RESULTS ON CLASSES OF FUNCTIONAL EQUATIONS

TRIBUTE TO ANTAL JÁRAI

by János Aczél and Che Tat Ng

While *individual* noncomposite functional equations in several variables had been solved at least since d'Alembert 1747 [9] and Cauchy 1821 [8], results on broad *classes* of such equations began appearing in the 1950's and 1960's. On general *methods of solution* see e.g. Aczél [1] and for *uniqueness of solutions* Aczél [2, 3], Aczél and Hosszú [6], Miller [20], Ng [21, 22], followed by several others. – Opening up and cultivating the field of *regularization* is *mainly Járai's achievement.* By regularization we mean assuming weaker regularity conditions, say measurability, of the unknown function and proving differentiability of several orders, for whole classes of functional equations. Differentiability of the unknown function(s) in the functional equation often leads to differential equations that are easier to solve.

For example, in Aczél and Chung [5] it was shown that locally Lebesgue integrable solutions of the functional equation

$$\sum_{i=1}^{n} f_i(x+\lambda_i y) = \sum_{k=1}^{m} p_k(x) q_k(y)$$

holding for x, y on open real intervals, with appropriate independence between the functions, are in fact differentiable infinitely many times. The differentiable solutions are then extracted using induced differential equations. Járai [11] showed that Lebesgue measurability and ordinary linear independence are sufficient to lead to the same solutions.

Aczél [4] called attention to some unsolved problems in the area of functional equations. One concerned Hilbert's fifth problem. Járai [15] formulated a problem that falls within that general call for non-composite functional equations in multiple variables. Here we exhibit the intricate problem he formulated and the sequence of results that led to its solution, and make references to his comprehensive book Járai [16].

Problem. *Let T and Z be open subsets of $\mathbb{R}^s$ and $\mathbb{R}^m$, respectively, and let D be an open subset of $T \times T$. Let $f : T \to Z$, $g_i : D \to T$ $(i = 1, ..., n)$, and $h : D \times Z^n \to Z$ be functions. Suppose that*

$$f(t) = h(t, y, f(g_1(t, y)), ..., f(g_n(t, y))) \quad \textit{for all} \quad (t, y) \in D\,;$$

$$h \quad \textit{is analytic}\,;$$

$g_1, ..., g_n$ are analytic and for each $t \in T$ there exists a y for which

$$(t,y) \in D \quad \text{and} \quad \frac{\partial g_i}{\partial y} \quad \text{has rank} \quad s \quad \text{for each} \quad i = 1, ..., n.$$

Is it true that every solution f which is measurable, or has the Baire property, is also analytic?

He proposed some incremental steps which may be taken to address the problem:

(I) Measurability implies continuity.

(II) Almost open solutions are continuous.

(III) Continuous solutions are locally Lipschitz.

(IV) Locally Lipschitz solutions are continuously differentiable.

(V) All p-times continuously differentiable solutions are $(p+1)$-times differentiable.

(VI) Infinitely many times differentiable solutions are analytic.

In [19] Járai and Székelyhidi outlined the above steps and gave a survey on the advances made. Many historic attributions were made to contributors in the field. Ng [23] contains results concerning the functional equation

$$f(x) + g(y) = h(T(x,y))$$

with given T. It is shown that under suitable assumptions, local boundedness of f implies the continuity of g.

Járai published a sequence of papers obtaining impressive results about that problem. [12] contains results regarding (I), (II), (IV), (V), and partially about (III). Step (III) is obtained for one variable in [13] and is generalized in [14]. In [15] Járai obtained the following result on the problem formulated above.

Theorem. *Suppose that the conditions of the Problem are satisfied and suppose that f has locally essentially bounded variation. Then f is infinitely many times differentiable.*

[16] contains, in Section 1, a summary account about the problem. We include some of it (abbreviated).

Theorem. (i) *If h is continuous and the functions g_i are continuously differentiable then every solution f, which is Lebesgue measurable or has Baire property, is continuous.*

(ii) *If h and g_i are p times continuously differentiable, then every almost everywhere differentiable solution f is p times continuously differentiable.*

(iii) *If* h *and* g_i *are* $\max\{2,p\}$ *times differentiable and there exists a compact subset* C *of* T *such that for each* $t \in T$ *there exists a* $y \in T$ *satisfying* $g_i(t,y) \in C$*, besides the other stated rank condition on* g_i*, then every solution* f*, which is Lebesgue measurable or has the Baire property, is* p *times continuously differentiable* $(1 \le p \le \infty;\ i = 1, ..., n)$.

Járai has deep insights and knowledge in the field of real analysis. He used the theorems reported in Giusti [10] swiftly, made fine and technical adaptations when necessary to get the above strong results.

In his book [16] many regularization theorems by him and others are assembled in a well organized way. For the convenience of the readers he has given several examples to illustrate how his general results can be applied to known functional equations. He devised and proved a general transfer principle which makes it possible to apply theorems concerning problems having only one unknown function also for cases with several unknown functions. A good example amongst many is the following

Theorem. *Let* $\alpha \neq \beta$ *be fixed real numbers,* $f, g_1, g_2 :]0,1[\to \mathbb{R}$*. Suppose that the functional equation*

$$\begin{aligned} &f(x) + (1-x)^{\alpha} g_1(u/(1-x)) + (1-x)^{\beta} g_2(u/(1-x)) \\ &= f(u) + (1-u)^{\alpha} g_1(x/(1-u)) + (1-u)^{\beta} g_2(x/(1-u)) \end{aligned}$$

is satisfied for all $x, u \in]0,1[$ *with* $x + u \in]0,1[$*. If the functions* f, g_1, g_2 *are Lebesgue measurable then they are* $\mathcal{C}^{\infty}$.

He offered readers some details which precede the applications of his regularization theorems. The functional equation has its source in the study of symmetric divergences and distance measures and the differentiable solutions have been reported by Sander [25]. A more elaborate example is their joint work in [18] connected to the Weierstrass sigma function (as in [7]). They extended the results of M. Bonk [7] on the functional equation

$$\chi(u+v)\phi(u-v) = \sum_{\nu=1}^{k} f_{\nu}(u) g_{\nu}(v)$$

and treated it under weaker regularity assumptions.

Section 16 of the book contains results on (VI), analyticity. Járai's results as well as those of Páles [24] are covered. In Járai, Ng and Zhang [17] a composite type functional equation is solved under different regularity assumptions. The uniqueness theorem of Ng [22] is applied to obtain continuous solutions in one case, and the differentiation steps are used to extract the differentiable solutions in another case.

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J. Aczél and C.T. Ng
Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario
Canada N2L 3G1
jdaczel@uwaterloo.ca
ctng@uwaterloo.ca