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ON THE MAXIMAL RUN-LENGTH FUNCTION IN CONTINUED FRACTIONS

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Dedicated to Professor János Galambos on the occasion of his 70^{th} birthday

Abstract. This paper is concerned with the metrical property and fractal structure of maximal run-length function in an infinite symbolic system: continued fraction dynamical system. More precisely, let $[a_1(x), a_2(x), \ldots]$ be the continued fraction expansion of $x \in [0, 1)$. Call

$$R_n(x) := \max_{i \geq 1} \left\{ k : a_{j+1}(x) = \cdots = a_{j+k}(x) = i, \text{ for some } 0 \leq j \leq n-k \right\}$$

the n-th maximal run-length function of x, which represents the longest run of same symbol in the first n partial quotients of x. We show that

$$\lim_{n\to\infty} \frac{R_n(x)}{\log_{\frac{\sqrt{5}+1}{2}} n} = \frac{1}{2}, \quad \text{a.e. } x \in [0, 1).$$

This extends a result of Erdős and Rényi in finite symbolic space. At the same time, fractal structure of exceptional sets with respect to above metrical result are also studied.

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1. Introduction

The run-length function was first raised in a teaching experiment in mathematics.The experiment goes like this [23]: The students are divided into two groups in a class and are asked to get a sequence of 1s and 0s with length two hundreds. In one group, the children obtain the sequence by tossing a cion and record the resulting heads and tails. In the other group, the children just write down a sequence as "random" as they may feel. The result is that one can easily distinguish the students from one group to another.

This is revealed by the following large number law given by Erdős and Rényi [5]. Denote by

$$Z_n := \max\left\{k \geq 1 : \epsilon_{i+1} = \cdots = \epsilon_{i+k} = 1, 1 \leq i \leq n-k\right\}$$

for the longest run of 1 in a Bernoulli trials.

Theorem 1.1. *Almost surely,*

$$\lim_{n\to\infty} \frac{Z_n}{\log_2 n} = 1.$$

See, [23] and reference therein, for a thorough investigation of metrical properties on above mentioned run-length function, and for the dimensional result see [20].

Above large number law gives a criterion to discern a random sequence from a non-random sequence. Similarly, we would like to ask what is a random real number should be. Essentially, the result of Erdős and Rényi discloses a property of a random number in dyadic expansion.

In this note, we consider the properties of run-length function in the continued fraction expansion of real numbers.

Let $[a_1(x), a_2(x), \cdots]$ be the continued fraction expansion of $x \in [0, 1)$. For any $n \geq 1$, define

$$R_n(x) := \max_{i\geq 1}\left\{k : a_{j+1} = \cdots = a_{j+k} = i, \text{ for some } 0 \leq j \leq n-k\right\}$$

and call it the n-th maximal run-length function of x. We show

Theorem 1.2.

$$\lim_{n\to\infty} \frac{R_n(x)}{\log_{\frac{\sqrt{5}+1}{2}} n} = \frac{1}{2}, \quad a.e.\ x \in [0, 1).$$

It is obvious that there exist points violating from above law by assuming on other asymptotic properties on R_n. At the same time, we study the size of such sets.

Let $\{\delta_n\}_{n=1}^{\infty}$ be a nondecreasing integer sequence with $\delta_n \to \infty$ as $n \to \infty$. Write

$$E(\{\delta_n\}_{n=1}^{\infty}) = \Big\{x \in [0,1) : \lim_{n\to\infty} \frac{R_n(x)}{\delta_n} = 1\Big\}.$$

and

$$F(\{\delta_n\}_{n=1}^{\infty}) = \Big\{x \in [0,1) : \limsup_{n\to\infty} \frac{R_n(x)}{\delta_n} = 1\Big\}.$$

Theorem 1.3. *Assume* $\lim_{n\to\infty} \frac{\delta_{n+\delta_n}}{\delta_n} = 1$. *Then*

$$\dim_H E(\{\delta_n\}_{n=1}^{\infty}) = 1.$$

Theorem 1.4. *Write* $\liminf_{n\to\infty} \frac{\delta_n}{n} = \alpha \in [0,1]$. *The dimension of* $F(\{\delta_n\}_{n=1}^{\infty})$ *is given by the solution to the pressure function*

$$P\left(-s\big(\log|T'| + \frac{\alpha}{1-\alpha}\log\tau(1)\big)\right) = 0,$$

where T is the Gauss map and $P(\phi)$ denotes the pressure function with the potential ϕ is defined as

$$P(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{y:T^n y = x} \exp\left\{\phi(y) + \cdots + \phi(T^{n-1}y)\right\}.$$

Remark. We will see (Lemma 2.4) that, except a countable set, for all $x \in [0,1)$,

$$\liminf_{n\to\infty} \frac{R_{n+R_n(x)}(x)}{R_n(x)} = 1.$$

Hence, the assumption in Theorem 1.3 is reasonable to some extent.

Remark. Theorem 1.4 can also serve as a complementary to Theorem 1.3, which indicates that there exists $\{\delta_n\}_{n=1}^{\infty}$ such that $\dim_H E(\{\delta_n\}_{n=1}^{\infty})$ is not always 1. More precisely, take $x_0 \in F(\{\delta_n\}_{n=1}^{\infty})$. Let $\delta'_n = R_n(x_0)$ and

$$E(\{\delta'_n\}_{n=1}^{\infty}) = \Big\{x \in [0,1) : \lim_{n\to\infty} \frac{R_n(x)}{\delta'_n} = 1\Big\}.$$

Then we have $x_0 \in E(\{\delta'_n\}_{n=1}^{\infty}) \subset F(\{\delta_n\}_{n=1}^{\infty})$. So, we have $E(\{\delta'_n\}_{n=1}^{\infty}) \neq \emptyset$ and $\dim_H E(\{\delta'_n\}_{n=1}^{\infty}) \neq 1$ whenever $\dim_H F(\{\delta_n\}_{n=1}^{\infty})$ does not.

The investigation on the fractal structure of sets arising in continued fractions can be traced back to Jarnik [14] in 1928, where he studied the set of badly approximable points, equivalently, the points with bounded partial quotients in continued fraction expansion. In 1941, Good [10] presented a rather overall exploration on the Hausdorff dimension of sets of numbers with general restrictions on their partial quotients. Within the last twenty years, with the flourish of the theory of dynamical systems, great importance is attached on continued fractions once again. Because continued fraction system can be viewed as a classical dynamical system with infinite iterated branches (see [16, 17, 22] and reference therein). For other dimensional results on the set arising in continued fraction, see [4, 7, 8, 12, 13, 19, 27] and reference therein.

It should be also mentioned that, run-length function can also be defined in other representations of numbers and maybe there will be more interesting results. For a rich study of the representation of real numbers, we refer to the monograph of J. Galambos [9].

2. Preliminaries

In this section, we collect some elementary properties shared by continued fractions and present some initial properties possessed by the run-length function R_n.

