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ON THE FOURIER COEFFICIENTS WITH RESPECT TO THE DISCRETE LAGUERRE SYSTEM

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Dedicated to the 70th birthday of Professor János Galambos

Abstract. The discrete Laguerre-functions play an important role in system identification. In this paper we investigate the Fourier coefficients with respect to the discrete Lagueerre system. Among others an explicit form is given for the Laguerre Fourier coefficients of rational functions. With the help of this formula we introduce a new transformation which can be used to reconstruct the pole of rational function. The domain of the transformation in question can be defined in the term of hyperbolic distance.

1. Introduction

In signal processing and image reconstruction the Fourier-, wavelet-, Gábor- transforms play important roles. There exists a common generalization of these transformations, the so-called *Voice-transformation*. The voice transforms are generated by unitary representations of locally compact groups. The wavelet transform can be obtained in this way from the affine group,the Gabor transform from the Heisenberg group. In the papers $[6, 7, 8]$ a new transform, called *hyperbolic wavelet transform* has been introduced starting from a unitary representation of the *Blaschke group*. We

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hope that this transformation will be useful in signal processing. In this paper, with the help of this transformation, we construct an algorithm to reconstruct transfer functions of systems.

In the definition of the voice-transform a *unitary representation* of the generating group $(G, \cdot)$ is used. Let us consider a Hilbert-space $(H, \langle\cdot,\cdot\rangle)$ and let $\mathcal{U}$ denote the set of unitary bijections $U : H \to H$. Namely, the elements of $\mathcal{U}$ are bounded linear bijections which satisfy $\langle Uf, Ug\rangle = \langle f, g\rangle$ $(f, g \in H)$. The set $\mathcal{U}$ with the composition operation $(U \circ V)f := U(Vf)$ $(f \in H)$ is a group, the neutral element of which is I, the identity operator on H. The inverse element of $U \in \mathcal{U}$ is the operator U^{-1}. It is equal to the adjoint operator U^*. The homomorphism $G \ni x \to U_x \in \mathcal{U}$ of the group $(G, \cdot)$ on the group $(\mathcal{U}, \circ)$ satisfying

$$\text{(1.1)} \qquad \begin{array}{l} \text{i) } U_{x\cdot y} = U_x \circ U_y \ \ (x, y \in G), \\ \text{ii) } G \ni x \to U_x f \in H \ \text{ is continuous for all } f \in H \end{array}$$

is called the unitary representation of $(G, \cdot)$ on H. The *voice transform* of $f \in H$ generated by the representation U and by the parameter $\phi \in H$ is the (complex-valued) function on G defined by

$$\text{(1.2)} \qquad (V_\phi f)(x) := \langle f, U_x\phi\rangle \quad (x \in G, f, \phi \in H).$$

For any representation $U : G \to \mathcal{U}$ and for each $f, \phi \in H$ the voice transform $V_\phi f$ is a continuous and bounded function on G.

In this section we introduce a *voice transform on the Blaschke group.* The Blaschke function are closely related to the hyperbolic geometry. Namely this group can be considered as the *transformation group of congruences* in the Poincaré model of the hyperbolic plain.

The so called *Blaschke functions* are defined as

$$\text{(1.3)} \qquad B_{\mathfrak{b}}(z) := \epsilon\frac{z - b}{1 - \bar{b}z} \qquad (z \in \mathbb{C}, \mathfrak{b} = (b, \epsilon) \in \mathbb{B} := \mathbb{D} \times \mathbb{T}),$$

where

$$\text{(1.4)} \quad \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, \ \ \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \le 1\}, \ \ \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}.$$

