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TAUBERIAN THEOREMS WITH APPLICATIONS TO ARITHMETICAL SEMIGROUPS AND PROBABILISTIC COMBINATORICS

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Dedicated to the 70-th birthday of Professor János Galambos

Abstract. In this paper we investigate functions Z and F holomorphic in the unit disk $\{y \in \mathbb{C} : |y| < 1\}$, which can be represented by $Z(y) = \sum_{n=0}^{\infty} \gamma(n) y^n = \exp\left(\sum_{m=1}^{\infty} \frac{\lambda(m)}{m} y^m\right)$ and $F(y) = \sum_{n=0}^{\infty} f(n) y^n = \exp\left(\sum_{m=1}^{\infty} \frac{\lambda_f(m)}{m} y^m\right)$, respectively, where $\lambda(m) \in \mathbb{R}_{\geq 0}$ and $\lambda_f(m) \in \mathbb{C}$ for all $m \in \mathbb{N}$. We define a class $\mathcal{F}$ of functions Z and characterize the asymptotic behaviour of the quotient $f(n)/\gamma(n)$ as $n \to \infty$ if, for example, $|\lambda_f| \leq \lambda$. The results are applied to the generating functions of additive arithmetical semigroups and of exp-log schemas in combinatorics. We notice that the definition of the functions $Z \in \mathcal{F}$ does not require any analytic continuation of $Z(y)$ over the boundary $|y| = 1$.

1. Introduction

In some recent papers ([14]–[19]) we investigated the mean behaviour of

$$M(x, f) := \sum_{n \leq x} f(n)$$

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for arithmetical functions $f:\mathbb{N}\to\mathbb{C}$ with $f(1)=1$. The underlying idea was to compare f with some function g, the mean behaviour of which is known. To be specific, we start with

$$M(x) := M(x, f - A_x g) = \sum_{n\le x} (f(n) - A_x g(n)), \tag{1.1}$$

where $A_x \in \mathbb{C}$ and f is "near" to g. To give an example, let f be multiplicative and $g(n) = n^{ia}$ for all $n \in \mathbb{N}$ with some $a \in \mathbb{R}$. Define λ_f and λ by the generating Dirichlet series ($s = \sigma + it,\ \ \sigma > 1$)

$$F(s) = \sum_{n=1}^{\infty} f(n) n^{-s} = \exp\left(\sum_{n=2}^{\infty} \frac{\lambda_f(m)}{\log m} m^{-s}\right)$$

and

$$G(s) = \sum_{n=1}^{\infty} g(n) n^{-s} = \zeta(s - ia) = \exp\left(\sum_{m=2}^{\infty} \frac{\lambda(m) m^{ia}}{\log m} m^{-s}\right),$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is Dirichlet's ζ-function. We assume

$$|\lambda_f(m)| \le \lambda(m)$$

($\lambda(m) = \Lambda(m)$ where Λ denotes von Mangoldt's function) and arrive at (cf. [14])

$$\frac{|M(x)|}{x} \le \left(\frac{1}{\log x} \int_{-\infty}^{\infty} \frac{|F(s) - A_x \zeta(s - ia)|^2}{|s|^2} dt\right)^{\frac{1}{2}} +$$

$$+ |A_x| \frac{1}{\log x} \sum_{m\le x} \frac{|\lambda_f(m) - \lambda(m) m^{ia}|}{m} + O\left(\frac{1}{\log x}\right),$$

where $s = 1 + \frac{1}{\log x} + it$. From this we conclude

Proposition. *Let f be multiplicative and $|f| \le 1$. Then the following assertions hold.*

(i) *Assume that the series*

$$\sum_{m=2}^{\infty} \frac{\lambda(m) - \operatorname{Re} \lambda_f(m) m^{-ia}}{m \log m} \tag{1.2}$$

converges for some $a \in \mathbb{R}$. *Put*

$$A_x := \exp\left(\sum_{m\le x}\frac{\lambda_f(m)m^{-ia}-\lambda(m)}{m\log m}\right). \tag{1.3}$$

Then

$$\frac{1}{x}\sum_{n\le x} f(n) = A_x\frac{1}{x}\sum_{n\le x} n^{ia} + o(1) = \\ = A_x\frac{x^{ia}}{1+ia} + o(1)$$

as $x \to \infty$.

(ii) *If the series* (1.2) *diverges for all* $a \in \mathbb{R}$ *then*

$$\frac{1}{x}\sum_{n\le x} f(n) = o(1)$$

as $x \to \infty$.

Remark 1. The Proposition is just a theorem of G. Halász [10]. The sum in (1.2) converges if and only if the same holds for

$$\sum_p \frac{1-\operatorname{Re} f(p)p^{-ia}}{p}.$$

In addition

$$A_x = \prod_{p\le x}\left(1-\frac{1}{p}\right)\left(1+\sum_{k=1}^{\infty} f(p^k)p^{-k(1+ia)}\right)\{1+o(1)\}$$

as $x \to \infty$.

The method used also leads to new proofs of the prime number theorem, Wirsing's theorem, etc. and to new quantitative estimates for multiplicative functions [9], [14], [16] and [18].

In this paper we apply the same idea in the case that the generating function is a power series.

Let us assume $f : \mathbb{N}_0 \to \mathbb{C}$ with $f(0) = 1$ and let $\gamma(n) \ge 0$ for $n \in \mathbb{N}$ and $\gamma(0) = 1$. Further, we assume that

$$F(y) := \sum_{n=0}^{\infty} f(n)y^n = \exp\left(\sum_{m=1}^{\infty}\frac{\lambda_f(m)}{m}y^m\right) \tag{1.4}$$

and

$$Z(y) := \sum_{n=0}^{\infty} \gamma(n) y^n = \exp\left(\sum_{m=1}^{\infty} \frac{\lambda(m)}{m} y^m\right) \tag{1.5}$$

converge for $|y| < 1$.

Analytically, the problem which we are confronted with here amounts to extracting information on the coefficients $\{f(n)\}$ by comparing it with the coefficients $\{\gamma(n)\}$.

Example 1. Let (G, ∂) be an additive arithmetical semigroup that is, by definition, G is a free abelian semigroup with identity element 1 such that G has a countable free generating set $\mathcal{P}$ of "primes" and $\partial : G \to \mathbb{N}_0$ is a "degree mapping" satisfying

(i) $\partial(g_1 g_2) = \partial(g_1) + \partial(g_2)$ for all $g_1, g_2 \in G$,

(ii) the total number $G(n)$ of elements of degree n in G is finite for each $n \geq 0$.

