be the Fourier-series of $f(x,y)$. Let $\Delta > 0$ be so small that $S^{(+\Delta)} \subseteq I$, and

$$f_\Delta(x,y) := \frac{1}{(2\Delta)^2} \int_{-\Delta}^{\Delta}\int_{-\Delta}^{\Delta} f(x+u)f(y+v)\,\mathrm{d}u\,\mathrm{d}v. \tag{2.3}$$

Since

$$\kappa(n) := \frac{1}{2\Delta}\int_{-\Delta}^{\Delta} e(nu)\,\mathrm{d}u = \frac{1}{4\pi i n\Delta}(e(n\Delta) - e(-n\Delta))$$

if $n\neq 0$, and $\kappa(0) = 1$, therefore the Fourier coefficients $b_{m,n}$ of f_Δ are

$$b_{m,n} = a_{m,n}\kappa(m)\cdot\kappa(n).$$

Assume that for some $\delta > 0$,

$$|a_{m,n}| \leq c\left(\frac{1}{1+|m|^\delta}\right)\left(\frac{1}{1+|n|^\delta}\right), \tag{2.4}$$

c is a constant. Thus

$$|b_{m,n}| \leq |a_{m,n}| \min\left(1, \frac{2}{|m|\Delta}\right) \min\left(1, \frac{2}{|n|\Delta}\right). \tag{2.5}$$

It is clear that $f_\Delta(u,v) = 1$ if $(u,v) \in S^{(-\Delta)}$, and $f_\Delta(u,v) = 0$ if $(u,v) \in$ $\in I \setminus S^{(+\Delta)}$.

Let $z = u + iv \in \mathbb{C}$. The fractional part of z is defined as $\{z\} = \{u\} + i\{v\}$.

Theorem 4. *Let $\gamma_j = \xi_j + i\eta_j$ $(j = 1, \ldots, k)$ be distinct nonzero numbers, $\mathcal{T} = \{\beta \mid \beta \in \mathbb{Z}[i], \{\gamma_j\beta\} \in S,\ j = 1, \ldots, k\}$. Assume that S satisfies the conditions stated above. Assume that $\xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_k$ are linearly independent over $\mathbb{Q}$. Then*

$$\lim_{x\to\infty} \sup_{g\in\mathcal{K}_1} \left| \frac{1}{|xW|} \sum_{\substack{\beta\in xW \\ \beta\in\mathcal{T}}} g(\beta) - \frac{a_{0,0}^k}{|xW|} \sum_{\beta\in xW} g(\beta) \right| = 0. \tag{2.6}$$

Here $a_{0,0,} = \lambda(S)$ =Lebesgue measure of S.

Theorem 5. *Let $S, \gamma_j, \mathcal{T}$ be as above, $\chi(u + iv) = e(Au + Bv)$. Let $\mathcal{L}$ be the lattice $\{m_1\xi_1 + \cdots + m_k\xi_k + n_1\eta_1 + \cdots n_k\eta_k\}$. Assume that either $nA \notin \mathcal{L}$ for $n \in \mathbb{Z} \setminus \{0\}$ or $nB \notin \mathcal{L}$ for $n \in \mathbb{Z} \setminus \{0\}$. Then*

$$\lim_{x\to\infty} \sup_{g\in\mathcal{K}_1} \left| \frac{1}{|xW|} \sum_{\substack{\beta\in xW \\ \beta\in\mathcal{T}}} g(\beta)\chi(\beta) \right| = 0. \tag{2.7}$$

Proof of Theorem 4. First we observe that

$$\begin{aligned} \#\{\beta \in xW \mid \{\gamma_j\beta\} \in S^{(+\Delta)} \setminus S^{(-\Delta)}\} &\le \\ \le c_1\lambda(S^{(+\Delta)} \setminus S^{(-\Delta)})\lambda(xW), \end{aligned} \tag{2.8}$$

and that $\lambda(S^{(+\Delta)} \setminus S^{(-\Delta)}) \le c_2\Delta$. c_2 may depend on S. Let $F(u + iv) =$ $= f(u,v)$, $F_\Delta(u + iv) = f_\Delta(u,v)$. In this notation

$$\begin{aligned} \sum_{\substack{\beta\in xW \\ \beta\in\mathcal{T}}} g(\beta) &= \sum_{\beta\in xW} g(\beta)F(\beta\gamma_1)\ldots F(\beta\gamma_k) = \\ &= \sum_{\beta\in xW} g(\beta)F_\Delta(\beta\gamma_1)\ldots F_\Delta(\beta\gamma_k) + \mathcal{O}(\Delta\lambda(xW)). \end{aligned} \tag{2.9}$$

Let K be so large that

$$\sum_{n\in\mathbb{Z}} \sum_{|m|\ge K} |b_{m,n}| + \sum_{|n|\ge K} \sum_{m} |b_{m,n}| \le \Delta. \tag{2.10}$$

Since $\sum b_{m,n}$ is absolutely convergent, therefore such a K exists. (See (2.5).)