Continued fraction expansion is induced by the Gauss map $T : [0,1) \to [0,1)$ given by

$$T(0) := 0, \quad T(x) := \frac{1}{x} \pmod 1 \quad \text{for } x \in (0,1). \tag{2.1}$$

Then every irrational number $x \in [0,1)$ can be uniquely expanded into an infinite form

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ddots}}} \tag{2.2}$$

where $a_1(x) = [\frac{1}{x}]$ and $a_n(x) = a_1(T^{n-1}(x))$ for $n \geq 2$ are called the partial quotients of x. For any $n \geq 1$ and $(a_1, \cdots, a_n) \in \mathbf{N}^n$, call

$$I(a_1, \cdots, a_n) = \begin{cases} \left[\dfrac{p_n}{q_n}, \dfrac{p_n + p_{n-1}}{q_n + q_{n-1}}\right), & \text{when } n \text{ is even;} \\ \left(\dfrac{p_n + p_{n-1}}{q_n + q_{n-1}}, \dfrac{p_n}{q_n}\right], & \text{when } n \text{ is odd} \end{cases}$$

an n-th order cylinder, where $p_k,\ q_k,\ 1 \leq k \leq n$, are determined by following recursive relations

$$p_k = a_k p_{k-1} + p_{k-2},\ q_k = a_k q_{k-1} + q_{k-2},\ 1 \leq k \leq n \tag{2.3}$$

with the conventions that $p_{-1} = 1, p_0 = 0,\ q_{-1} = 0, q_0 = 1$. It is well known, see [18], that $I(a_1, \cdots, a_n)$ just represents the set of points in $[0, 1)$ which have a continued fraction expansions begin with $a_1, \cdots, a_n$, i.e.,

$$I(a_1, \cdots, a_n) := \big\{x \in [0, 1) : a_1(x) = a_1, \cdots, a_n(x) = a_n\big\}.$$

Proposition 2.1. ([18]) *For any $n \geq 1$ and $(a_1, \cdots, a_n) \in \mathbb{N}^n$, one has*

$$\big|I(a_1, \cdots, a_n)\big| = \frac{1}{q_n(q_n + q_{n-1})}, \tag{2.4}$$

where $|I(a_1, \cdots, a_n)|$ denotes the length of $I(a_1, \cdots, a_n)$.

Proposition 2.2. ([28]) *For any $n \geq 1$ and $1 \leq k \leq n$,*

$$1 \leq \frac{q_n(a_1, \cdots, a_n)}{q_\ell(a_1, \cdots, a_k) q_{n-k}(a_{k+1}, \cdots, a_n)} \leq 2. \tag{2.5}$$

$$\frac{a_k + 1}{2} \leq \frac{q_n(a_1, a_2, \cdots, a_n)}{q_{n-1}(a_1, \cdots, a_{k-1}, a_{k+1}, \cdots, a_n)} \leq a_k + 1.$$

If $a_k = i$, for all $1 \leq k \leq n$, then

$$\tau^n(i) \leq q_n(i, \cdots, i) = \frac{\tau^{n+1}(i) - \varsigma^{n+1}(i)}{\tau(i) - \varsigma(i)} \leq 2\tau^n(i), \tag{2.6}$$

where $\tau(i) = \frac{i + \sqrt{i^2 + 4}}{2}$ and $\varsigma(i) = \frac{i - \sqrt{i^2 + 4}}{2}$.

For the Gauss map T, it is known that Gauss measure μ given as

$$d\mu = \frac{1}{\log 2} \frac{1}{1 + x} dx$$

is T-invariant and ergodic. Besides this, the following ψ-mixing properties is quite essential in proving the metrical theory on the run-length function R_n.

Lemma 2.3. ([1, 15]) *For any $k \geq 1$, let $\mathbb{B}_1^k = \sigma(a_1, \cdots, a_k)$ and $\mathbb{B}_k^\infty = \sigma(a_k, a_{k+1} \cdots)$ denote by the σ-algebras generated by the random variables $a_1, \cdots, a_k$, respectively, $a_k, a_{k+1}, \cdots$. One has, for any $A \in \mathbb{B}_1^k$ and $B \in \mathbb{B}_{k+n}^\infty$,*

$$\mu(A \cap B) = \mu(A)\mu(B)(1 + \theta\rho^n),$$

with $|\theta| \leq K$, where K, ρ are positive constants, $\rho < 1$, independent of A, B, n, k.

For a wealth of classical results about continued fractions, we recommend the books Khintchine [18], Hardy and Wright [11], Schmidt [24] and Bugeaud [2]. The books of Billingsley [1], Cornfeld, Fomin and Sinai [3] and Schweiger [25] contain an excellent introduction to the dynamics of the Gauss transformations and its connections with Diophantine approximation.

Now we present some feature possessed by the run-length function R_n. Since $R_n(x)$ depends only on the first n partial quotients, so sometimes we write $R_n(a_1, \cdots, a_n)$ for $R_n(x)$ when necessary.

Write $I = [0, 1)$. Denote by

$$U(I) = \{x \in I : a_n(x) = a_{n+1}(x), \text{ ultimately}\}.$$

It is clear that $U(I)$ is countable.

Lemma 2.4. *For any $x \in I \setminus U(I)$, we have*

$$\liminf_{n\to\infty} \frac{R_{n+R_n(x)}(x)}{R_n(x)} = 1.$$

Proof. For any $x \in I$ and $n \geq 1$, if $a_n(x) \neq a_{n+1}(x)$

$$\begin{aligned} R_{n+R_n}(x) &= \max\{R_n(a_1(x), \cdots, a_n(x)),\ R_{R_n}(a_{n+1}(x), \cdots, a_{n+R_n}(x))\} \\ &\leq \max\{R_n(x), R_n\} = R_n. \end{aligned}$$

Thus we have, for any $x \in I$, $R_{n+R_n} = R_n$ for infinitely many n's. ■

This serves the reason why we assume, in Theorem 1.3, that

$$\lim_{n\to\infty} \frac{\delta_{n+\delta_n}}{\delta_n} = 1.$$

The following is an equivalent condition to the assumption on $\{\delta_n\}_{n=1}^\infty$ in Theorem 1.3.