In particular, if we assume $G(n) \ll q^n n^{\varrho}$ with some ϱ and $q > 1$ then

$$\hat{Z}(z) := \sum_{n=0}^{\infty} G(n) z^n = \prod_{m=1}^{\infty} (1 - z^m)^{-P(m)}$$

is the zeta function associated with G, where $P(m)$ denotes the total number of primes of degree m in G. Obviously

$$\log \prod_{m=1}^{\infty} (1 - z^m)^{-P(m)} = \sum_{m=1}^{\infty} P(m) \sum_{j=1}^{\infty} j^{-1} z^{jm} =$$

$$= \sum_{m=1}^{\infty} \frac{1}{m} \sum_{d|m} dP(d) z^m = \sum_{m=1}^{\infty} \frac{\bar{\Lambda}(m)}{m} z^m,$$

where

$$\bar{\Lambda}(m) = \sum_{d|m} dP(d).$$

Then, since $P(d) \leq G(d) \ll q^d d^{\varrho}$,

$$\begin{aligned} \bar{\Lambda}(m) &= mP(m) + O\left(mG\left(\frac{m}{2}\right) \sum_{r \leq m} \frac{1}{r}\right) = \\ &= mP(m) + O\left(mq^{\frac{m}{2}} \left(\frac{m}{2}\right)^{\varrho} \log m\right). \end{aligned} \tag{1.6}$$

Putting $y = qz$, $\lambda(m) = q^{-m}\bar{\Lambda}(m)$ and $\gamma(n) = q^{-n}G(n)$ leads to

$$Z(y) := \hat{Z}(yq^{-1}) = \sum_{n=0}^{\infty} \gamma(n)y^n = \exp\left(\sum_{m=1}^{\infty} \frac{\lambda(m)}{m} y^m\right). \tag{1.7}$$

Observe

$$\frac{\lambda(m)}{m} = q^{-m} \sum_{\substack{p\in\mathcal{P}\\ \partial(p)=m}} 1 + O\left(q^{-m/2} m^{\varrho} \log m\right). \tag{1.8}$$

Now, let $\tilde{f} : G \to \mathbb{C}$ be multiplicative and let

$$f(n) := q^{-n} \sum_{\substack{g\in G\\ \partial(g)=n}} \tilde{f}(g). \tag{1.9}$$

Then the generating function of f is given by

$$\begin{aligned} F(y) :&= \sum_{n=0}^{\infty} f(n)y^n = \\ &= \sum_{g\in G} \tilde{f}(g) q^{-\partial(g)} y^{\partial(g)} = \\ &= \prod_{p} \left(1 + \sum_{k=1}^{\infty} \tilde{f}(p^k) q^{-k\partial(p)} y^{k\partial(p)}\right) = \\ &= \exp\left(\sum_{m=1}^{\infty} \frac{\lambda_f(m)}{m} y^m\right). \end{aligned}$$

This holds at least in a *formal* sense since $f(0) = 1$ ($\neq 0$). It is also valid for complex values $y, |y| < 1$ in terms of *ordinary* convergence if, for example, the function $\tilde{f}$ is completely multiplicative of modulus ≤ 1. Then $|\lambda_f(m)| \leq \lambda(m)$ and $|f(n)| \leq \gamma(n)$ for all $m, n \in \mathbb{N}$.

Further

$$\begin{aligned} \frac{\lambda_f(m)}{m} &= q^{-m} \sum_{d|m} d \sum_{\substack{p\in\mathcal{P}\\ \partial(p)=d}} \tilde{f}(p) = \\ &= q^{-m} \sum_{\substack{p\in\mathcal{P}\\ \partial(p)=m}} \tilde{f}(p) + O\left(q^{-m/2} m^{\varrho} \log m\right). \end{aligned} \tag{1.10}$$

Example 2. In the field of combinatorial structures, for example, *multisets* and *selections* (cf. [2], p. 45. ff) can be considered as additive arithmetical semigroups and subsets of such semigroups, respectively. Many types of combinatorial objects decompose as sets of simpler basic objects known as "prime", "irreducible" or "connected" components. For instance, a permutation decomposes as a set of cyclic permutations and a graph as a set of connected components. Such situations are combinatorial analogues of the fact that the elements of an additive arithmetical semigroup uniquely decompose as products of prime elements.

Especially combinatorial structures which belong to the exp − log schemas are of great interest. Following Flajolet and Soria [7], [8] we introduce

Definition 1 (see [23])**.** Let $\Delta(\nu,\theta)$ be the region $|z| \le 1+\nu$ minus the region $|\arg(z-1)| \le \theta$, with $\nu > 0$ and $0 < \theta < \pi/2$. We say that $C(z)$ is of *logarithmic type* with multiplicity constant $\delta > 0$ if

$$C(z) = \delta \log\left(\frac{1}{1-z/\varrho}\right) + R(z),$$

where $R(z)$ is analytic in $\Delta(\nu,\theta)$, and as $z \to \varrho$ in $\Delta(\nu,\theta)$,

$$R(z) = K + O\left(\left(1-\frac{z}{\varrho}\right)^{\alpha}\right)$$

with $0 < \alpha < 1$ and K a complex constant. We say that $L(z) = e^{C(z)}$ describes the *exp-log schemas* if $C(z)$ is of logarithmic type. Hence,

$$L(z) = e^{R(z)}\left(\frac{1}{1-z/\varrho}\right)^{\delta} = e^{K}\left(\frac{1}{1-z/\varrho}\right)^{\delta} + O\left(\left(1-\frac{z}{\varrho}\right)^{-\delta+\alpha}\right).$$

For a more recent definition in the connection with exp-log schemas see [6], p. 446.

Remark 2. Consider additive arithmetical semigroups satisfying Axiom $A^{\#}$ of Knopfmacher ([20]), i.e.

$$G(n) = Aq^n + O(q^{\alpha n}),$$

where $A > 0$, $0 < \alpha < 1$, $q > 1$. Then

$$Z(z) = \frac{H(z)}{1-qz},$$

where $H(z)$ is holomorphic for $|z| < q^{-\alpha}$ and $H(q^{-1}) = A$.

In [7], Example 3, and [23], Example 7, for instance, Flajolet-Soria and Panario-Richmond, respectively, assert that the arithmetical semigroups give a family of examples in the $\exp - \log$ schemas. But this is not correct as it stands. Indlekofer, Manstavicius and Warlimont [19] gave an example of an additive arithmetical semigroup G with

$$Z(z) = \frac{1+qz}{1-qz} H_2(z),$$

where $H_2(z)$ is holomorphic and $\neq 0$ for $|z| < q^{-\frac{1}{2}}$. Then

$$Z(z) = \exp(C(z))$$

is obviously not in the class described above. Furthermore, Indlekofer, Manstavicius and Warlimont showed, if $H(z) = Z(z)(1-qz)$ is holomorphic for $|z| \leq q^{-\frac{1}{2}}$ then $Z(z) \neq 0$ for $|z| \leq q^{-1}$ and $\log H(z)$ is holomorphic for $|z| \leq q^{-1+\varepsilon}$ with some $\varepsilon > 0$ and then $Z(z)$ fits in the Definition 1. This ends Remark 2.

