ON THE UNIQUENESS OF SOLUTIONS TO RELATIONS BASED ON PRODUCTS AND MIXED SUMS OF RANDOM VARIABLES

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To Professor Janos Galambos on his 70th birthday

Abstract. We discuss uniqueness of solutions to $Z \stackrel{d}{=} XY$ and $Z \stackrel{d}{=} U_1X_1 + +U_2X_2 + \cdots U_lX_l$, where the random variables are independent and $Z, X_1, \cdots, X_l$ are identically distributed. We also discuss symmetry of products and quotients of independent random variables.

1. Introduction

In the present paper we continue the work of Galambos and Simonelli [7] on characterization of probability distributions by properties of products of random variables through two related problems.

Let Z, U, and V be independent random variables satisfying

$$Z \stackrel{d}{=} UV. \tag{1}$$

Clearly, if the distribution functions of U and V are given, then the distribution function of Z is uniquely determined. The natural question is whether this property continues to hold if one prescribes the distribution functions of Z and U (or of Z and V). It is easy to see that in general the answer is no. For example, let $P(Z = 0) = P(U = 0) = 1$, then (1) holds for any random variable V. More interesting examples will be given in Section 2. Special cases in which the knowledge of any two factors in (1) uniquely determines the remaining variable were discussed by Kotz and Steutel [10], Huang and Chen [9], and Yeo and Milne [16, 17].

By letting V in (1) be some function of independent copies of Z, the previous question specializes into a variety of characterization problems. For example, let Z_i, $1 \le i \le l$, X_k, $1 \le k \le l + m$, $m, j \ge 1$, be independent identically distributed (iid) random variables, $U_1, \cdots U_{l+m}$, $1 \le k \le m+l$ be iid random variables, the U_i's independent of the X_i's and Z_j's. An interesting problem consists of characterizing all solutions to

$$Z_1 \stackrel{d}{=} U_1(X_1 + X_2), \tag{2}$$

or, more generally, to

$$Z_1 + \cdots + Z_l \stackrel{d}{=} U_1X_1 + U_2X_2 + \cdots + U_{l+m}X_{l+m}. \tag{3}$$

Several authors have investigated the case $U_1 = U^\delta$, $\delta > 0$, where U is a given random variable concentrated on $(0, 1)$. For U uniformly distributed on $(0, 1)$, the characterization of the solutions to (2) was given by Shanbhag [15] for $\delta = 1$, Artikis [4] for $0 < \delta < 1$, and Alamatsaz [1] for $\delta > 1$. The general case was solved by Pakes [13, 14]. In [14] Pakes states that these results could be generalized to arbitrary non-negative bounded random variables with prescribed $E(U^\delta)$.

In this paper we further discuss the uniqueness of the solutions to these problems. In dealing with the first question, we derive necessary conditions under which any two of the variables in (1) determine the remaining variable as well as necessary conditions for the product or quotient of two independent random variables to be symmetric about zero. Then we are going to consider problems related to the solutions to (2) and (3). Our setting will not be as general but our results will hold for a larger class of equations. One of our main tools of investigation is the Mellin transform, which we define next. Our presentation follows that in [6].

Let Y be a non-negative random variable with distribution function F_Y. The Mellin transform of Y is defined by

$$M_Y(s) = \int_0^{+\infty} y^s \, dF_Y(y).$$

M_Y always exists for all values of the parameter s in some strip $D = \{s : -a \le \le Re(s) \le b\}$, where $a, b \ge 0$, and $Re(z)$ denotes the real part of the complex number z. In the above definition we assume $0^s = 0$ for all $s \in D$. Next we extend the definition of Mellin transform to arbitrary random variables.

Let X be an arbitrary random variable with distribution function G_X, and put

$$X^+ = \begin{cases} X & \text{if } X \geq 0 \\ 0 & \text{otherwise,} \end{cases} \qquad X^- = \begin{cases} -X & \text{if } X < 0 \\ 0 & \text{otherwise.} \end{cases}$$

The Mellin transforms of X^+ and X^- are defined by

$$M_{X^+}(s) = \int_0^{+\infty} x^s \, dG_X(x) \qquad \text{and} \qquad M_{X^-}(s) = \int_{-\infty}^{0} (-x)^s \, dG_X(x).$$

Let η be a formal indeterminate with the property that $\eta\,\eta = \eta^2 = 1$. Then the function

$$M_X(s) = M_{X^+}(s) + \eta\, M_{X^-}(s),$$

defined in some strip D, completely determines the distribution function of X. We call M_X the Mellin transform of X. Whenever we say that M_X satisfies some given property we mean that M_{X^+} and M_{X^-} satisfy that property.