Lemma 2.5. *Let $\{\delta_n\}_{n=1}^\infty$ be an integer sequence with $\delta_n \to \infty$ as $n \to \infty$ and $\lim_{n\to\infty} \frac{\delta_{n+\delta_n}}{\delta_n} = 1$. Then, for any $M > 1$, we have*

$$\lim_{n\to\infty} \frac{\delta_{n+M\delta_n}}{\delta_n} = 1.$$

Proof. For any $\epsilon > 0$, by the assumption, there exists an integer N such that for any $n \geq N$, $\delta_{n+\delta_n} \leq (1+\epsilon)\delta_n$. Note that δ_n is increasing, so, for any $0 \leq j < M$,

$$\delta_{n+j\delta_n+\delta_{n+j\delta}} \geq \delta_{n+(j+1)\delta}.$$

As a result, for any $0 \leq j < M$ and $n \geq N$, we have

$$\delta_{n+(j+1)\delta} \leq (1+\epsilon)\delta_{n+j\delta}.$$

Therefore, for any $n \geq N$,

$$\frac{\delta_{n+M\delta_n}}{\delta_n} = \prod_{j=0}^{M-1} \frac{\delta_{n+(j+1)\delta}}{\delta_{n+j\delta}} \leq (1+\epsilon)^M.$$

This gives the desired result. ■

To end this section, we cite two tools to give a bound estimation on the Hausdorff dimension of a fractal set, namely Hölder properties and Billingsley Theorem [1, 6, 26].

Lemma 2.6. *Let $E \in \mathbb{R}^n$. If $f : E \to \mathbb{R}^m$ is α-Hölder, i.e., there exists constant $c > 0$ such that for all $x, y \in E$,*

$$|f(x) - f(y)| \leq c|x-y|^\alpha,$$

then $\dim_H f(E) \leq \frac{1}{\alpha} \dim_H E$.

Lemma 2.7. *Let $E \subset (0,1]$ be a Borel set and μ be a measure with $\mu(E) >$ > 0. If for any $x \in E$,*

$$\liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \geq s,$$

where $B(x,r)$ denotes the ball with center x and radius r, then $\dim_{\mathrm{H}} E \geq s$.

3. Metric property on R_n

Proof of Theorem 1.2. We show Theorem 1.2 in this section by following the ideas presented in [23] Page 71.

(1) We show that for almost all $x \in [0,1)$,

$$\limsup_{n \to \infty} \frac{R_n(x)}{\log_{\tau(1)} n} < \frac{1+\epsilon}{2}, \text{ for all } \epsilon > 0.$$

It suffices to show that

$$\mu\left\{x \in I : R_n(x) > \left[\frac{1+\epsilon}{2}\log_{\tau(1)n}\right] + 1 := u_n, \text{i.o.}\right\} = 0$$

where μ is the Gauss measure, and i.o. means infinitely often.

Borel-Cantelli Lemma will be applied to present this assertion. So we will estimate the measure of the set $\{R_n > u_n\}$.

Note that for any $x \in [0,1)$ with $R_n(x) = k$, there would exist integers $i \geq 1$ and $0 \leq j \leq n-k$ such that $a_{j+1}(x) = \cdots = a_{j+k}(x) = i$. Thus,

$$\begin{aligned}
\mu\{R_n > u_n\} &= \sum_{k=u_n+1}^{\infty} \mu\{R_n = k\} \leq \\
&\leq \sum_{k=u_n+1}^{\infty}\sum_{i=1}^{\infty}\sum_{j=0}^{n-k} \mu\{x \in I : a_{j+1}(x) = \cdots = a_{j+k}(x) = i\} = \\
&= \sum_{k=u_n+1}^{\infty}\sum_{i=1}^{\infty}\sum_{j=0}^{n-k} \mu\{x \in I : a_1(x) = \cdots = a_k(x) = i\} \leq \\
&\leq \sum_{k=u_n+1}^{\infty}\sum_{i=1}^{\infty} \frac{n}{\tau^{2k}(i)} \leq \sum_{i=1}^{\infty}\int_{u_n}^{\infty} \frac{n}{\tau^{2x}(i)}dx \leq \\
&\leq \frac{n}{2\log_{\tau(1)} n}\sum_{i=1}^{\infty}\frac{1}{\tau^{2u_n}(i)}.
\end{aligned}$$

Choose $N_0 \in \mathbb{N}$ such that $N_0^{\frac{1+\epsilon}{\log \tau(1)}} \geq 2e$. For any $n \geq N_0$,

$$\begin{aligned}
\sum_{i=1}^{\infty}\frac{n}{\tau^{2u_n}(i)} &\leq \frac{3n}{\tau^{2u_n}(1)} + \sum_{i=4}^{\infty}\frac{1}{\tau^{2u_n}(i)} \leq \\
&\leq \frac{3n}{n^{1+\epsilon}} + \sum_{k=1}^{\infty}\sum_{e^k < i \leq e^{k+1}} \frac{n}{n^{\frac{1+\epsilon}{\log \tau(1)}k}} \leq \frac{3}{n^{\epsilon}} + \sum_{k=1}^{\infty}\frac{e^{k+1}n}{n^{\frac{1+\epsilon}{\log \tau(1)}k}} \leq \\
&\leq \frac{3}{n^{\epsilon}} + \frac{2ne^2}{n^{\frac{1+\epsilon}{\log \tau(1)}}} \leq \frac{3}{n^{\epsilon}} + \frac{2ne^2}{n^{1+\epsilon}} \leq \frac{21}{n^{\epsilon}}.
\end{aligned}$$

Thus, choose $L > 0$ such that $L\epsilon > 1$, we have

$$\sum_{m=1}^{\infty}\mu\{R_{m^L} > u_{m^L}\} \leq N_0 + \sum_{m=1}^{\infty}\frac{21}{m^{L\epsilon}} < \infty.$$

So, for almost all $x \in [0,1)$, $R_{m^L} \leq u_{m^L}$ ultimately. Thus,

$$\limsup_{n\to\infty}\frac{R_n(x)}{u_n} \leq \limsup_{m\to\infty}\frac{R_{(m+1)^L}}{u_{m^L}} \leq \limsup_{m\to\infty}\frac{u_{(m+1)^L}}{u_{m^L}}\limsup_{m\to\infty}\frac{R_{(m+1)^L}}{u_{(m+1)^L}} \leq 1.$$

So, we get, for almost all $x \in [0,1)$,

$$\limsup_{n\to\infty} \frac{R_n(x)}{\log_{\tau(1)} n} \le \frac{1}{2}.$$

(2) We show that for almost all $x \in [0,1)$,

$$\liminf_{n\to\infty} \frac{R_n(x)}{\log_{\tau(1)} n} \ge \frac{1-\epsilon}{2},$$

for any $\epsilon > 0$.