The basic conditions in this paper will be (see (1.4) and (1.5))

$$0 \leq \lambda(m) = O(1) \quad (m \in \mathbb{N}) \tag{1.11}$$

and

$$|Z(y)| \ll Z(|y|) \left| \frac{1-|y|}{1-y} \right|^{\varepsilon} \qquad (|y| < 1) \tag{1.12}$$

for some $\varepsilon > 0$. Let

$$B(n) = \exp\left(\sum_{m \leq n} \frac{\lambda(m)}{m} \right). \tag{1.13}$$

Then we assume that

$$n\gamma(n) \asymp B(n) \tag{1.14}$$

and

$$B(m) = o(B(n)) \quad \text{if } m = o(n) \ (n \to \infty). \tag{1.15}$$

Definition 2. We say that the function Z given in (1.5) belongs to the $\exp - \log$ class $\mathcal{F}$ in case (1.11), (1.12), (1.14) and (1.15) hold.

Example 3. Let $Z(y)$, defined in (1.7), have the form

$$Z(y) = \sum_{n=0}^{\infty} \gamma(n) y^n = \frac{H(y)}{(1-y)^\delta} \quad (|y| < 1), \tag{1.16}$$

where $\gamma(n) \geq 0$, $\delta > 0$ and $H(y) = O(1)$ for $|y| < 1$ and

$$\lim_{y \to 1^-} H(y) = A > 0. \tag{1.17}$$

Then, of course, the behaviour of $H(y)$ as $y \to 1^-$ dictates the behaviour of the coefficients $\gamma(n)$. For example, if (1.17) holds then, since $\gamma(n) \geq 0$,

$$\sum_{n \leq N} \gamma(n) \sim \frac{AN^\delta}{\Gamma(\delta + 1)} \tag{1.18}$$

by the Hardy-Littlewood Tauberian theorem.

Further, since

$$Z'(y) \sim \frac{\delta A}{(1-y)^{\delta+1}} \qquad \text{as} \quad y \to 1^-$$

we have

$$\sum_{m=1}^{\infty} \lambda(m) y^m \sim \frac{\delta}{1-y} \qquad \text{as} \quad y \to 1^-$$

and again, since $\lambda(m) \geq 0$,

$$\sum_{m \leq N} \lambda(m) \sim \delta N \tag{1.19}$$

and

$$\sum_{m \leq N} \frac{\lambda(m)}{m} \sim \delta \log N \tag{1.20}$$

as $N \to \infty$.

If in addition, it is assumed that $H(y)$ is continuous on the closed disc and the derivative of $(H(y) - H(1))(1-y)^{-\delta}$ is bounded for $|y| < 1$, then

$$\gamma(n) \sim \frac{An^{\delta-1}}{\Gamma(\delta)}$$

which by summation yields the estimate (1.18). Therefore it is convenient to assume

$$\gamma(n) \asymp n^{\delta-1}.$$

Observe, that if $Z(y)$ is defined by (1.5) and (1.16) with (1.17), respectively, and $0 \le \lambda(m) \ll 1$ then, for $r = 1 - 1/n$,
(1.21)

$$B(n) = \exp\left(\sum_{m\le n} \frac{\lambda(m)}{m}\right) \asymp \exp\left(\sum_{m\le n} \frac{\lambda(m)}{m} r^m\right) \asymp Z(r) \asymp (1-r)^{-\delta} = n^\delta,$$

which implies

$$\frac{B(m)}{B(n)} \asymp \left(\frac{m}{n}\right)^\delta = o(1) \qquad \text{if } m = o(n)$$

as $n \to \infty$ and (1.15) is satisfied.

Example 4. Assume that

$$0 < c_1 \le \lambda(m) \le c_2 < \infty \qquad (m \in \mathbb{N}).$$

Then, obviously

$$\begin{aligned} |Z(y)| &= Z(|y|) \exp\left(\sum_{m=1}^\infty \frac{\lambda(m)}{m} |y|^m (\cos(mt) - 1)\right) \le \\ &\le Z(|y|) \exp\left(c_1 \sum_{m=1}^\infty \frac{|y|^m}{m} (\cos(mt) - 1)\right) = \\ &= Z(|y|) \left|\frac{1-|y|}{1-y}\right|^{c_1} \end{aligned}$$

and

$$\frac{B(m)}{B(n)} = \exp\left(-\sum_{m<l\le n} \frac{\lambda(l)}{l}\right) \ll \exp\left(c_1 \log \frac{m}{n}\right) = o(1)$$

if $m = o(n)$ $(n \to \infty)$. Elementary estimates immediately yield

$$n\gamma(n) \asymp B(n),$$

where the constants involved in $\asymp$ only depend on c_1 and c_2 (see Manstavicius [22], Lemma 3.1).

Let us now come back to the coefficient asymptotics. To compare the asymptotic behaviour of $f(n)$ with $\gamma(n)$ we shall use a Tauberian condition which says that $\{\lambda_f(m)\}$ is "near" to $\{\lambda(m)\}$. To start with we shall assume that λ_f splits into

$$\lambda_f = \lambda_{f,1} + \lambda_{f,2} \tag{1.22}$$

such that

$$|\lambda_{f,1}(m)| \le \lambda(m) \quad (m \le n) \qquad \text{and} \qquad \sum_{m=1}^{\infty} \frac{|\lambda_{f,2}(m)|}{m} \le c_1 < \infty. \tag{1.23}$$

We may assume that $\lambda_{f,1}(m) = 0$ if $m > n$ since these values do not influence $f(n)$. Put (cf. (1.4))

$$F(y) := F_n(y) = F_I(y) F_{II}(y), \tag{1.24}$$

where
(1.25)

$$F_I(y) := \exp\left(\sum_{m=1}^{n} \frac{\lambda_{f,1}(m)}{m} y^m\right), \qquad F_{II}(y) := \exp\left(\sum_{m=1}^{\infty} \frac{\lambda_{f,2}(m)}{m} y^m\right).$$

With these notations we prove

Theorem 1. *Let Z be an element of the exp-log class $\mathcal{F}$ and let $F(y)$ in* (1.4) *satisfy* (1.24) *and* (1.25)*. Further assume $\lambda_f(m) = O(1)$ $(m \in \mathbb{N})$. Then the following two assertions hold.*

(i) *Let, for some $a \in \mathbb{R}$, and every $n \in \mathbb{N}$*

$$\sum_{m \le n} \frac{\lambda(m) - \operatorname{Re} \lambda_{f,1}(m) e^{ima}}{m} \le c_2 < \infty \tag{1.26}$$

and

$$\sum_{m \le n} |\lambda(m) - \lambda_{f,1}(m) e^{ima}| = o(n) \tag{1.27}$$

as $n \to \infty$. Put

$$A_n := \exp\left(-ina + \sum_{m \le n} \frac{\lambda_{f,1}(m) e^{ima} - \lambda(m)}{m}\right) F_{II}(1).$$