If $\mathfrak{b} \in \mathbb{B}$, then $B_{\mathfrak{b}}$ is 1-1 map on $\mathbb{T}$ and $\mathbb{D}$. The restrictions of the Blaschke functions on the set $\mathbb{D}$ or on $\mathbb{T}$ with the operation $(B_{\mathfrak{b}_1} \circ B_{\mathfrak{b}_2})(z) := B_{\mathfrak{b}_1}(B_{\mathfrak{b}_2}(z))$ form a group. In the set of the parameters $\mathbb{B} := \mathbb{D} \times \mathbb{T}$ let us define the operation induced by the function composition in the following way $B_{\mathfrak{b}_1} \circ B_{\mathfrak{b}_2} = B_{\mathfrak{b}_1 \circ \mathfrak{b}_2}$. The group $(\mathbb{B}, \circ)$ will be isomorphic with the group $(\{B_{\mathfrak{b}}, \mathfrak{b} \in \mathbb{B}\}, \circ)$. The neutral element of the group $(\mathbb{B}, \circ)$ is $\mathfrak{e} := (0, 1) \in \mathbb{B}$ and the inverse element of $\mathfrak{b} = (b, \epsilon) \in \mathbb{B}$ is $\mathfrak{b}^{-1} = (-b\epsilon, \bar{\epsilon})$.

It can be prowed that the map

$$\text{(1.5)} \qquad \rho(z_1, z_2) := \frac{|z_1 - z_2|}{|1 - \bar{z}_1\, z_2|} = |B_{z_1}(z_2)| \ \ (B_{z_1} := B_{(z_1,1)}, z_1, z_2 \in \mathbb{D})$$

is a metric on $\mathbb{D}$. Moreover the Blaschke functions B_b $(b \in \mathbb{D})$ are isometries with respect to this metric, i.e.

$$\rho(B_b(z_1), B_b(z_2)) = \rho(z_1, z_2) \ \ (b \in \mathbb{D}, z_1, z_2 \in \mathbb{D}). \tag{1.6}$$

The lines in this model are the sets

$$\mathcal{L}_b := \{B_b(r) : -1 < r < 1\} \ \ (b \in \mathbb{D}),$$

i.e. circles crossing perpendicularly the unit circle.

The voice transform will be constructed on the Hardy space $H = H^2(\mathbb{T})$, where the inner product is given by

$$\langle f, g \rangle := \frac{1}{2\pi} \int_{\mathbb{I}} f(e^{it})\overline{g(e^{it})}\, dt \ \ (f, g \in H). \tag{1.7}$$

The system $h_n(z) := z^n$ $(z \in \overline{\mathbb{D}}, n \in \mathbb{N})$ is orthonormal and complete with respect to this scalar product.

For the definition of the hyperbolic wavelets we use the following representation of the Blaschke-group on $H^2(\mathbb{T})$:

$$(U_{\mathfrak{b}} f) := F_{\mathfrak{b}-1}\, f \circ B_{\mathfrak{b}^{-1}}, \tag{1.8}$$

where

$$F_{\mathfrak{b}}(z) := \frac{\sqrt{\epsilon(1 - |b|^2)}}{1 - \bar{b}z} \ \ (\mathfrak{b} = (b, \epsilon) \in \mathbb{B}, z \in \overline{\mathbb{D}}). \tag{1.9}$$

The representation $U_{\mathfrak{b}}$ $(\mathfrak{b} \in \mathbb{B})$ given by (1.8) is unitary on $H^2(\mathbb{T})$. The voice transform generated by $(U_b)_{b\in\mathbb{B}}$ is given by the following formula

$$(\mathcal{V}_\phi f)(\mathfrak{b}) := \langle f, U_{\mathfrak{b}}\phi \rangle \ \ (f, \phi \in H^2(\mathbb{T})). \tag{1.10}$$

The map $\mathcal{V}_\phi$ is called *hyperbolic wavelet transform,* the parameter function ϕ is the *hyperbolic mother wavelet* of the transform.