It suffices to show that

$$\mu\left\{x \in I : R_n(x) < \left[\frac{1-\epsilon}{2} \log_{\tau(1)n}\right] - 1 := u_n, \text{i.o.}\right\} = 0.$$

Borel-Cantelli Lemma is used again. We fix some notation at first. Write $R_{m,n}(x) = R_{n-m}(a_{m+1}, \cdots, a_n)$ and $k_n = [\frac{n}{u_n^{1+\epsilon}}]$. Then

$$\{R_n < u_n\} \subset \{R_{iu_n^{1+\epsilon},\ iu_n^{1+\epsilon}+u_n}, 0 \le i < k_n\} =$$

$$= \{R_{iu_n^{1+\epsilon},\ iu_n^{1+\epsilon}+u_n} < u_n, 0 \le i < k_n - 1\} \cap \{R_{(k_n-1)u_n^{1+\epsilon},\ (k_n-1)u_n^{1+\epsilon}+u_n} < u_n\}.$$

Apply Lemma 2.3, we have

$$\begin{aligned}
&\mu\{R_n < u_n\} \le \\
&\le \mu\big\{R_{iu_n^{1+\epsilon},\ iu_n^{1+\epsilon}+u_n} < u_n, 0 \le i < k_n - 1\big\}\mu\big\{R_{u_n} < u_n\big\}(1 + \theta\rho^{u_n^{1+\epsilon}-u_n}) \le \\
&\le \Big(\mu\{R_{u_n} < u_n\}\Big)^{k_n} (1 + \theta\rho^{u_n^{1+\epsilon}-u_n})^{k_n} = \\
&= \left(1 - \sum_{i=1}^{\infty} \mu(I_{u_n}(i, \cdots, i))\right)^{k_n} (1 + \theta\rho^{u_n^{1+\epsilon}-u_n})^{k_n} \le \\
&\le e^{-k_n \mu(I_{u_n}(1,\cdots,1))} e^{k_n\theta\rho^{u_n^{1+\epsilon}-u_n}} \le e^{-\frac{1}{8}\frac{1}{\tau^{2u_n}(1)}\frac{n}{u_n^{1+\epsilon}}} e^{n\theta\rho^{u_n^{1+\epsilon}-u_n}} = \\
&= e^{-\frac{1}{8}\frac{n^\epsilon}{u_n^{1+\epsilon}}} e^{n\theta\rho^{u_n^{1+\epsilon}-u_n}} \le Me^{-\frac{1}{8}\frac{n^\epsilon}{u_n^{1+\epsilon}}},
\end{aligned}$$

where the last assertion follows from $n\theta\rho^{u_n^{1+\epsilon}-u_n} \to 0$ as $n \to \infty$.

So,

$$\sum_{n=1}^{\infty} \mu\left\{R_n < u_n\right\} \le \sum_{n=1}^{\infty} Me^{-\frac{1}{8}\frac{n^\epsilon}{u_n^{1+\epsilon}}} < \infty.$$

Thus, we have, for almost all $x \in [0,1)$,

$$\liminf_{n\to\infty} \frac{R_n(x)}{\log_{\tau(1)} n} \ge \frac{1}{2}.$$

This finishes the proof. ■

4. Dimensional results on run-length function

4.1. Proof of Theorem 1.3

We cite a lemma in [10] at first.

Lemma 4.1. *Let* $\{\mathcal{C}_n, n \geq 1\}$ *be a sequence of sets of positive integers. Let*

$$C = \{x \in I : a_n(x) \in \mathcal{C}_n, \text{ for all } n \geq 1\}.$$

Then $\dim_H(C \cap I(b_1, \cdots, b_n)) = \dim_H C$ *provided* $b_k \in \mathcal{C}_k$ *for all* $1 \leq k \leq n$*.*

This lemma indicates that any finite change on the restrictions on the partial quotients will not change the dimension. Recall that

$$E(\{\delta_n\}_{n=1}^{\infty}) = \Big\{x \in [0,1) : \lim_{n\to\infty} \frac{R_n(x)}{\delta_n} = 1\Big\}.$$

Since $\delta_n \to \infty$ as $n \to \infty$, we assume $\delta_n \geq 4$ for all $n \geq 1$.

Proof of Theorem 1.3. Fix $M \in \mathbb{N}$. Define an integer sequence recursively by setting

$$N_0 = 0, \quad \delta_{N_0} = \delta_1, \quad N_{k+1} = N_k + (M+1)\delta_{N_k}, \quad \text{for} \quad k \geq 0.$$

For any $B \geq 2$, define a subset $E_B^M(\{\delta_n\}_{n=1}^{\infty})$ of $E(\{\delta_n\}_{n=1}^{\infty})$ as follows.

$$E_B^M(\{\delta_n\}_{n=1}^{\infty}) = \Big\{x \in I \ : (a_{N_k+M\delta_{N_k}+4}(x), \ldots, a_{N_{k+1}}(x)) = (1, \ldots, 1),$$
$$\Big(a_{N_k+i\delta_{N_k}+j}(x)\Big)_{j=1}^{3} = (2,1,2), \quad k \geq 0, \ 1 \leq i \leq M;$$
$$1 \leq a_n(x) \leq B, \text{ for all other } n \geq 1\Big\}.$$

By Lemma 2.5, it is direct to check that

$$E_B^M(\{\delta_n\}_{n=1}^{\infty}) \subset E(\{\delta_n\}_{n=1}^{\infty}).$$

In the sequel, we estimate the dimension of $E_B^M(\{\delta_n\}_{n=1}^{\infty})$.

For any $k \geq 0$, write $C_k = 3km + \delta_{N_0} + \delta_{N_1} + \cdots + \delta_{N_{k-1}}$. Lemma 2.5 also implies

$$\lim_{k\to\infty} \frac{\delta_{N_k}}{N_k} = 0, \quad \text{and} \quad \lim_{k\to\infty} \frac{C_k}{N_k - C_k} = \frac{1}{M}.$$

So, for any $\epsilon > 0$, we can choose k_0 large such that for any $k \geq k_0$,

$$\delta_{N_k} \leq \frac{N_k - 1}{2}\epsilon, \text{ and } \frac{C_k}{N_k - C_k - 1} < \frac{2}{M}. \tag{4.1}$$

Take $x_0 \in E_B^M(\{\delta_n\}_{n=1}^{\infty})$, write

$$E_B^M\Big(\{\delta_n\}_{n=1}^{\infty}, x_0, k_0\Big) = E_B^M\Big(\{\delta_n\}_{n=1}^{\infty}\Big) \cap I(a_1(x_0), \ldots, a_{N_{k_0}}(x_0)).$$

Define

$$f_\epsilon : E_B^M(\{\delta_n\}_{n=1}^{\infty}, x_0, k_0) \to f_\epsilon\Big(E_B^M(\{\delta_n\}_{n=1}^{\infty}, x_0, k_0)\Big),$$
$$x = [a_1, a_2, \cdots] \to y = [b_1, b_2, \cdots],$$

where the sequence $(b_1, b_2, \cdots)$ is obtained by eliminating the terms

$$N_k + i\delta_{N_k} + 1, N_k + i\delta_{N_k} + 2, N_k + i\delta_{N_k} + 3, \text{ and } N_k + M\delta_{N_k} + 4, \ldots, N_{k+1}$$

for all $0 \leq i \leq M$ and $k \geq 1$ in the sequence $(a_1, a_2, \cdots)$.