Then

$$f(n) = A_n \gamma(n) + o(\gamma(n)) \qquad \text{as } n \to \infty,$$

where $o(\cdot)$ *depends only on* c_1 *and* c_2*.*

(ii) *Assume that*

$$\sum_{m\le n} \frac{\lambda(m) - \operatorname{Re}\lambda_{f,1}(m)e^{ima}}{m} =: c_2(n) \to \infty \qquad (n \to \infty)$$

uniformly in $a \in \mathbb{R}$*. Then*

$$f(n) = o(\gamma(n))$$

as $n \to \infty$*, where* $o(\cdot)$ *depends only on* c_1 *and* $c_2(n)$*.*

Since

$$yF'(y) = \left(\sum_{m=1}^{\infty} \lambda_f(m) y^m\right) F(y) =: \Lambda_f(y) F(y)$$

and

$$yZ'(y) = \left(\sum_{m=1}^{\infty} \lambda(m) y^m\right) Z(y) =: \Lambda(y) Z(y)$$

we get

$$\begin{aligned} H_1(y) :&= \sum_{m=0}^{\infty} h_1(m) y^m := \\ :&= yF'(y) - A_n yZ'(y) = \\ &= \Lambda_f(y)(F(y) - A_n Z(y)) + \\ &+ A_n Z(y)(\Lambda_f(y) - \Lambda(y)) \end{aligned} \tag{1.28}$$

and because of $\lambda_f(m) = O(1)$ $(m \in \mathbb{N})$

$$\begin{aligned} |h_1(n)| = |nf(n) - A_n n\gamma(n)| &= \Big| \sum_{m\le n} \lambda_f(m)\{f(n-m) - A_n\gamma(n-m)\} + \\ &+ A_n \sum_{m\le n} (\lambda_f(m) - \lambda(m))\gamma(n-m)\Big| \ll \\ &\ll \sum_{m\le n} |f(m) - A_n\gamma(m)| + |A_n| \sum_{m\le n} |\lambda_f(m) - \\ &- \lambda(m)|\gamma(n-m) =: \\ &=: \Sigma_1 + \Sigma_2. \end{aligned} \tag{1.29}$$

Our aim is to establish the estimates $\Sigma_1 = o(B(n))$ and $\Sigma_2 = o(Bn)$ as $n \to \infty$. This will prove Theorem 1.

We observe that, in Theorem 1, f and λ_f, respectively, may depend on n or on other parameters. If we turn away from this general situation we can formulate the following

Theorem 2. *Let Z be an element of the exp-log class $\mathcal{F}$ and let $F(y)$ in* (1.4) *satisfy* (1.22), (1.23) *and* (1.24) *with*

$$\lambda_f(m) = O(1), \quad |\lambda_{f,1}(m)| \le \lambda(m) \qquad \text{for all } m \in \mathbb{N}$$

and

$$\sum_{m=1}^{\infty} \frac{|\lambda_{f,2}(m)|}{m} < \infty.$$

Put

$$F(y) = F_I(y) F_{II}(y),$$

where

$$F_I(y) := \exp\left(\sum_{m=1}^{\infty} \frac{\lambda_{f,1}(m)}{m} y^m\right), \qquad F_{II}(y) := \exp\left(\sum_{m=1}^{\infty} \frac{\lambda_{f,2}(m)}{m} y^m\right)$$

for $|y| < 1$. Then the following two assertions hold.

(i) *Let*

$$\sum_{m=1}^{\infty} \frac{\lambda(m) - \operatorname{Re} \lambda_{f,1}(m) e^{ima}}{m} \tag{1.30}$$

converge for some $a \in \mathbb{R}$. Put

$$A_n = \exp\left(-ina + \sum_{m \le n} \frac{\lambda_{f,1}(m) e^{ima} - \lambda(m)}{m}\right) F_{II}(1).$$

Then

$$f(n) = A_n \gamma(n) + o(\gamma(n)) \qquad \text{as} \quad n \to \infty.$$

(ii) *Let* (1.30) *diverge for all $a \in \mathbb{R}$. Then*

$$f(n) = o(\gamma(n)) \qquad \text{as} \quad n \to \infty. \tag{1.31}$$

Remark 3. Obviously

$$\lim_{n\to\infty} \frac{f(n)}{\gamma(n)} \text{ exists and is } \neq 0$$

if and only if

$$\sum_{m=1}^{\infty} \frac{\lambda_f(m) - \lambda(m)}{m} \quad \text{converges.}$$

If (1.30) converges for some $a \in \mathbb{R}$ then we may write

$$\exp\left(\sum_{m\le n} \frac{\lambda_{f,1}(m)e^{ima} - \lambda(m)}{m}\right) = c(a)L_a(n) + o(1), \tag{1.32}$$

where

$$c(a) := \exp\left(\sum_{m=1}^{\infty} \frac{\operatorname{Re}\lambda_{f,1}(m)e^{ima} - \lambda(m)}{m}\right) \tag{1.33}$$

and

$$L_a(n) := \exp\left(i\sum_{m\le n} \frac{\operatorname{Im}\lambda_{f,1}(m)e^{ima}}{m}\right). \tag{1.34}$$

With these notations we have

Theorem 3. *Assume that the conditions of Theorem 2 hold. Then either*

(i) *there exists $a \in \mathbb{R}$ such that,*

$$F(y) = c(a)L_a\left(\frac{1}{1-|y|}\right) F_{II}(1)Z(e^{-ia}y) + o(Z(|y|)) \tag{1.35}$$

uniformly as $|y| \to 1^-$, where $c(a)$, L_a are given by (1.33), (1.34) and $L_a(u)$ is slowly varying as $u \to \infty$,
or

(ii)
$$F(y) = o(Z(|y|)).$$

Remark 4. The conclusion (i) of Theorem 3 shows that there can be at most one real number a for which the series (1.30) is convergent. Indeed, assume that to the contrary there are two distinct values a_1 and a_2, say, for which this series is convergent. Then there are (non-zero) constants $c(a_1)$ and $c(a_2)$ such that, uniformly as $|y| \to 1^-$

$$|c(a_1)||Z(e^{-ia_1}y)| = |c(a_2)||Z(e^{-ia_2}y)| + o(Z(|y|))$$

and the choice $y = |y|e^{ia_1}$ leads to an impossible situation as $|y| \to 1^-$.

We remark, that we consider functions $Z(y) = \sum_{n=0}^{\infty} \gamma(n)y^n$ satisfying the conditions of Theorem 1 because of their connections with the generating functions of additive arithmetical semigroups and of combinatorial structures from the exp-log schemas, respectively. In the first case (see §5) we obtain new and known results (see [4], [17] and [21]) for a much wider class of additive arithmetical semigroups and in the second case (see §6) we generalize the most popular "singularity analysis" by Flajolet and Odlydzko [5], since we do not require analytic continuation of the generating functions outside the disk of convergence.