2. The discrete Laguerre system

For any $n \in \mathbb{B}$ and any $b \in \mathbb{D}$ the discrete Laguerre function $L_{n,b}$ is defined by

$$L_{n,b}(z) := \frac{\sqrt{1 - |b|^2}}{1 - \bar{b}z} \left(\frac{z - b}{1 - \bar{b}z} \right)^n \ \ (z \in \overline{\mathbb{D}}, n \in \mathbb{N}, b \in \mathbb{D}). \tag{2.1}$$

It is known that the system $(L_{n,b}, n \in \mathbb{N})$ is orthonormal and complete in $H^2(\mathbb{T})$ with respect to the scalar product (1.7). This is a consequence of the fact that these functions can be obtained from the power functions $h_n(z) := z^n \ \ (n \in \mathbb{N}, z \in \mathbb{C})$ by the unitary representation (1,10):

$$L_{n,b} = U_{\mathfrak{b}^{-1}} h_n \ \ (n \in \mathbb{N}, \mathfrak{b} = (b,1) \in \mathbb{B}). \tag{2.2}$$

$U_{\mathfrak{b}}$ is unitary, therefore $U_{\mathfrak{b}}^* = U_{\mathfrak{b}}^{-1} = U_{\mathfrak{b}^{-1}}$. Consequently for any $m, n \in \mathbb{N}$

$$\begin{aligned} &\text{i)} \quad \langle L_{n,b}, L_{m,b}\rangle = \langle U_{\mathfrak{b}^{-1}} h_n, U_{\mathfrak{b}^{-1}} h_m\rangle = \langle h_n, h_m\rangle = \delta_{mn}, \\ &\text{ii)} \qquad \langle f, L_{n,b}\rangle = \langle f, U_{\mathfrak{b}^{-1}} h_n\rangle = \langle U_{\mathfrak{b}} f, h_n\rangle. \end{aligned} \tag{2.3}$$

Thus the discrete Laguerre Fourier coefficients of f are equal to the trigonometric Fourier coefficients of the function $U_{\mathfrak{b}} f$. This relation can be used to compute the discrete Laguerre Fourier coefficients.

Introducing the notation (see (1.3) and (1.9))

$$\alpha_b := \frac{F_b'}{F_b}, \beta_b := B_b' \ \ (b \in \mathbb{D}) \tag{2.4}$$

the derivative of $L_{n,b}$ can be expressed in the form

$$L_{n,b}' = (F_b B_b^n)' = F_b' B_b^n + n B_b' F_b B_b^{n-1} = \frac{F_b'}{F_b} F_b B_b^n + n B_b' F_b B_b^{n-1}.$$

Consequently

$$L_{n,b}' = \alpha_b L_{n,b} + \beta_b n L_{n-1,b} \ \ (n \geq 1, b \in \mathbb{D}). \tag{2.5}$$

We show that for the derivatives of higher order the following recursion holds.

Thorem 1. *For every* $n \geq i, i, n \in \mathbb{N}$

$$L_{n,b}^{(i)} = \sum_{j=0}^{i} \gamma_{i,j,b} \binom{n}{j} L_{n-j,b} \ \ (b \in \mathbb{D}), \tag{2.6}$$

where

$$\begin{aligned} \gamma_{0,0,b} = 1, \ \ \gamma_{i,j,b} = 0 \ \ \text{if } j > i \text{ or } j < 0 \ \ (i \in \mathbb{N}) \ \ \text{and} \\ \gamma_{i+1,j,b} = \alpha_b \gamma_{i,j,b} + \gamma_{i,j,b}' + j \beta_b \gamma_{i,j-1,b} \ \ (j = 0, 1, \cdots, i+1). \end{aligned} \tag{2.7}$$

Proof. By definition and by (2.5)

$$L_{n,b} = \sum_{j=0}^{0} \binom{n}{j} L_{n,b}\gamma_{0,0,b} \ \ (\gamma_{0,0,b} := 1)$$

$$L'_{n,b} = \alpha_b L_{n,b} + \beta_b n L_{n-1,b} = \sum_{j=0}^{1} \binom{n}{j} L_{n-j,b}\gamma_{i,j,b} \ \ (\gamma_{1,0,b} = \alpha_b, \ \gamma_{1,1,b} = \beta_b)$$

and Theorem 1 is true for $i = 0, 1$. Applying induction suppose that (2.6) holds. Then by (2.5)