Denote by

$$E_B = \{x \in I : 1 \leq a_n(x) \leq B, \text{ for all } n \geq 1\}.$$

By Lemma 4.1, it follows directly that

$$\dim_H f_\epsilon\Big(E_B^M(\{\delta_n\}_{n=1}^{\infty}, x_0, k_0)\Big) = \dim_H E_B.$$

Write

$$\alpha_1 = 1 + \epsilon(M+1)\log_2(B+1), \quad \alpha_2 = \big(1 + 4M^{-1}\big)\log_2(B+1).$$

In the following we will check that f_ϵ is $\frac{1}{\alpha_1\alpha_2}$-Hölder, which will give, by Lemma 2.7, that

$$\dim_H E_B^M(\{\delta_n\}_{n=1}^{\infty}) \geq \frac{1}{\alpha_1\alpha_2}\dim_H E_B. \tag{4.2}$$

For any $x_1, x_2 \in E_B^M(\{\delta_n\}_{n=1}^{\infty})$, let $y_1 = f_\epsilon(x_1), y_2 = f_\epsilon(x_2)$. Let n be the smallest integer such that $a_{n+1}(x_1) \neq a_{n+1}(x_2)$ and $N_k \leq n < N_{k+1}$ with some $k \geq k_0$. By the definition of f_ϵ, we know that

$$b_j(y_1) = b_j(y_2), \ 1 \leq j \leq N_k - C_k.$$

Since

$$x_i \in I(a_1(x_i), \ldots, a_{n+1}(x_i), a_{n+2}(x_i)) \subset \bigcup_{1 \leq a_{n+2} \leq B} I(a_1(x_i), \ldots, a_{n+1}(x_i), a_{n+2})$$

the difference of x_1 and x_2 is larger than the gap between

$$\bigcup_{1\le a_{n+2}\le B} I(a_1(x_i),\cdots,a_{n+1}(x_i),a_{n+2}),\ \ i=1,2.$$

Assume that $x_1>x_2$, then

$$|x_1-x_2|\ge\left|\frac{(B+1)p_{n+1}(x_1)+p_n(x_1)}{(B+1)q_{n+1}(x_1)+q_n(x_1)}-\frac{p_{n+1}(x_1)}{q_{n+1}(x_1)}\right|\ge\frac{1}{2(B+1)q^2_{N_{k+1}}(x_1)}.$$

By Proposition 2.2, we have

$$q_{N_{k+1}}(x_1)\le(B+1)^{(M+1)\delta_{N_k}}q_{N_k}(x_1)\le q^{\alpha}_{N_k}(x_1)$$

$$q_{N_k}(x_1)\le(B+1)^{C_k}q_{N_k-C_k}(y_1)\le q^{\alpha_2}_{N_k-C_k}(y_1).$$

So, we get

$$|x_1-x_2|\ge\frac{1}{2(B+1)}\Big(\frac{1}{q^2_{N_k-C_k}}\Big)^{\alpha_1\alpha_2}\ge\frac{1}{2(B+1)}\big|y_1-y_2\big|^{\alpha_1\alpha_2},$$

this gives the assertion (4.2). Thus,

$$\dim_H E(\{\delta_n\}_{n=1}^{\infty})\ge\frac{1}{\alpha_1\alpha_2}\dim_H E_B.$$

Letting $\epsilon\to0$, then $M\to\infty$ and finally $B\to\infty$, we get

$$\dim_H E(\{\delta_n\}_{n=1}^{\infty})\ge1.$$

■

4.2. Proof of Theorem 1.4

We first define a sequence of real numbers so-called pre-dimensional number as done in [27]. Let $\mathcal{A}\subseteq\mathbb{N}$ be a finite or infinite subset. For any $n\ge1$, $0<\alpha<1$ and $\rho\ge0$, define

$$f_n(\rho)=\sum_{a_1,\cdots,a_{n(1-\alpha)}\in\mathcal{A}}\left(\frac{1}{q_n(a_1,\cdots,a_{n(1-\alpha)},1,\cdots,1)}\right)^{2\rho}.$$

It is easy to see that $f_n(\cdot)$ is decreasing. By Proposition 2.1, we have $f_n(\rho)<1$ when ρ is large enough, define

$$s_n(\mathcal{A},\alpha)=\inf\{\rho\ge0:\ f_n(\rho)\le1\}.$$

Remark. If $\mathcal{A} \subseteq \mathbb{N}$ is finite, we have $f_n(s_n(\mathcal{A}, \alpha)) = 1$, that is

$$\sum_{a_1,\cdots,a_n \in \mathcal{A}} \frac{1}{q_n^{2s_n(\mathcal{A},\alpha)}(a_1, \cdots, a_{n(1-\alpha)}, 1, \cdots, 1)} = 1.$$

If $\mathcal{A} \subseteq \mathbb{N}$ is infinite, we have $f_n(s_n(\mathcal{A})) \leq 1$, that is

$$\sum_{a_1,\cdots,a_n \in \mathcal{A}} \frac{1}{q_n^{2s_n(\mathcal{A},\alpha)}(a_1, \cdots, a_{n(1-\alpha)}, 1, \cdots, 1)} \leq 1.$$

Proposition 4.2. $\lim_{n\to\infty} s_n(\mathcal{A}, \alpha)$ *exists.*

Write $\lim_{n\to\infty} s_n(\mathcal{A}, \alpha) = s(\mathcal{A}, \alpha)$. From Proposition 2.1 and the definition of $s_n(\mathcal{A}, \alpha)$, we have $0 \leq s(\mathcal{A}, \alpha) \leq 1$. For any $B \in \mathbb{N}$, take $\mathcal{A}_{\mathcal{B}} = \{1, 2, \cdots, B\}$. For simplicity, write $s_n(B, \alpha)$ for $s_n(\mathcal{A}_{\mathcal{B}}, \alpha)$, $s(B, \alpha)$ for $s(\mathcal{A}_{\mathcal{B}}, \alpha)$, $s_n(\alpha)$ for $s_n(\mathbb{N}, \alpha)$ and $s(\alpha)$ for $s(\mathbb{N}, \alpha)$.

Proposition 4.3. $\lim_{B\to\infty} s(B, \alpha) = s(\alpha)$.

Now we list some properties shared by $s(\alpha)$, which will be used late.

Proposition 4.4. *For any* $0 < \alpha < 1$, $s(\alpha) > \frac{1}{2}$. $s(\alpha)$ *is non-decreasing and continuous with respect to* α. *Moreover,*

$$\lim_{\alpha\to 0} s(\alpha) = \frac{1}{2} \quad \text{and} \quad \lim_{\alpha\to 1} s(\alpha) = 1.$$

We remark that the original idea for the proof of above propositions is coming from I. J. Good [10], and a detailed establishment of Good's idea is presented in [27], where a similar function is discussed. From a point of view of dynamical system, $s(\alpha)$ can be also given as the solution to the pressure function

$$P\left(-s\big(\log|T'| + \frac{\alpha}{1-\alpha}\log\tau(1)\big)\right) = 0,$$

where the pressure function $P(\phi)$ with the potential ϕ is defined as

$$P(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{y:T^n y = x} \exp\left\{\phi(y) + \cdots + \phi(T^{n-1}y)\right\}.$$

In such a way, the convergence of Proposition 4.3 comes from a result by Mauldin and Ubránski [21].