Moreover, our method is differing, since we do not prove an asymptotic formula for $nf(n)$ via Cauchy's theorem

$$nf(n) = \frac{1}{2\pi} \int\limits_{|y|=r<1} \frac{F'(y)}{y^{n+1}}\, dy$$

but compare $nf(n)$ with $n\gamma(n)$ as formulated in (1.29). This leads via Parseval's equality to an estimate of the distance between $nf(n)$ and $A_n n\gamma(n)$, a procedure which is also effective for quantitative investigations of occuring remainder terms.

Remark 5. Recently E. Manstavicius [22] obtained the assertion (i) of Theorem 1 for $a = 0$ in the case of Example 4 and under the restriction $|\lambda_f(m)| \le \lambda(m)$ $(m \in \mathbb{N})$.

Remark 6. The assumption $\lambda_f(m) = O(1)$ is essentially used in the estimate (1.29) and in the proof of Theorem 1 (cf. Lemma 4 and Lemma 5). The possibility to omit this condition is shortly discussed in Remark 9 of 5.

2. Some lemmas

The technical details of the proof of Theorem 1 (and Theorem 2) will be collected in several Lemmata. In the following we assume that (1.26) and (1.27) hold for $a = 0$.

Lemma 1. *There exist* $\varepsilon_1(n) \searrow 0$, $\varepsilon_2(n) \searrow 0$ $(n \to \infty)$ *such that* $\varepsilon_1(n) \le \varepsilon_2(n)$ *and*

$$\sum_{m \le n} |\lambda(m) - \lambda_{f,1}(m)| \le \varepsilon_1(n)n \tag{2.1}$$

and

$$\sum_{m \le n} |\lambda(m) - \lambda_f(m)| \le \varepsilon_2(n)n \tag{2.2}$$

as $n \to \infty$.

Proof. We observe that (1.23) implies

$$\sum_{m \le n} |\lambda_{f,2}(m)| = o(n)$$

which together with (1.27) proves Lemma 1. ■

Let us define $\varepsilon(n)$ by

$$\varepsilon(n) = (\varepsilon_2(n))^{\frac{1}{2}}$$

and prove

Lemma 2. *If* $\varepsilon(n)n \le u \le n$ *then*

$$A_n - A_u = o(1) \qquad as \quad n \to \infty.$$

Proof. Let $\varepsilon(n)n \le u \le n$. Then

$$\frac{A_n}{A_u} = \exp\left(\sum_{u < m \le n} \frac{\lambda_{f,1}(m) - \lambda(m)}{m} \right)$$

and

$$\left| \sum_{u < m \le n} \frac{\lambda_{f,1}(m) - \lambda(m)}{m} \right| \le u^{-1}\varepsilon_1(n)n \le \varepsilon(n).$$

Thus

$$A_n = A_u(1 + o(1)) = A_u + o(1).$$

Without loss of generality we can take $\lambda_{f,1}(m) = \lambda(m) = 0$ if $m > n$. Then we introduce, for $2 \le u \le n$, the functions

$$F_u(y) = \exp\left(\sum_{m\le u} \frac{\lambda_{f,1}(m)}{m} y^m\right) \cdot F_{II}(y) \tag{2.3}$$

and

$$Z_u(y) = \exp\left(\sum_{m\le u} \frac{\lambda(m)}{m} y^m\right). \tag{2.4}$$

Lemma 3. *For $u \ge 2$ put $y = \left(1 - \frac{1}{u}\right) e^{it}$. Then*

$$\max_{|t|\le\pi} |F_u(y) - A_n Z_u(y)| \ll Z_u(|y|) \le B(u) \qquad uniformly\ for\ 2 \le u \le n$$

and, if $\varepsilon(n)n \le u \le n$,

$$\max_{|t|\le\pi} |F_u(y) - A_n Z_u(y)| = o(Z_u(|y|)) = o(B(u)) \qquad as \quad u \to \infty.$$

Proof. The first assertion is obvious since

$$|F_u(y)| \ll Z_u(|y|)|F_{II}(y)| \ll B(u).$$

In the other case, we assume first that $|t| \le \frac{K}{u}$ with $K > 0$. Then with $r = 1 - \frac{1}{u}$

$$F_u(y) A_n Z_u(y) = \exp\left(\sum_{m\le u} \frac{\lambda_{f,1}(m) - \lambda(m)}{m}(r^m e^{imt} - 1) - \right.$$

$$\left. - \sum_{u<m\le n} \frac{\lambda_{f,1}(m) - \lambda(m)}{m}\right) \frac{F_{II}(y)}{F_{II}(1)} =$$

$$= \exp(\Sigma' + \Sigma'') \frac{F_{II}(y)}{F_{II}(1)}.$$

It is wellknown that $F_{II}(y) \to F_{II}(1)$ in the Stoltz angle $\left\{re^{it} : |t| \le \frac{K}{u}\right\}$ as $n \to \infty$. Further

$$\Sigma' \le \sum_{m\le u} \frac{|\lambda_{f,1}(m) - \lambda(m)|}{m}(|r^m - 1| + |e^{itm} - 1|).$$

Now

$$|r^m - 1| + |e^{imt} - 1| \leq \frac{m}{u} + m\frac{K}{u} \qquad \text{if} \quad m \leq u$$

and

$$\Sigma' \leq \left(\frac{1}{u} + \frac{K}{u}\right) \cdot \varepsilon_2(n)n = o(1).$$

Obviously

$$\Sigma'' \leq u^{-1} \sum_{m \leq n} |\lambda_{f,1}(m) - \lambda(m)| = o(1)$$

which implies

$$|F_u(y) - A_n Z_u(y)| = o(1) Z_u(|y|) = o(B(u)) \qquad \text{for} \quad |t| \leq \frac{K}{u}. \tag{2.5}$$

Assume now that $|t| \geq \frac{K}{u}$. Then

$$\begin{aligned}
\frac{|F_u(y)|^2}{Z_u(|y|)|Z_u(\bar{y})|} = \left|\frac{F_u(y)}{Z_u(|y|)}\right|^2 \cdot \frac{Z_u(|y|)}{|Z_u(\bar{y})|} \ll \\
&\ll \exp\Bigg(-2 \sum_{\substack{m=1 \\ \lambda(m) \neq 0}}^{u} \frac{\lambda(m)}{m} \left(1 - \frac{\operatorname{Re}(\lambda_{f,1}(m) e^{imt})}{\lambda(m)}\right) r^m + \\
&\quad + \sum_{m=1}^{u} \frac{\lambda(m)}{m} \left(1 - \operatorname{Re} e^{-imt}\right) r^m \Bigg) \ll \\
&\ll \exp\left(2 \sum_{m=1}^{u} \frac{\lambda(m) - \operatorname{Re} \lambda_{f,1}(m)}{m} r^m\right) \ll \\
&\ll 1
\end{aligned}$$

because of (1.26) (for $a = 0$) and since, if $\lambda(m) \neq 0$,

$$\begin{aligned}
2(1 - \operatorname{Re} e^{-itm}) = |1 - e^{-itm}|^2 \leq \\
&\leq 2\left|1 - \frac{\lambda_{f,1}(m)}{\lambda(m)}\right|^2 + 2\left|\frac{\lambda_{f,1}(m)}{\lambda(m)} - e^{-itm}\right|^2 \leq \\
&\leq 4\left(1 - \frac{\operatorname{Re} \lambda_{f,1}(m)}{\lambda(m)}\right) + 4\left(1 - \frac{\operatorname{Re}(\lambda_{f,1}(m) e^{itm})}{\lambda(m)}\right).
\end{aligned}$$