This transform shares many of the properties of the Fourier transform, for example, M_X uniquely determines the distribution functions of X, but it has an additional multiplicative property that makes it preferable to use in studying products and quotients of independent random variables.

Let X and Y be independent random variables. Then

$$\begin{aligned} M_{XY}(s) &= M_{(XY)^+}(s) + \eta\, M_{(XY)^-}(s) = \\ &= M_{X^+}(s) M_{Y^+}(s) + M_{X^-}(s) M_{Y^-}(s) + \\ &\qquad + \eta\, \left(M_{X^+}(s) M_{Y^-}(s) + M_{X^-}(s) M_{Y^+}(s)\right) = \\ &= \left(M_{X^+}(s) + \eta\, M_{X^-}(s)\right) \left(M_{Y^+}(s) + \eta\, M_{Y^-}(s)\right). \end{aligned}$$

Moreover

$$M_{\frac{1}{Y}}(s) = M_Y(-s),$$

which gives

$$M_{\frac{X}{Y}}(s) = \left(M_{X^+}(s) + \eta\, M_{X^-}(s)\right) \left(M_{Y^+}(-s) + \eta\, M_{Y^-}(-s)\right).$$

2. Characterization by products

We mentioned in the introduction that the knowledge of Z and U or V it is not enough to uniquely determine the remaining quantity. When this happens we say that

(1) does not have a unique solution. The next three examples shed some light on this problem, and justify the conditions in Theorem 1 and Theorem 4 below.

Example 1. Let U be a random variable uniformly distributed on $(-1, 1)$, $V = YX$, where $Y = 1, -1$, $P(Y = 1) = p$, $P(Y = -1) = 1 - p = q$, X a gamma random variable with parameters $(2, 1)$, and U, X, Y independent. Then direct calculation gives (see [6]) for details)

$$\begin{aligned} M_U(s) &= \frac{1}{2(s+1)} + \eta \frac{1}{2(s+1)}, \\ M_V(s) &= M_Y(s) M_Y(s) = (p + \eta\, q)\, \Gamma(2+s) \\ &= p\, \Gamma(2+s) + \eta\, q\, \Gamma(2+s), \quad \text{and} \\ M_Z(s) &= \frac{\Gamma(s+1)}{2} + \eta \frac{\Gamma(s+1)}{2}. \end{aligned}$$

M_Z does not depend on p, $0 < p < 1$. In other words, (1) does not have a unique solution.

The property that Z is both positive and negative is not needed. Moreover if the product is non-negative, characterization of V in (1), given the distribution functions of Z and U are known, is equivalent to the characterization of Y in $W = X + Y$ given X and W are known.

Example 2. Let us consider the following functions

$$f(t) = \begin{cases} 1 - |t| & |t| \le 1 \\ 0 & |t| > 1, \end{cases}$$

$$g(t) = \begin{cases} 1 - \frac{|t|}{2} & |t| \le 1 \\ \frac{1}{2|t|} & |t| > 1, \end{cases}$$

$$h(t) = \begin{cases} 1 - \frac{|t|}{2} & |t| \le 1 \\ \frac{1}{2\sqrt{|t|}} & |t| > 1. \end{cases}$$

Now f, g, h satisfy Polya's conditions (see [12], p. 83) and therefore they are characteristic functions of absolutely continuous distribution functions. Moreover

$$f(t)g(t) = f(t)h(t) \quad \text{for all } t \in R.$$

Let X, Y, W be independent random variables with characteristic functions f, g, and h, respectively. Then $X + Y \stackrel{d}{=} X + W$, and consequently, $e^X e^Y \stackrel{d}{=} e^X e^W$. That is, (1) does not have a unique solution.

The next example shows that the product of two independent random variables can be symmetric with neither of the two factors being symmetric. Here and throughout the rest of the paper we only deal with symmetry about zero. This example is taken from [6].

Example 3. Consider the characteristic function $f(t)$ in Example 2, and let $g(t) = f(t)$, if $|t| \leq 1$, g periodic with period 2. Then for any $0 < a < 1$, there exist independent random variables X and Y such that

$$\begin{aligned} M_X(s) &= \frac{1}{2}f(s) + \eta\,\frac{g(s)}{2}, \qquad \text{and} \\ M_Y(s) &= af(s) + \eta\,(1-a)f(s) \end{aligned}$$

(see [6], pp. 29-31, for details). Clearly X is not symmetric, and Y is symmetric if, and only if, $a = 1/2$. However

$$M_{XY}(s) = \frac{f^2(s)}{2} + \eta\,\frac{f^2(s)}{2},$$

which implies XY is a symmetric random variable for all $0 < a < 1$.