At last, we extend the definition of $s(\alpha)$ on $(0,1)$ to $[0,1]$ by :

$$S(\alpha)=\begin{cases}1, & \alpha=0;\\ s(\alpha), & 0<\alpha<1;\\ \frac{1}{2}, & \alpha=1.\end{cases}$$

We first study the dimension of

$$F(\alpha)=\left\{x\in I:\limsup_{n\to\infty}\frac{R_n(x)}{n}=\alpha\right\},\quad 0\le\alpha\le 1,$$

and then extend it to the general set $F(\{\delta_n\}_{n=1}^{\infty})$.

When $\alpha=0$, there is nothing to prove, since $F(\alpha)$ is a full set. So we only care for the case $0<\alpha\le 1$. We give the upper bound estimation first.

Lemma 4.5. *For any $0<\alpha\le 1$, we have $\dim_H F(\alpha)\le S(\alpha)$.*

Proof. Note that for any $0<\beta<\alpha$,

$$F(\alpha)\subset\{x\in I:R_n(x)>\beta n, i.o.\}=\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\{x\in I:R_n(x)>\beta n\}.$$

Moreover,

$$\{x\in I:R_n(x)>\beta n\}\subset\bigcup_{i=1}^{\infty}\bigcup_{0\le j<n-\beta n}\{x\in I:a_{j+1}(x)=\cdots=a_{j+\beta n}(x)=i\}\subset$$

$$\subset\bigcup_{i=1}^{\infty}\bigcup_{0\le j<n-\beta n}\bigcup_{a_1,\cdots,a_j,a_{j+\beta n+1},\cdots,a_n}I(a_1,\cdots,a_j,i,\cdots,i,a_{j+\beta n+1},\cdots,a_n).$$

By Proposition 2.2, the following geometry structurer will used frequently.

$$\frac{1}{8}\le\frac{|I(a_1,\cdots,a_j,i,\cdots,i,a_{j+\beta n+1},\cdots,a_n)|}{|I(a_1,\cdots,a_j,a_{j+\beta n+1},\cdots,a_n,i,\cdots,i)|}\le 8.$$

For any $s>S(\beta)>\frac{1}{2}$, let $\epsilon=\frac{s-S(\beta)}{2}$. There exists $n_0\ge 2$ such that for all $n\ge n_0$, $s>s_n+\epsilon$ and $2^{\frac{n-1}{2}\epsilon}>32n$. So,

$$H^s(F_\alpha) \le \liminf_{m\to\infty} \sum_{n=m}^{\infty}\sum_{i=1}^{\infty} \sum_{0\le j<n-\beta n} \sum_{b_1,\cdots,b_{n-\beta n}} 8|I(b_1,\cdots,b_{n-\beta n},i,\cdots,i|^s \le$$

$$\le \liminf_{m\to\infty} \sum_{n=m}^{\infty}\sum_{i=1}^{\infty} \sum_{b_1,\cdots,b_{n-\beta n}} 8n\left(\frac{1}{q_n(b_1,\cdots,b_{n-\beta n},i,\cdots,i)}\right)^{2s} \le$$

$$\le \liminf_{m\to\infty} \sum_{n=m}^{\infty}\sum_{i=1}^{\infty} \sum_{b_1,\cdots,b_{n-\beta n}} 32n\left(\frac{1}{q_n(b_1,\cdots,b_{n-\beta n},1,\cdots,1)}\right)^{2s} \times$$

$$\times \left(\frac{q_{n\beta}(1,\cdots,1)}{q_{n\beta}(i,\cdots,i)}\right)^{2s} \le$$

$$\le \liminf_{m\to\infty} \sum_{n=m}^{\infty}\sum_{i=1}^{\infty} \left(\frac{q_{n\beta}(1,\cdots,1)}{q_{n\beta}(i,\cdots,i)}\right)^{2s} \le \liminf_{m\to\infty} \sum_{n=m}^{\infty}\sum_{i=1}^{\infty} \frac{q_{n\beta}(1,\cdots,1)}{q_{n\beta}(i,\cdots,i)} \le$$

$$\le 3 + \liminf_{m\to\infty} \sum_{n=m}^{\infty}\sum_{i=4}^{\infty} \left(\frac{2}{\tau(i)}\right)^n < \infty.$$

Thus, we get $\dim_H F(\alpha) \le S(\beta)$. Letting $\beta \to \alpha$, we have

$$\dim_H F(\alpha) \le S(\alpha).$$

■

For the lower bound, we will distinguish two cases according as $\alpha = 1$ or not.

Lemma 4.6. $\dim_H F(1) \ge \frac{1}{2}$.

Proof. For any $k \ge 1$, let $N_k = k!$, $B_k = 2^{\sqrt{N_k}}$ and $M_k = \frac{N_{k+1}}{\log N_k}$. Set

$$F = \left\{ x \in I : a_n(x) \left\{ \begin{array}{ll} \in [1, B_k], & N_k < n \le N_k + M_k; \\ = 1, & \text{otherwise.} \end{array} \right. \right\}.$$

It is easy to see that $F \subset F(\alpha)$. First, we define a sequence of pre-dimensional numbers $\{s_k, k \ge 1\}$: let s_k be the real solution to the equation

$$\sum_{1\le a_1,\cdots,a_{M_k}\le B_k} \left(\frac{1}{q_{N_{k+1}-N_k}(a_1,\cdots,a_{M_k},1,\cdots,1)}\right)^{2s} = 1. \tag{4.3}$$

Let $\underline{s} = \liminf_{k\to\infty} s_k$. We claim that $\underline{s} = \frac{1}{2}$.

(i) It is evident that $\lim_{k\to\infty} \frac{\log B_k}{\log N_k} = 0$. So for any $\epsilon > 0$, we can choose k_0 such that for all $k \ge k_0$, $B_k^\epsilon - 2^\epsilon > B_k^{\frac{1}{2}\epsilon}$ and $\frac{1}{4} B_k^{\frac{1}{2}\epsilon M_k} \ge 2^{N_{k+1}}$.

Replace s in (4.3) by $\frac{1-\epsilon}{2}$, we have

$$\sum_{1\le a_1,\cdots,a_{M_k}\le B_k}\left(\frac{1}{q_{N_{k+1}-N_k}(a_1,\cdots,a_{M_k},1,\cdots,1)}\right)^{1-\epsilon}\ge$$

$$\ge\frac{1}{4}\sum_{1\le a_1,\cdots,a_{M_k}\le B_k}\left(\frac{1}{q_{M_k}(a_1,\cdots,a_{M_k})}\right)^{1-\epsilon}\frac{1}{q_{N_{k+1}-N_k-M_k}(1,\cdots,1)}\ge$$

$$\ge\frac{1}{4}\left(\sum_{i=1}^{B_k}\frac{1}{(i+1)^{1-\epsilon}}\right)^{M_k}\frac{1}{2^{N_{k+1}}}\ge\frac{1}{4}(B_k^{\epsilon}-2^{\epsilon})^{M_k}\frac{1}{2^{N_{k+1}}}\ge 1.$$

So, for all $k\ge k_0$, $s_k\ge\frac{1-\epsilon}{2}$. Thus $\underline{s}\ge\frac{1}{2}$.