$$L_{n,b}^{(i+1)} = \left(L_{n,b}^{(i)}\right)' = \sum_{j=0}^{i} \binom{n}{j} (L'_{n-j,b}\gamma_{i,j,b} + L_{n-j,b}\gamma'_{i,j,b}) =$$

$$= \sum_{j=0}^{i} \binom{n}{j} \left((\alpha_b L_{n-j,b} + \beta_b(n-j)L_{n-j-1,b})\gamma_{i,j,b} + L_{n-j,b}\gamma'_{i,j,b}\right) =$$

$$= \sum_{j=0}^{i} \binom{n}{j} (\alpha_b\gamma_{i,j,b} + \gamma'_{i,j,b})L_{n-j,b} + \sum_{j=0}^{i} \binom{n}{j} (n-j)L_{n-j-1,b}\beta_b\gamma_{i,j,b} =$$

$$= \sum_{j=0}^{i} \binom{n}{j} (\alpha_b\gamma_{i,j,b} + \gamma'_{i,j,b})L_{n-j,b} + \sum_{j=0}^{i} \binom{n}{j+1} L_{n-j-1,b}(j+1)\beta_b\gamma_{i,j,b} =$$

$$= \sum_{j=0}^{i+1} (\alpha_b\gamma_{i,j,b} + \gamma'_{i,j,b} + j\beta_b\gamma_{i,j-1,b}) \binom{n}{j} L_{n-j,b}$$

and by (2.7) the claim holds for $i + 1$ instead of i. ■

We denote by $\mathfrak{R}$ the set of rational functions analytic in the closed disc $\overline{\mathbb{D}}$. The rational functions of the form

$$r_{i,a}(z) := \frac{z^i}{(1 - \overline{a}z)^{i+1}} \ \ (z \in \overline{\mathbb{D}}, \ \ a \in \mathbb{D}, \ \ i \in \mathbb{N}) \tag{2.8}$$

generates the set $\mathfrak{R}$. Namely every function $f \in \mathfrak{R}$ can be written in the form

$$f(z) = \sum_{k=1}^{N} \sum_{i=0}^{m_k - 1} \frac{\lambda_{k,i}\, z^i}{(1 - \overline{a}_k z)^{i+1}} =: \sum_{k=1}^{N} R_k(z), \tag{2.9}$$

where $a_k^* := 1/\overline{a}_k$ $(k = 1, 2, \cdots, N)$ are the poles of f with the multiplicity m_k and the $\lambda_{k,i}$'s are complex numbers and $\lambda_{k,m_k-1} \neq 0$.

In order to get the Fourier coefficients of f with respect to the discrete Laguerre system we shall use the next

Lemma 1. *For every function* $g \in \mathfrak{R}$

$$\langle g, r_{n,a}\rangle = \frac{g^{(n)}(a)}{n!} \quad (n \in \mathbb{N}, a \in \mathbb{D}). \tag{2.10}$$

Proof. By definition

$$\begin{aligned}\langle g, r_{n,a}\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{g(e^{it})e^{-int}}{(1-ae^{-it})^{n+1}}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{g(e^{it})e^{it}}{(e^{it}-a)^{n+1}}\,dt = \\ &= \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{g(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta.\end{aligned}$$

Hence by Cauchy's integral formula we get (2.10). ■

Now we compute the conjugate of the Laguerre Fourier coefficients of R_k. In the case $m_k = 1$

$$\langle L_{n,b}, R_k\rangle = \overline{\lambda}_{k,0} L_{n,b}(a_k). \tag{2.11}$$

If $m_k = 2$ then by (2.5) and (2.10)

$$\begin{aligned}\langle L_{n,b}, R_k\rangle &= \overline{\lambda}_{k,0} L_{n,b}(a_k) + \overline{\lambda}_{k,1} L'_{n,b}(a_k) = \\ &= (\overline{\lambda}_{k,0} + \overline{\lambda}_{k,1}\alpha_b(a_k))L_{n,b}(a_k) + \overline{\lambda}_{k,1}\beta_b(a_k) n L_{n-1,b}(a_k) := \\ &:= c_{k,0} L_{n,b}(a_k) + c_{k,1} n L_{n-1,b}(a_k) \quad (c_{k,1} \neq 0).\end{aligned} \tag{2.12}$$

A similar formula holds in the general case.