We are now ready to state our first theorem.

Theorem 1. *Let Z, U, V be independent random variables, $P(Z = 0) \neq 1$, and $Z \stackrel{d}{=} UV$. Let $D = \{s :\ -a \leq Re(s) \leq b\}$, a, $b \geq 0$, be a complex strip contained in the domains of the Mellin transforms of Z, U and V. We further assume that at least one of U or V is non-negative, or that U and V are both symmetric random variables.*

(i) *If for all $s \in D$, $M_U(s) \neq 0$, then the distribution functions of Z and U uniquely determine the distribution function of V.*

(ii) *Suppose $M_U(s) \neq 0$ for all s in some subset B of D containing a limit point. If the Mellin transform of V is analytic in D, then the distribution functions of Z and U uniquely determine the distribution function of V.*

Proof. Let Z, U, V be independent random variables, $P(Z = 0) \neq 1$, and $Z \stackrel{d}{=} UV$. For all s in some strip $D = \{s :\ -a \leq \text{Re}(s) \leq b\}$, a, $b \geq 0$, $M_Z(s)$ is given by

$$M_{U^+}(s)M_{V^+}(s) + M_{U^-}(s)M_{V^-}(s) + \eta\,\left(M_{U^+}(s)M_{V^-}(s) + M_{U^-}(s)M_{V^+}(s)\right).$$

Since the proofs of the cases $P(U \geq 0) = 1$ and $P(V \geq 0) = 1$ are essentially the same, we only give the details for the first case. We start by proving (i). In this case one immediately obtains that for any $s \in D$,

$$M_{V^+}(s) = \frac{M_{Z^+}(s)}{M_{U^+}(s)} \qquad \text{and} \qquad M_{V^-}(s) = \frac{M_{Z^-}(s)}{M_{U^+}(s)}$$

if $P(U \geq 0) = 1$, and

$$M_{V^+}(s) = M_{V^-}(s) = \frac{M_{Z^+}(s)}{2\, M_{U^+}(s)}$$

if U and V are symmetric random variables. In either case, the above equations are well defined and uniquely determine the positive and negative parts of the Mellin transform of V for all $s \in D$, and by the uniqueness property of the Mellin transform, this uniquely determines the distribution function of V. This proves part (i).

Next we let U, V_1, and V_2 be random variables satisfying the conditions in (ii), and either $P(U \geq 0) = 1$ or U, V_1 and V_2 symmetric. Suppose $Z \stackrel{d}{=} UV_1 \stackrel{d}{=} UV_2$. Then from the equation $M_{UV_1}(s) = M_{UV_2}(s)$ for all $s \in D$, it is easy to see that for all $s \in B$,

$$M_{V_1^+}(s) - M_{V_2^+}(s) = M_{V_1^-}(s) - M_{V_2^-}(s) = 0. \tag{4}$$

Since B contains a limit point, then by the Uniqueness Theorem of analytic functions, see Hille [8] for example, we further have that (4) continues to hold for all $s \in D$, thus giving $V_1 \stackrel{d}{=} V_2$. This completes the proof of (ii). ■

Whenever random variables Z, U, and V satisfy the conditions of Theorem 1, this theorem provides a simple, effective way to determine any one of the factors in (1) given the other two are known.

Let X be a gamma random variable with parameters (a_1, a_2), written as $X \sim \gamma(a_1, a_2)$, and let Y be a (type 1) beta random variable with parameters r and t, written as $Y \sim \beta_1(r, t)$. By direct computation (see [6]) one obtains

$$\begin{aligned} M_X(s) &= \frac{\Gamma(s + a_1)}{\Gamma(a_1)\, a_2^s} \quad \text{and} \\ M_Y(s) &= \frac{\Gamma(r+t)\Gamma(s+r)}{\Gamma(r)\,\Gamma(r+t+s)}, \end{aligned}$$

and therefore, Theorem 1 always holds if at least one of U or V is a gamma or a beta random variable. For example, one has the following result:

Lemma 2. Let Z, U, V be independent random variables, and let $Z \stackrel{d}{=} UV$. Then for arbitrary positive l and m, any two of the following three conditions imply the third.

$$\text{(i) } Z \sim \gamma(l, r); \qquad \text{(ii) } U \sim \beta(l, m); \qquad V \sim \gamma(l+m, r).$$

The above result is similar to a result of Yeo and Milne [17]. In their statement, however, it is further assumed that U and V are absolutely continuous and that U has bounded support. These assumptions are not needed here.

Next we are going to impose some conditions on the random variables in (2), and extend a result by Alzaid and Al-osh [3].