Let

$$F_{\Delta}^{(K)}(u+iv) = \sum_{\substack{|m|\le K\\ |n|\le K}} b_{m,n} e(mu+nv). \tag{2.11}$$

Since

$$|F_{\Delta}(u+iv) - F_{\Delta}^{(K)}(u+iv)| \le \Delta,$$

from (2.9) we have

$$\sum_{\substack{\beta\in xW\\ \beta\in\mathcal{T}}} g(\beta) = \sum_{\substack{m_1,\dots,m_k\\ n_1,\dots,n_k}}{}^{*} b_{m_1,n_1}\dots b_{m_k,n_k} \sum_{\beta\in xW} g(\beta)\chi_{m_1,\dots,n_k}(\beta).$$

The star indicates that we sum over those m_j, n_j for which $|m_j| \le K$, $|n_j| \le \le K$ $(j = 1,\dots,k)$, where $\chi_{m_1,\dots,n_k}(\beta) = e(\lambda \mathrm{Re}\,\beta + \mu \mathrm{Im}\,\beta)$,

$$\lambda = \sum_{j=1}^{k} (m_j\xi_j + n_j\eta_j), \quad \mu = \sum_{j=1}^{k} (n_j\xi_j - m_j\eta_j).$$

From the assumption of the theorem we have that either λ or μ is irrational, consequently, by Theorem D we have that

$$\sum_{\substack{\beta\in xW\\ \beta\in\mathcal{T}}} g(\beta) = a_{0,0}^{k} \sum_{\beta\in xW} g(\beta) + o_x(|xW|) + \mathcal{O}(\Delta|xW|).$$

Hence we obtain that

$$\lim_{x\to\infty} \sup_{g\in\mathcal{K}_1} \left| \frac{1}{|xW|} \sum_{\substack{\beta\in xW\\ \beta\in\mathcal{T}}} g(\beta) - \frac{a_{0,0}^{k}}{|xW|} \sum_{\beta\in xW} g(\beta) \right| \le c\Delta.$$

Since Δ is arbitrary, therefore our theorem is true. ■

The proof of Theorem 5 is similar. We omit it.

References

[1] **Daboussi, H. and H. Delange,** Quelques proprietes des functions multiplicatives de module au plus egal 1, *C. R. Acad. Sci. Paris, Ser. A,* **278** (1974), 657–660.

[2] **Daboussi, H. and H. Delange,** On multiplicative arithmetical functions whose module does not exceed one, *J. London Math. Soc.,* **26** (1982), 245–264.

[3] **Delange, H.,** Generalization of Daboussi's theorem, In: *Colloq. Math. Soc. János Bolyai* **34**, Topics in Classical Number Theory, Budapest, 1981, 305–318.

[4] **Dupain, Y., R.R. Hall and G. Tenenbaum,** Sur l'equirepartition modulo 1 de certaines fonctions de diviseurs, *J. London Math. Soc. (2),* **26** (1982), 397–411.

[5] **Goubain, L.,** Sommes d'exponentielles et principe de l'hyperbole, *Acta Arith.,* **73** (1995) 303–324.

[6] **Montgomery, H.L. and R. C. Vaughan,** Exponential sums with multiplicative coefficients, *Invent. Math.,* **43** (1977), 69–82.

[7] **Kátai, I.,** A remark on a theorem of H. Daboussi, *Acta Math. Hungar.,* **47(1-2)** (1986), 223–225.

[8] **Kátai, I.,** Uniform distribution of sequences connected with arithmetical functions, *Acta Math. Hungar.,* **51(3-4)** (1988), 401–408.

[9] **Indlekofer, K.-H. and I. Kátai,** Exponential sums with multiplicative coefficients, *Acta Math. Hungar.,* **53(3-4)** (1989), 263–268.

[10] **Indlekofer, K.-H. and I. Kátai,** On a theorem of Daboussi, *Publ. Math. Debrecen,* **57(1-2)** (2000), 145–152.

[11] **Bassily, N.L., J.-M. De Koninck and I. Kátai,** On a theorem of Daboussi related to the set of Gaussian integers, *Mathematica Pannonica,* **14/2** (2003), 267–272.

[12] **Indlekofer, K.-H. and I. Kátai,** A note on a theorem of Daboussi, *Acta Math. Hungar.,* **101(3)** (2003), 211–216.

[13] **De Koninck, J.-M. and I. Kátai,** On the distribution modulo 1 on the values of $F(n)+\alpha\sigma(n)$, *Publ. Math. Debrecen,* **66** (2005), 121–128.

[14] **Kátai, I.,** A remark on trigonometric sums, *Acta Math. Hungar.,* **112(3)** (2006), 227–231.

[15] **Indlekofer, K.-H. and I. Kátai,** Some remarks on trigonometric sums, *Acta Math. Hungar.,* **118(4)** (2008), 313–318.

[16] **Huixue Lao,** A remark on trigonometric sums, *Acta Arithmetica,* **134** (2008), 127–131.

[17] **Daboussi, H.,** On some exponential sums, In: *Analytic Number Theory,* Proceedings of a Conference in honour of Paul T. Bateman, Birkhäuser, Boston, 1990, 111–118.

[18] **Indlekofer, K.-H.,** Properties of uniformly summable multiplicative functions, *Periodica Math. Hungar.,* **17** (1986), 143–161.

[19] **Kátai, I.,** Some remarks on a theorem of H. Daboussi, *Mathematica Pannonica,* **19** (2008), 71–80.

[20] **Kátai, I.,** On the sum of bounded multiplicative functions over some special subsets of integers, *Uniform Distribution Theory,* **3** (2008), 37–43.

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