(ii) For any $\epsilon>0$, choose k_0 large such that for all $k\ge k_0$, $(1+\frac{1}{\epsilon})^{M_k}<$ $<\tau(1)^{N_{k+1}-N_k-M_k}$. Replace s in (4.3) by $\frac{1+\epsilon}{2}$, we have

$$\sum_{1\le a_1,\cdots,a_{M_k}\le B_k}\left(\frac{1}{q_{N_{k+1}-N_k}(a_1,\cdots,a_{M_k},1,\cdots,1)}\right)^{1+\epsilon}\le$$

$$\le\sum_{1\le a_1,\cdots,a_{M_k}\le B_k}\left(\frac{1}{q_{M_k}(a_1,\cdots,a_{M_k})}\right)^{1+\epsilon}\frac{1}{q_{N_{k+1}-N_k-M_k}(1,\cdots,1)}\le$$

$$\le\left(\sum_{i=1}^{\infty}\frac{1}{(i)^{1+\epsilon}}\right)^{M_k}\left(\frac{1}{\tau(1)}\right)^{N_{k+1}-N_k-M_k}\le$$

$$\le(1+\frac{1}{\epsilon})^{M_k}\left(\frac{1}{\tau(1)}\right)^{N_{k+1}-N_k-M_k}\le 1.$$

So, $\underline{s}\le\frac{1}{2}$. This proves the claim.

For any $n\ge 1$, set

$$D_n=\left\{(a_1,\cdots,a_n)\in\mathbb{N}^n: a_i(x)\left\{\begin{array}{ll}\in[1,B_k], & N_k<i\le N_k+M_k;\\ =1, & \text{otherwise.}\end{array}\right.\right\}.$$

For any $(a_1,\cdots,a_n)\in D_n$, we call $I(a_1,\cdots,a_n)$ is an n-th order admissible interval (with respect to F). Then it is evident that

$$F=\bigcap_{n=1}^{\infty}\bigcup_{(a_1,\cdots,a_n)\in D_n}I(a_1,\cdots,a_n).$$

Now we define a probability measure μ supported on F. Since $\sharp D_1=1$, we set $\mu(I(1))=1$. For any $(a_1,\cdots a_{N_2})\in D_{N_2}$, let

$$\mu\big(I(a_1,\cdots a_{N_2})\big)=\left(\frac{1}{q_{N_2-N_1}(a_2,\cdots,\sigma_{N_2})}\right)^{s_1}$$

and for any $N_1 < n < N_2$ and $(\sigma_1, \cdots \sigma_n) \in D_n$, let

$$\mu\big(I(a_1, \cdots, a_n)\big) = \sum_{a_{n+1}, \cdots, a_{N_2}} \mu\big(I(a_1, \cdots, a_n, a_{N_2})\big).$$

where the summation is taken over all $(a_{n+1}, \cdots, a_{N_2})$ such that $(a_1, \cdots, a_{N_2}) \in$ $\in D_{N_2}$. Suppose for some $k \geq 2$, $\mu\big(I(a_1, \cdots, a_{N_k})\big)$ has been defined for any $(a_1, \cdots, a_{N_k}) \in D_{N_k}$. For any $(a_1, \cdots, a_{N_{k+1}}) \in D_{N_{k+1}}$, set

$$\mu\big(I(\sigma_1, \cdots, \sigma_{N_{k+1}})\big) = \left(\frac{1}{q_{N_{k+1}-N_k}(a_{N_k+1}, \cdots, a_{N_{k+1}})}\right)^{s_k} \mu\big(I(a_1, \cdots a_{N_k})\big),$$

and for any $N_k < n < N_{k+1}$ and $(a_1, \cdots a_n) \in D_n$, let

$$\mu\big(I(a_1, \cdots, a_n)\big) = \sum_{a_{n+1}, \cdots, a_{N_{k+1}}} \mu\big(I(a_1, \cdots, a_n, a_{n+1}, \cdots, a_{N_{k+1}})\big),$$

where the summation is over all $(a_{n+1}, \cdots, a_{N_{k+1}})$ with $(a_1, \cdots, a_{N_{k+1}}) \in$ $\in D_{N_{k+1}}$.

Until now, the set function $\mu : \{I(a), a \in D \backslash D_0\} \to \mathbb{R}^+$ is well defined. By definition of s_k, it is easy to check that for any $n \geq 1$ and $(a_1, \cdots, a_n) \in D_n$, we have

$$\mu\big(I(a_1, \cdots, a_n)\big) = \sum_{a_{n+1}} \mu\big(I(a_1, \cdots, a_{n+1})\big),$$

where the summation is taken over all a_{n+1} such that $(a_1, \cdots, a_n, a_{n+1}) \in$ $\in D_{n+1}$. Notice that

$$\sum_{a_1 \in D_1} \mu\big(J(a_1)\big) = 1,$$

by Kolmogorov extension theorem, the set function μ can be extended into a probability measure supported on F, which is still denoted by μ. From the definition of μ, we have for any $k \geq 1$ and $(a_1, \cdots, a_{N_k}) \in D_{N_k}$

$$\mu\big(I(a_1, \cdots, a_{N_k})\big) = \prod_{j=0}^{k-1} \left(\frac{1}{q_{N_{j+1}-N_j}(a_{N_j+1}, \cdots, a_{N_{j+1}})}\right)^{s_j}. \tag{4.4}$$

In order to apply the mass distribution principle to give a lower bound estimation of $\dim_H F$, we will estimate the measure of arbitrary balls. We estimate the measure of admissible intervals first.

For any $\epsilon > 0$, there exists k_0 such that for all $k \geq k_0$,

$$s_k > \underline{s} - \epsilon, 2^{2k} \leq 2^{\frac{N_k - 1}{2}\epsilon}, 2^{k+3} 2^{M_k} \leq 2^{(N_k-1)\epsilon}.$$

Denote

$$c_0 = 2^{2k_0+4}(B_{k_0}+1)^{N_{k_0}+2} \geq$$
$$\geq \max\left\{q^2_{N_{k_0}+2}(a_1,\ldots,a_{N_{k_0}+2}) : (a_1,\ldots,a_{N_{k_0}} \in D_{k_0+2})\right\}.$$

Then for any $k \geq k_0$ and $(a_1,\ldots,a_{N_k}) \in D_{N_k}$,

$$\begin{aligned} \mu\big(I(a_1,\ldots,a_{N_k})\big) &\leq c_0\prod_{j=0}^{k-1}\left(\frac{1}{q_{N_{j+1}-N_j}(a_{N_j+1},\ldots,a_{N_{j+1}})}\right)^{\underline{s}-\epsilon} \leq \\ &\leq c_0 2^{2k}\left(\frac{1}{q^2_{N_k}}\right)^{\underline{s}-\epsilon} \leq c_0\left(\frac{1}{q^2_{N_k}}\right)^{\underline{s}-2\epsilon} \leq \\ &\leq 2c_0|I(a_1,\ldots,a_{N_k})|^{\underline{s}-2\epsilon}. \end{aligned} \tag{4.5}$$

When $N_k < n < N_{k+1}$ for some $k \geq k_0$ and $(a_1,\ldots,a_n) \in D_n$, we divide it into two cases.