Let U, $U_1, \cdots U_l$ be given independent random variables, non-degenerate at zero. We say that the random variable Z, non-degenerate at zero, is a solution to

$$Z \stackrel{d}{=} U\Big(X_1 + \cdots + X_l\Big), \tag{5}$$

if Z, $X_1, \cdots, X_l$ are iid random variables independent of U. Similarly we say that Z is a solution to

$$Z \stackrel{d}{=} U_1 X_1 + \cdots + U_k X_k, \tag{6}$$

if Z, $X_1, \cdots, X_l$ are iid random variables independent of U_j, $1 \leq j \leq l$.

Theorem 3. *Suppose all random variables in* (5) *and* (6) *are defined by their moments and are non-negative. If there exists a solution* Z *to* (I)*,* $(I) = (5)$ *or* (6)*, then* (I) *admits an infinite number of solutions. Moreover if* W *is another solution to* (I)*, then* $W \stackrel{d}{=} cZ$*, for some real number c.*

Proof. Let us assume the random variables Z and W are solutions to (I)= (5) or (6), and let c be such that $E(W) = cE(Z)$. Clearly, cZ is also a solution to (I). We are going to show that $W \stackrel{d}{=} cZ$ by proving that all moments of W are uniquely determined by c and the moments of Z and U or U_j, $1 \leq j \leq l$. The proof is by induction on n. Clearly, our claim holds for $n = 1$. So let us assume that our claim holds for all moments of order less than or equal to n. If (I)=(5) and the X_j's are independent copies of W, we have

$$E\Big[W^{n+1}\Big] \quad = \quad E\Big[\Big(U(X_1 + \cdots + X_l)\Big)^{n+1}\Big]. \tag{7}$$

By using independence and our induction hypothesis, the right hand side of (7) can be written as

$$E[U^{n+1}] \, \left(l\, E[W^{n+1}] + h(Z,n)\right),$$

where $h[Z,n]$ is an expression containing moments of Z only of order less than or equal to n. Hence

$$E[W^{n+1}] = E[U^{n+1}] \, \left(l\, E[W^{n+1}] + h(Z,n)\right).$$

Since we are assuming that W and U are non-negative and non-degenerate at zero, we can divide both sides of the equation above by the quantity inside the parentheses obtaining

$$\frac{E[W^{n+1}]}{l\, E[W^{n+1}] + h(Z,n)} = E[U^{n+1}]. \tag{8}$$

The above equation uniquely determines $E[W^{n+1}]$. To see this we consider the function

$$f(x) = \frac{x}{lx + h(Z,n)}, \quad x \geq 0.$$

Clearly $f(x) = E[U^{n+1}]$ has a solution given by $x = c^{n+1}E[Z^{n+1}]$. Moreover, since

$$f'(x) = \frac{h(Z,n)}{(lx + h(Z,n))^2},$$

f is strictly increasing and therefore there is at most one value of x which satisfies $f(x) = E[U^{n+1}]$. Hence $E[W^{n+1}]$ is uniquely determined and this gives that $E[W^n] = c^n E[Z^n]$ for all $n \geq 1$. Since we are assuming that W is characterized by its moments, this further gives that $W \stackrel{d}{=} cZ$.

If W and Z are solutions to (6), by proceeding as above we obtain that

$$E[W^{n+1}] = E[W^{n+1}] \sum_{j=1}^{l} E[U_j^{n+1}] + q(Z, U_{\cdot}, n),$$

where $q(Z, U_{\cdot}, n)$ is an expression that contains moments of Z and U_j's of order at most n. The positivity of the terms above justify rewriting the above equation as

$$\frac{E[W^{n+1}]}{E[W^{n+1}] + e(Z, U_{\cdot}, n)} = \sum_{j=1}^{l} E[U_j^{n+1}],$$

where $e(Z, U_{\cdot}, n) = q(Z, U_{\cdot}, n) / \sum_{j=1}^{l} E[U_j^{n+1}]$. By reasoning as before one again obtains that the above equation uniquely determines $E[W^{n+1}]$, which implies $W \stackrel{d}{=}$ $\stackrel{d}{=} cZ$. ∎

Theorems 1 and 3 can be combined to deduce characterization results.

Lemma 4. *Suppose for some positive integer $k \geq 1$, $U, X_1, X_2, \cdots, X_{k+1}$ are independent non-negative random variables, $X_1, \cdots, X_{k+1}$ identically distributed, and*

$$Z \stackrel{d}{=} U(X_1 + \cdots + X_{k+1}). \tag{9}$$

Then any two of the following conditions imply the third.