Thus, by (1.12)

$$|F_u(y) - A_n Z_u(y)| \ll |F_u(y)| + |Z_u(y)| \ll \\ \ll (Z_u(|y|)|Z_u(y)|)^{\frac{1}{2}} + |Z_u(y)| \ll \\ \ll (K^{-\varepsilon/2} + K^{-\varepsilon}) Z_u(|y|) \ll \\ \ll K^{-\varepsilon/2} B(u) \tag{2.6}$$

for $|t| \geq \frac{K}{u}$. Collecting (2.5) and (2.6) gives

$$\max_{|t| \leq \pi} |F_u(y) - A_n Z_u(y)| = o(Z_u(|y|)) = o(B(u))$$

which ends the proof of Lemma 3. ■

Lemma 4. *As* $n \to \infty$

$$\sum_{m \leq n} \gamma(m) |\lambda_f(n-m) - \lambda(n-m)| = o(B(n)). \tag{2.7}$$

Proof. We first observe

$$\Sigma := \sum_{m \leq u} B(m) |\lambda_f(n-m) - \lambda(n-m)| = \\ = B(u) \sum_{m \leq u} |\lambda_f(n-m) - \lambda(n-m)| - \\ - \int_1^u \sum_{m \leq t} |\lambda_f(n-m) - \lambda(n-m)| \, dB(t) = \\ = O(uB(u)) \qquad \text{uniformly for } 2 \leq u \leq n. \tag{2.8}$$

Further, if $\varepsilon(n) n \leq u \leq n$,

$$\Sigma \ll \varepsilon_2(n) n B(u) \leq \\ \leq \varepsilon_2(n) \frac{u}{\varepsilon(n)} B(u) = \\ = o(uB(u)).$$

Now

$$\sum_{m\le n}\gamma(m)|\lambda_f(n-m)-\lambda(n-m)| \ll \sum_{m\le n} B(m)\frac{|\lambda_f(n-m)-\lambda(n-m)|}{m} =$$

$$=\frac{1}{n}\sum_{m\le n} B(m)|\lambda_f(n-m)-\lambda(n-m)|+$$

$$+\int\limits_1^{\varepsilon(n)n}\frac{O(uB(u))}{u^2}\,du+\int\limits_{\varepsilon(n)n}^{n}\frac{o(uB(u))}{u^2}\,du=$$

$$=o(B(n))+O\left(\int\limits_1^{\varepsilon(n)n}\gamma([u])\,du\right)+$$

$$+o(1)\int\limits_{\varepsilon(n)n}^{n}\gamma([u])\,du=$$

$$=o(B(n))+O(B(\varepsilon(n)n))+o(B(n))=$$

$$=o(B(n))$$

because of (1.15). Thus Lemma 4 holds. ■

Lemma 5. *Let* $0<r<1$. *Then*

$$\int\limits_{-\pi}^{\pi}|\Lambda_f(re^{it})|^2dt=O\left(\frac{1}{1-r}\right)$$

and

$$\int\limits_{-\pi}^{\pi}|\Lambda_f(re^{it})-\Lambda(re^{it})|^2dt=o\left(\frac{1}{1-r}\right)$$

as $r\to 1^-$.

Proof. Using Parseval's equality gives the proof of Lemma 5 by (2.2) of Lemma 1 and since $\lambda_f(m)=O(1)$. ■

3. Proof of Theorem 1

We shall prove that Σ_1 and Σ_2 in (1.29) satisfy $\Sigma_1 = o(B(n))$ and $\Sigma_2 = = o(B(n))$ as $n \to \infty$. Obviously

$$\Sigma_2 = o(B(n)) \tag{3.1}$$

by Lemma 4. Further, using the notation of (1.28)

$$\sum_{m\le n} |f(m) - A_n\gamma(m)| = \sum_{m\le n} \frac{|h_1(m)|}{m} = = \frac{1}{n}\sum_{m\le n}|h_1(m)| + \int_1^n \frac{\sum_{m\le u}|h_1(m)|}{u^2}du. \tag{3.2}$$

Putting $r = 1 - \frac{1}{u}$ $\quad (2 \le u \le n)$ we get

$$\sum_{m\le u}|h_1(m)| \le u^{1/2}\left(\sum_{m\le u}|h_1(m)|^2\right)^{\frac{1}{2}} \ll$$

$$\ll u^{1/2}\left(\sum_{m=1}^{\infty}|h_1(m)|^2 r^{2m}\right)^{\frac{1}{2}} \ll$$

$$\ll u^{1/2}\left(\int_{-\pi}^{\pi}|H_{1,u}(re^{it})|^2 dt\right)^{\frac{1}{2}},$$

where we may choose (cf. (2.3) and (2.4))

$$H_{1,u}(y) = \Lambda_f(y)(F_u(y) - A_nZ_u(y)) + A_nZ_u(y)(\Lambda_f(y) - \Lambda(y)).$$

Since

$$|H_{1,u}(re^{it})|^2 \le 2|\Lambda_f(re^{it})|^2|F_u(re^{it}) - A_nZ_u(re^{it})|^2 +$$

$$+2|Z_u(re^{it})|^2|\Lambda_f(re^{it}) - \Lambda(re^{it})|^2,$$

we have

$$\int_{-\pi}^{\pi} |H_{1,u}(re^{it})|^2 dt \ll \max_{|t|\le\pi} |F_u(re^{it}) - A_n Z_u(re^{it})|^2 \int_{-\pi}^{\pi} |\Lambda_f(re^{it})|^2 dt +$$

$$+ \max_{|t|\le\pi} |Z_u(re^{it})|^2 \int_{-\pi}^{\pi} |\Lambda_f(re^{it}) - \Lambda(re^{it})|^2 dt.$$

Thus, by Lemma 5

$$\sum_{m\le u} |h_1(m)|^2 \ll u \cdot \max_{|t|\le\pi} |F_u(re^{it}) - A_n Z_u(re^{it})|^2 + \\ + o\left(uB^2(u)\right).$$

This implies, by Lemma 3

$$\sum_{m\le u} |h_1(m)| = o(uB(u)) \qquad \text{if} \quad \varepsilon(n)n \le u \le n \tag{3.3}$$

and

$$\sum_{m\le u} |h_1(m)| = O(uB(u)) \qquad \text{if} \quad 2 \le u \le n. \tag{3.4}$$

Using (3.2), (3.3) and (3.4) give

$$\begin{aligned} &\sum_{m\le n} |f(m) - A_n\gamma(m)| = \\ =&o(B(n)) + \int_{1}^{\varepsilon(u)\cdot n} O\left(\frac{B(u)}{u}\right) du + \int_{\varepsilon(u)n}^{n} o\left(\frac{B(u)}{u}\right) du = \\ =&o(B(n)) \qquad \text{as} \quad n\to\infty \end{aligned}$$

by (1.15) which proves (i) of Theorem 1 for $a = 0$. In the case $a \ne 0$ we replace $\lambda_f(n)$ by $\lambda_f(n)e^{ina}$ to end the proof of assertion (i) in Theorem 1.