(i) $N_k < n \leq N_k + M_k$ for some $k \geq k_0$. By the definition μ and similar estimation as (4.4), we have

$$\mu(I(a_1,\ldots,a_n)) \leq$$
$$\leq c_0 2^{k+1}\left(\frac{1}{q_n^2}\right)^{\underline{s}-\epsilon}\sum_{a_{n+1},\ldots,a_{N_k+M_k}}\left(\frac{1}{q^2_{N_{K+1}-n}(a_{n+1},\ldots,a_{N_k+M_k},1,\ldots,1)}\right)^{s_k}.$$

Note that

$$1 =$$
$$= \sum_{\substack{b_{N_k+1},\ldots,b_n,\\ a_{n+1},a_{N_k+M_k}}}\left(\frac{1}{q^2_{N_{k+1}-N_k}(b_{N_k+1},\ldots,b_n,a_{n+1},\ldots,a_{N_k+M_k},1,\ldots,1)}\right)^{s_k} \geq$$
$$\geq \frac{1}{4}\sum_{b_{N_k+1},\ldots,b_n}\left(\frac{1}{q^2_{n-N_k}(b_{N_k+1},\ldots,b_n)}\right)^{s_k}\times$$
$$\times\sum_{a_{n+1},\ldots,a_{N_k+M_k}}\left(\frac{1}{q^2_{N_{k+1}-n}(a_{n+1},\ldots,a_{N_k+M_k},1,\ldots,1)}\right)^{s_k} \geq$$
$$\geq \frac{1}{4}\frac{1}{q_{n-N_k}(1,\ldots,1)}\sum_{a_{n+1},\ldots,a_{N_k+M_k}}\left(\frac{1}{q^2_{N_{k+1}-n}(a_{n+1},\ldots,a_{N_k+M_k},1,\ldots,1)}\right)^{s_k}.$$

So, we have

$$\mu(I(a_1,\ldots,a_n)) \leq c_0 2^{k+1}\left(\frac{1}{q_n^2}\right)^{\underline{s}-\epsilon} 2^{n-N_k+2} \leq c_0\left(\frac{1}{q_n^2}\right)^{\underline{s}-2\epsilon}. \tag{4.6}$$

(ii) $N_k + M_k < n < N_{k+1}$. This case is simple, because for any $(a_1, \dots, a_n) \in$ $\in D_n$

$$\mu(I(a_1, \dots, a_n)) = \mu(I(a_1, \dots, a_{N_{k+1}})).$$

So, by (4.5), for any $(a_1, \dots, a_n) \in D_n$, it holds trivially that

$$\mu(I(a_1, \dots, a_n)) \leq 2c_0 |I(a_1, \dots, a_n)|^{\underline{s}-2\epsilon}. \tag{4.7}$$

Now we estimate the measure $B(x, r)$ with x as center and r the radius. Let $r_0 = \frac{1}{c_0}$. For any $x \in F$, there exist $a_1, a_2, \dots$ such that $x \in I(a_1, \dots, a_n)$ and $(a_1, \dots, a_n) \in D_n$ for all $n \geq 1$. For any $0 < r < r_0$, there exist $n \geq N_{k_0} + 2$ such that

$$I(a_1, \dots, a_n, a_{n+1}) \leq r < I(a_1, \dots, a_n).$$

So, $I(a_1, \dots, a_{n+1}) \subset B(x, r) \subset I(a_1, \dots, a_{n-2})$. Thus, by (4.5) (4.6) and (4.7), we have

$$\mu(B(x, r)) \leq \mu(I(a_1, \dots, a_{n-2})) \leq 2c_0 \left(\frac{1}{q_{n-2}^2} \right)^{\underline{s}-2\epsilon}.$$

Then it is routine to check that $q_{n-2}^{1+\epsilon} \geq q_{n+1}$. So we have

$$\mu(B(x, r)) \leq 2c_0 \left(\frac{1}{q_{n+1}^2} \right)^{\underline{s}-3\epsilon} \leq 4c_0 r^{\underline{s}-3\epsilon}.$$

As a consequence, we get $\dim_H F \geq \underline{s}$. ∎

Lemma 4.7. *For any* $0 < \alpha < 1$, $\dim_H F(\alpha) \geq S(\alpha)$.

Proof. Given a sequence $\{N_k, k \geq 1\}$ with $N_k << N_{k+1}$. Write

$$N_{k+1} = \frac{\alpha(N_k - N_{k-1})\ell_k}{1 - \alpha} + N_k, M_k = \alpha(N_k - N_{k-1}).$$

Define

$$F(\alpha, B) =$$
$$= \{x \in I : 1 \leq a_n(x) \leq B, a_{N_{k+1} - M_{k+1} + 1} = \dots = a_{N_{k+1}} = 1, \forall k \geq 1, n \geq 1\}.$$

For any $x \in F(\alpha, B)$, we will construct an element $x' \in F(\alpha)$: Insert the digit string $(2, 1, 2)$ after each position $N_k + iM_k$, $0 \leq i \leq \ell_k$ in the continued fraction expansion of x. Denote by $F'(\alpha, B)$ the collection of all elements got in this way.

With a similar method established in Theorem 1.3, we can show that

$$\dim_H F'(\alpha, B) = \dim_H F(\alpha, B).$$

So, we only need to show

$$\dim_H F(\alpha, B) \geq S(\alpha, B).$$

The proof is almost the same as given in [27], where we studied the dimension of the set

$$\{x \in I : a_n(x) \geq \phi(n), \text{ i.o. } n \in \mathbb{N}\}.$$

So we refer to it with no details. ■

Proof of Theorem 1.4. It is a consequence from the proof of Lemma 4.7 by choosing the sequence N_k such that $\lim_{k\to\infty} \frac{\delta_{N_k}}{N_k} = \alpha$. ■

Also it can be checked with the same idea that

$$\dim_H\{x \in I : R_n(x) \geq \delta_n, \text{ i.o. } n \in \mathbb{N}\} = S(\alpha),$$

where $\alpha = \liminf_{n\to\infty} \frac{\delta_n}{n} \in [0, 1]$.

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