(i) *Z has the same distribution, determined by its moments, as each of $X_1, \cdots, X_{k+1}$;*

(ii) *$U \sim \beta(l, m)$;*

(iii) *Z is exponentially distributed.*

The above result is similar to a result of Yeo and Milne [17]. In their statement, however, the authors assume that the random variable U is of bounded support, an assumption not needed here.

Note that the proof of Lemma 4 immediately follows from Theorems 1 and 3 and elementary properties of a beta random variable.

Next we turn to symmetry of products of independent random variables. Let X and Y be independent, and let $Z = XY$ and $W = X/Y$. Clearly, if X or Y is symmetric, then Z and W are also symmetric. As we have seen in Example 3, the symmetry of Z is not enough to characterize the symmetry of X or Y. The next result provides necessary conditions under which the symmetry of the product or quotient would necessarily imply the symmetry of at least one of the factors.

Theorem 5. *Let X and Y be independent random variables with continuous density functions $f(x)$ and $g(y)$, respectively.*

(i) *Let $Z = XY$. Suppose that there exists $\delta > 0$ such that for all $0 \le t \le \delta$, $E[|X|^t] < +\infty$ and $E[|Y|^t] < +\infty$. Then Z is symmetric if, and only if, at least one of X or Y is symmetric.*

(ii) *Let $W = X/Y$ and let us further assume that $X \overset{d}{=} Y$. Suppose that there exists $\delta > 0$ such that for all $-\delta \le t \le \delta$, $E[|X|^t] < +\infty$. Then W is symmetric if, and only if, X is symmetric.*

A question related to (ii) was discussed by Laha [11]. He proved that whenever $W = X/Y$ is a Cauchy random variable, and X and Y are iid, then X must be symmetric. Note that (i) is interesting only in the case X and Y are not identically distributed. If X and Y are iid, $Z = XY$ is symmetric if, and only if, X is symmetric with no further assumption (see [6]).

Proof. We start by proving (i). Suppose that X and Y are independent random variables, $Z = XY$, and $E[|X|^t]$ and $E[|Y|^t]$ are both finite for $0 \le t \le \delta$. We claim that $M_X(s)$ and $M_Y(s)$ are defined on the strip $D = \{s : 0 \le Re(s) \le \delta\}$, and analytic in its interior. That these functions are both defined in D is clear. To prove analyticity, we let $0 < a < b < \delta$ and consider s with $a < Re(s) < b$. Since for every $c > 0$, $|x|^s f(x)$ is a continuous function of x for fixed s, $0 < x < c$, and an analytic function of s, $a < Re(s) < b$, for fixed x,

$$\int_0^c x^s f(x)$$

is analytic in $a < Re(s) < b$.

Moreover, for any $c > 0$,

$$\begin{aligned}\left|\int_0^c x^s f(x)\,dx\right| &\leq \int_0^1 x^{Re(s)} f(x)\,dx + \int_1^{+\infty} x^{Re(s)} f(x)\,dx \\ &\leq \int_0^1 f(x)\,dx + \int_0^{+\infty} x^{\delta} f(x)\,dx,\end{aligned}$$

which shows that $\int_0^c x^s f(x)dx$ converges uniformly as $c \to +\infty$. Hence, $E[|X^+|^s]$ is analytic in $\{s : a < Re(s) < b\}$. Since a and b were arbitrary, this shows that $M_{X^+}(s)$ is analytic in the interior of D. Clearly the same result applies to $M_{X^-}(s)$ and $M_Y(s)$.

If Z is symmetric, then for all s in D, $M_{Z^+}(s) = M_{Z^-}(s)$, and this and the independence of X and Y give

$$(M_{X^+}(s) - M_{X^-}(s))\ (M_{Y^+}(s) - M_{Y^-}(s)) = 0.$$

Since M_X and M_Y are analytic, this implies that at least one of the equations

$$\begin{aligned}M_{X^+}(s) &= M_{X^-}(s), \\ M_{Y^+}(s) &= M_{Y^-}(s)\end{aligned}$$

must hold for all s in the interior of D. By continuity, this also holds for all s with $Re(s) = 0$. Hence, either X or Y must be symmetric.

Next we turn to the proof of (ii). Let us assume that X and Y are iid, $E[|X|^t] < < +\infty$ and $E[|Y|^{-t}|] < +\infty$ for all $0 \leq t \leq \delta$, and $W = X/Y$ is symmetric. From the proof of part (i) our assumptions on X and Y imply that the Mellin transforms of X and $1/Y$ are analytic for all $-\delta < t < \delta$. Therefore, by part (i), X must be symmetric. ∎

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