Lemma 2. *The conjugate of the Laguerre Fourier coefficients of* R_k *are of the form*

$$\langle L_{n,b}, R_k\rangle = \sum_{j=0}^{m_k-1} \binom{n}{j} L_{n-j,b}(a_k) c_{k,j}, \tag{2.13}$$

where the coefficients $c_{k,j}$ *do not depend on* n *and* $c_{k,m_k-1} \neq 0$.

Proof. Applying (2.6) and (2.10) for the conjugate of the Laguerre Fourier coefficients of R_k we get

$$\begin{aligned}\langle L_{n,b}, R_k\rangle &= \sum_{i=0}^{m_k-1} \frac{\overline{\lambda}_{k,i}}{i!} L_{n,b}^{(i)}(a_k) = \\ &= \sum_{i=0}^{m_k-1} \frac{\overline{\lambda}_{k,i}}{i!} \sum_{j=0}^{i} \gamma_{i,j,b}(a_k) \binom{n}{j} L_{n-j,b}(a_k) = \\ &= \sum_{j=0}^{m_k-1} \binom{n}{j} L_{n-j,b}(a_k) \sum_{i=j}^{m_k-1} \frac{\overline{\lambda}_{k,i}}{i!} \gamma_{i,j,b}(a_k) = \\ &= \sum_{j=0}^{m_k-1} \binom{n}{j} L_{n-j,b}(a_k) c_{k,j},\end{aligned}$$

where the coefficients

$$c_{k,j} =: \sum_{i=j}^{m_k-1} \frac{\overline{\lambda}_{k,i}}{i!} \gamma_{i,j,b}(a_k) \;\; (j = 0, 1, \cdots, m_k - 1)$$

are independent from n. Hence by (2.7) we get

$$c_{k,m_k-1} = \frac{\overline{\lambda}_{k,m_k-1}}{(m_k-1)!} \gamma_{m_k-1,m_k-1,b}(a_k) = \overline{\lambda}_{k,m_k-1} \beta_b^{m_k-1}(a_k) \neq 0,$$

and Lemma 2 is proved. ■

3. Reconstruction algorithm

Let us fix the parameters (the inverse poles) $a_1, a_2, \cdots, a_N \in \mathbb{D}$ of $f \in \mathfrak{R}$. Depending on this set of inverse poles and using the hyperbolic distance ρ defined in (1.5) for $i = 1, 2, \cdots, N$ we introduce the following domains of $\mathbb{D}$:

$$\begin{aligned} D_i :&= \{b \in \mathbb{D} : \rho(b, a_i) > \max_{1 \le j \le N, i \neq j} \rho(b, a_j)\}, \\ D_0 :&= \bigcup_{j=1}^{N} D_i. \end{aligned} \tag{3.1}$$

We show that on set D_0 the limit

$$(\mathcal{Q}f)(b) := \lim_{n\to\infty} \frac{\langle L_{n+1,b}, f\rangle}{\langle L_{n,b}, f\rangle} \;\; (f \in \mathfrak{R}) \tag{3.2}$$

exists and the function $\mathcal{Q}f$ can be used to reconstruct the poles of $f \in \mathfrak{R}$. We remark that the operator $\mathcal{Q}$ defined on $\mathfrak{R}$ in this way is not linear.