If

$$\sum_{m\le n} \frac{\lambda(m) - \operatorname{Re}\lambda_{f,1}(m)e^{ima}}{m} = c_2(n) \to \infty \quad (n\to\infty)$$

uniformly in $a \in \mathbb{R}$, then

$$\frac{F_u(re^{it})}{Z_u(r)} = o(1) \qquad \text{uniformly in } t \ \text{ as } r \to 1^-.$$

For $|y| = r = 1 - \frac{1}{u}$ we choose

$$H_{1,u}(y) := \Lambda_f(y) F_u(y)$$

and conclude as above

$$\sum_{m \le u} |h_1(m)| = o(uB(u)) \qquad \text{as } \ u \to \infty$$

which implies

$$f(n) = o(B(n)n^{-1}) = o(\gamma(n)).$$

This ends the proof of Theorem 1. ■

4. Proofs of Theorem 2 and Theorem 3

If the series (1.30) converges then obviously (1.26) is valid. Further

$$\sum_{m \le n} |\lambda(m) - \lambda_{f,1}(m) e^{ima}| =$$

$$= \sum_{\substack{m \le n \\ \lambda(m) \ne 0}} \lambda(m) \left| 1 - \frac{\lambda_{f,1}(m) e^{ima}}{\lambda(m)} \right| \le$$

$$\le \left(\sum_{m \le n} \lambda^2(m) \left| 1 - \frac{\lambda_{f,1}(m) e^{ima}}{\lambda(m)} \right|^2 \right)^{1/2} n^{1/2} \ll \tag{4.1}$$

$$\ll \left(\sum_{\substack{m \le n \\ \lambda(m) \ne 0}} \lambda(m) \left(1 - \frac{\operatorname{Re} \lambda_{f,1}(m) e^{ima}}{\lambda(m)} \right)^2 \right)^{1/2} n^{1/2} =$$

$$= o(n)$$

since (1.30) converges. Thus, (1.27) holds and assertion (i) of Theorem 2 follows from (i) of Theorem 1.

Now, if (1.30) diverges for all $a \in \mathbb{R}$, then, by Dini's theorem, the divergence is uniform for all a and thus the conclusion (ii) of Theorem 2 is an immediate consequence of Theorem 1.

Let the series (1.20) be convergent for $a = 0$. Then, similarly to Lemma 3 one can prove

Lemma 6. *Let* $|y| = 1 - \frac{1}{n}$. *Then*

$$F(y) = A_n F_{II}(1) Z(y) + o(Z(y))$$

uniformly as $|y| \to 1^-$.

In the case $a \neq 0$ we replace $f(n)$ by $f(n)e^{ina}$ and $\lambda_f(m)$ by $\lambda_f(m)e^{ima}$ and the proof of Lemma 3 shows, using the notations of (1.32), (1.33) and (1.34),

$$F(y) - c(a) L_a\left(\frac{1}{1-|y|}\right) Z(e^{-ia}y) = o(Z(|y|)).$$

We only have to prove that, if $c > 0$

$$\frac{L_a(cu)}{L_a(u)} \to 1 \qquad \text{as} \;\; u \to \infty.$$

Without loss of generality we may assume $c > 1$. Then

$$\frac{L_a(cu)}{L_a(u)} = \exp\left(i \sum_{u<m\le cu} \frac{\text{Im } \lambda_{f,1}(m)e^{ima}}{m} \right).$$

Obviously

$$\sum_{u<m\le cu} \frac{\text{Im } \lambda_{f,1}(m)e^{ima}}{m} \le u^{-1} \sum_{u<m<\le cu} |\text{Im } \lambda_{f,1}(m)e^{ima}| \ll$$

$$\ll u^{-1} \left\{ \sum_{m\le cu} (\lambda(m) - \text{Re}\, \lambda_{f,1}(m)e^{ima}) + \sum_{m\le cu} |\lambda_{f,1}(m)e^{ima} - \lambda(m)| \right\},$$

and as in (4.1) the right hand side may be estimated by

$$u^{-1} o(cu) = o(1)$$

which proves Theorem 3. ■

5. Applications to additive arithmetical semigroups

An immediate consequence of Theorem 2 is

Theorem 4. *Let (G, ∂) be an additive arithmetical semigroup such that*

$$\hat{Z}(z) = \sum_{n=0}^{\infty} G(n) z^n = \exp\left(\sum_{m=1}^{\infty} \frac{\bar{\Lambda}(m)}{m} z^m\right) = \frac{\hat{H}(z)}{(1-qz)^{\delta}},$$

where $\hat{H}(z) = O(1)$ for $|z| < q^{-1}$, $\hat{H}(r) \asymp 1$ for $0 < r < q^{-1}$ and $\delta > 0$. Assume that $\bar{\Lambda}(m) = O(q^m)$ and $G(n) \asymp q^n n^{\delta-1}$. Suppose $|\tilde{f}(g)| \le 1$ for all $g \in G$ and either

(i) *$\tilde{f}$ is a completely multiplicative function on G, or*

(ii) *$\tilde{f}$ is a multiplicative function such that $\tilde{f}(p^k) = 0$ for each prime power p^k with $\partial(p) \le \frac{\log 2}{\log q}$.*

If there exists a real number a such that

$$\sum_{p \in \mathcal{P}} q^{-\partial(p)} \left(1 - \mathrm{Re}(\tilde{f}(p) q^{-i\vartheta\partial(p)})\right) \tag{5.1}$$

converges for $\vartheta = a$, then

$$\sum_{\substack{g \in G \\ \partial(g) = n}} \tilde{f}(g) =$$

$$= q^{ina} \prod_{\partial(p) \le n} (1 - q^{-\partial(p)}) \left(1 + \sum_{k=1}^{\infty} \tilde{f}(p^k) q^{-k\partial(p)(1+ia)}\right) G(n) + o(G(n)).$$

If (5.1) *diverges for all $\vartheta \in \mathbb{R}$ then*

$$\sum_{\substack{g \in G \\ \partial(g) = n}} \tilde{f}(g) = o(G(n)).$$

Proof. We use Theorem 2. The case (i) of completely multiplicative functions is obvious (see (1.10)). If $\tilde{f}$ is described by (ii) we write

$$F(y) = \prod_{\substack{p \\ \partial(p) > \frac{\log 2}{\log q}}} \left(1 + \sum_{k=1}^{\infty} \tilde{f}(p^k) q^{-k\partial(p)} y^{k\partial(p)}\right).$$