Theorem 2. *For any function $f \in \mathfrak{R}$ of the form* (2.9) *the limit* (3.2) *exists for all points in D_0 and*

$$(\mathcal{Q}f)(b) = B_b(a_i) \text{ if } b \in D_i \; (i = 1, 2, \cdots, N). \tag{3.3}$$

Proof. Suppose that $b \in D_i$. Then $B_b(a_i) \neq 0$ and by (2.9) we have

$$\begin{aligned}
&\frac{\langle L_{n+1,b}, f\rangle}{\langle L_{n,b}, f\rangle} = \frac{\sum_{k=1}^{N}\langle L_{n+1,b}, R_k\rangle}{\sum_{k=1}^{N}\langle L_{n,b}, R_k\rangle} = \\
&= \frac{\langle L_{n+1,b}, R_i\rangle + \sum_{k=1,k\neq i}^{N}\langle L_{n+1,b}, R_k\rangle}{\langle L_{n,b}, R_i\rangle + \sum_{k=1,k\neq i}^{N}\langle L_{n,b}, R_k\rangle} = \\
&= B_b(a_i)\frac{\langle L_{n+1,b}, R_i\rangle/B_b^{n+1}(a_i) + \sum_{k=1,k\neq i}^{N}\langle L_{n+1,b}, R_k\rangle/B_b^{n+1}(a_i)}{\langle L_{n,b}, R_i\rangle/B_b^{n}(a_i) + \sum_{k=1,k\neq i}^{N}\langle L_{n+1,b}, R_k\rangle/B_b^{n}(a_i)} = \\
&= \frac{u_{n+1}(b) + v_{n+1}(b)}{u_n(b) + v_n(b)}.
\end{aligned}$$

Set $m := \max\{m_k : k = 1.2. \cdots, N\}$. Then using $L_{j,b} = F_b B_b^j$, by (2.13) we get

$$\begin{aligned}
&\frac{\langle L_{n+m,b}, R_k\rangle}{B_b^{n+m}(a_i)} = \\
&= \sum_{j=0}^{m_k-1} c_{k,j}(a_k)\binom{n+m}{j} L_{n+m-j,b}(a_k)/B_b^{n+m}(a_i) = \\
&= \frac{F_b(a_k)}{B_b^m(a_i)}\sum_{j=0}^{m_k-1} c_{k,j}(a_k)B_b^{m-j}(a_k)\binom{n+m}{j}\frac{B_b^n(a_k)}{B_b^n(a_i)}.
\end{aligned} \tag{3.4}$$

By the definition of the hyperbolic metric for $b \in D_i$ and $j = 0, 1, \cdots, m_k - 1$ we have

$$\left|\binom{n+m}{j}\frac{B_b^n(a_k)}{B_b^n(a_i)}\right| = \binom{n+m}{j}\left|\frac{\rho(a_k, b)}{\rho(a_i, b)}\right|^n \to 0 \; (n \to \infty, k = 1, \cdots, N, k \neq i).$$

Consequently

$$v_{n+m}(b) := \sum_{k=1,k\neq i}^{N}\langle L_{n+m,b}, R_k\rangle/B_b^{n+m}(a_i) \to 0 \; (n \to \infty).$$

Moreover by (3.4) for any $q_i(b) > \max\{\rho(b,a_k)/\rho(b,a_i) : k = 1,\cdots,N, k \neq i\}$ and $b \in D_i$ we have

$$v_{n+m}(b) = O(q_i^n(b)) \;\; (n \to \infty). \tag{3.5}$$

Furthermore by (2.13)

$$u_{n+m}(b) := \langle L_{n+m,b}, R_i\rangle / B_b^{n+m}(a_i) = \frac{F_b(a_i)}{B_b^m(a_i)} \sum_{j=0}^{m_i-1} c_{i,j}(a_i) \binom{n+m}{j} B_b^{m-j}(a_i).$$

By (2.13) $c_{i,m_i-1,b}(a_i) \neq 0$, consequently for the sequence $w_n(b) := u_{n+1}(b)/u_n(b)$ we get

$$\lim_{n\to\infty} w_n(b) = \lim_{n\to\infty} \frac{u_{n+1}(b)}{u_n(b)} = \lim_{n\to\infty} \frac{u_{n+m+1}(b)}{u_{n+m}(b)} = 1, \;\; \lim_{n\to\infty} |u_n(b)| = \infty. \tag{3.6}$$

By the definition of $u_n(b)$ and $v_n(b)$

$$\frac{\langle L_{n+1,b}, f\rangle}{\langle L_{n,b}, f\rangle} = B_b(a_i) \frac{u_{n+1}(b) + v_{n+1}(b)}{u_n(b) + v_n(b)}.$$

We show that

$$\lim_{n\to\infty} \frac{u_{n+1}(b) + v_{n+1}(b)}{u_n(b) + v_n(b)} = \lim_{n\to\infty} \frac{u_n(b)}{v_n(b)} = 1$$

and (3.3) is proved. Indeed by (3.5) and (3.6)

$$\left| w_n(b) - \frac{u_{n+1}(b) + v_{n+1}(b)}{u_n(b) + v_n(b)} \right| = \left| w_n(b) - \frac{w_n(b) + v_{n+1}(b)/u_n(b)}{1 + v_n(b)/u_n(b)} \right| =$$
$$= \frac{|w_n(b)||v_n(b)/u_n(b) - v_{n+1}(b)/u_n(b)|}{|1 + v_n(b)/u_n(b)|} = O(q_i^n(b) \to 0 \;\; (n \to \infty)$$

and theorem is proved. ■

For the rate of convergence we have the estimation

$$\Delta_n(b) : = \left| \frac{\langle L_{n+1,b}, f\rangle}{\langle L_{n,b}, f\rangle} - B_b(a_i) \right| \leq$$
$$\leq \left(|1 - w_n(b)| + \left| w_n(b) - \frac{u_{n+1}(b) + v_{n+1}(b)}{u_n(b) + v_n(b)} \right| \right) |B_b(a_i)| =$$
$$= |1 - w_n(b)| + O(q_i^n(b)) \;\; (b \in D_i).$$

If $m_i = 1$ then $w_n(b) = 1$ $(n \in \mathbb{N})$, else $w_n(b) - 1 = O(1/n)$ and consequently we have

Corollary. *For $b \in D_i$ the rate of convergence*

$$\Delta_n(b) = \begin{cases} O(q_i^n) & \text{if } m_i = 1 \\ O\left(\dfrac{1}{n}\right), & \text{if } m_i > 1. \end{cases}$$

In connection with (3.3) we introduce the following notion. Denote $\mathfrak{S}$ the region vise step function on $\mathbb{D}$.

Definition. *The map $\mathcal{S} : \mathfrak{R} \to \mathfrak{S}$ defined by*

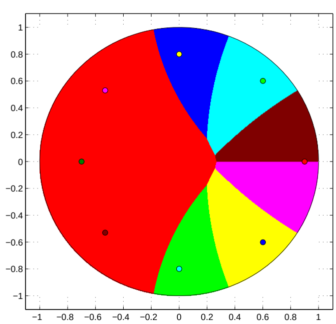
$$(\mathcal{S}f)(b) := B_b^{-1}((\mathcal{Q}f)(b)) \ \ (f \in \mathfrak{R}, b \in D_0)$$

is called spectral operator.

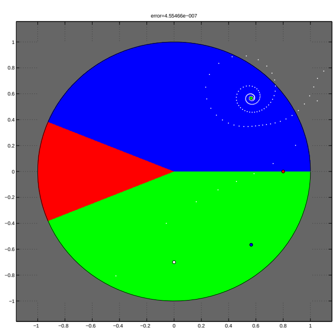
By Theorem 2 with the operator $\mathcal{S}$ we can reconstruct the poles a_i of $f \in \mathfrak{R}$ for which $D_i \neq \emptyset$:

$$(\mathcal{S}f)(b) = a_i \ \ (f \in \mathfrak{R}, \ \ b \in D_i, \ \ i = 1, 2, \cdots, N).$$

Domains



$$D = \bigcup_{i=1}^{7} D_i$$



$$D = \bigcup_{i=1}^{3} D_i$$

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