Put

$$F(y) = \Pi_1(y)\Pi_2(y),$$

where

$$\Pi_1(y) := \prod_{\partial(p) > \frac{\log 2}{\log q}} (1 - \tilde{f}(p)(q^{-1}y)^{\partial(p)})^{-1},$$

$$\Pi_2(y) := \prod_{\partial(p) > \frac{\log 2}{\log q}} \left(1 + \sum_{k=2}^{\infty} (\tilde{f}(p^k) - \tilde{f}(p^{k-1}))\tilde{f}(p)(q^{-1}y)^{k\partial(p)}\right)$$

and define $\lambda_{f,1}$ and $\lambda_{f,2}$, respectively, by

$$\Pi_1(y) = \exp\left(\sum_{m=1}^{\infty} \frac{\lambda_{f,1}(m)}{m} y^m\right), \tag{5.2}$$

$$\Pi_2(y) = \exp\left(\sum_{m=1}^{\infty} \frac{\lambda_{f,2}(m)}{m} y^m\right).$$

Obviously (cf. (1.10)), $|\lambda_{f,1}(m)| \le \lambda(m)$ $(m \in \mathbb{N})$, and we show

$$\sum_{m=1}^{\infty} \frac{|\lambda_{f,2}(m)|}{m} < \infty. \tag{5.3}$$

Consider the product $\Pi_2(y)$ for $|y| < q^{1/2}$. We have

$$\sum_{\partial(p) > \frac{\log 2}{\log q}} \sum_{k=2}^{\infty} \left|(\tilde{f}(p^k) - \tilde{f}(p^{k-1}))\tilde{f}(p)(q^{-1}y)^{k\partial(p)}\right| =: \Sigma_1 + \Sigma_2,$$

where

$$\Sigma_1 := \sum_{\frac{\log 2}{\log q} < \partial(p) < \frac{\log 3}{\log q}} \sum_{k=2}^{\infty} \left|(\tilde{f}(p^k) - \tilde{f}(p^{k-1}))\tilde{f}(p)(q^{-1}y)^{k\partial(p)}\right|$$

and

$$\Sigma_2 := \sum_{\partial(p) \ge \frac{\log 3}{\log q}} \sum_{k=2}^{\infty} \left|(\tilde{f}(p^k) - \tilde{f}(p^{k-1}))\tilde{f}(p)(q^{-1}y)^{k\partial(p)}\right|.$$

The sum Σ_1 can be estimated by

$$\Sigma_1 \ll \sum_{k=2}^{\infty} (q^{-1}|y|)^k \ll \frac{1}{1-(q^{-1}|y|)} < \infty$$

if $|y| < q^{1/2}$ since there are only finitely many primes p with $\frac{\log 2}{\log q} < \partial(p) < \frac{\log 3}{\log q}$.

Concerning Σ_2 we have

$$\Sigma_2 \leq \sum_{\partial(p) \geq \frac{\log 3}{\log q}} \frac{2(q^{-1}|y|)^{2\partial(p)}}{1-3^{-1/2}} \ll \sum_{n=0}^{\infty} G(n) q^{-2n} |y|^{2n} \ll 1$$

for $|y| < q^{1/2}$. Hence the infinite product $\Pi_2(y)$ converges absolutely for $|y| < < q^{1/2}$. Further $\Pi_2(y) \neq 0$ for $|y| \leq 1$, since

$$\left| 1 + \sum_{k=2}^{\infty} (\tilde{f}(p^k) - \tilde{f}(p^{k-1}))\tilde{f}(p)(q^{-1}y)^{k\partial(p)} \right| \geq 1 - 2\sum_{k=2}^{\infty} (q^{-1}|y|)^{k\partial(p)} > 0$$

if $\partial(p) > \frac{\log 2}{\log q}$ and $|y| \leq 1$. Thus (5.3) holds. Applying Theorem 2 gives Theorem 4. ■

Remark 7. The condition in (ii) may be weakened. Then we arrive at

Corollary 1. *Let (G, ∂) be as in Theorem 4. Put $\gamma(n) := G(n)q^{-n}$ and assume $\gamma(n) - \gamma(n-1) = o(\gamma(n))$ as $n \to \infty$. Then, if $|\tilde{f}| \leq 1$ is multiplicative and $\tilde{f}(p^k) \neq 0$ for some prime power p^k with $\partial(p) \leq \frac{\log 2}{\log q}$, the assertions of Theorem 4 hold.*

Since the (finite) product

$$\prod_{\substack{p \\ \partial(p) \leq \frac{\log 2}{\log q}}} \left(1 + \sum_{k=1}^{\infty} \tilde{f}(p^k) q^{-k\partial(p)} y^{k\partial(p)} \right)$$

is holomorphic for $|y| < q$ the assertion of Corollary 1 follows immediately.

Let now $\tilde{g} : G \to \mathbb{R}$ be a (real-valued) additive function. Then, by the continuity theorem of Lévy, the distribution functions

$$(5.4) \qquad G_n(x) := \frac{1}{G(n)} \#\{g \in G : \partial(g) = n, \tilde{g}(g) \leq x\}$$

tend to a limit distribution $G(x)$,

$$G_n \Rightarrow G, \tag{5.5}$$

if and only if there exists a function $\varphi(t)$ which is continuous at $t = 0$ such that

$$\frac{1}{G(n)} \sum_{\substack{g \in G \\ \partial(g)=n}} e^{it\tilde{g}(g)} \to \varphi(t)$$

as $n \to \infty$ for $t \in \mathbb{R}$. Moreover, $\varphi(t)$ is the characteristic function of $G(x)$. We note that the function $\tilde{f}(g) := e^{it\tilde{g}(g)}$ is multiplicative and $|\tilde{f}(g)| = 1$ since $\tilde{g}(g)$ is real-valued and additive.

A direct application of Theorem 4 will lead to an analogue of the Erdős-Wintner theorem giving necessary and sufficient conditions for the weak convergence (5.5).

Here we consider the problem to determine when a given additive function $\tilde{g}$ may be renormalised by sequences $\alpha(n)$ and $\beta(n)$ so that as $n \to \infty$ the frequencies

$$\frac{1}{G(n)} \# \left\{ g \in G, \quad \partial(g) = n : \frac{\tilde{g}(g) - \alpha(n)}{\beta(n)} \le x \right\}$$

possess a weak limit. In particular we obtain an analogue of the celebrated theorem of Erdős and Kac.

Remark 8. A simpler and a little bit easier problem is to characterize all additive functions $\tilde{g}$ which, after a suitable translation, possess a limiting distribution. Indeed, in order that there exists a sequence $\{\alpha(n)\}$, $n \in \mathbb{N}$, for which the frequencies

$$\frac{1}{G(n)} \# \{ g \in G, \quad \partial(g) = n : \tilde{g}(g) - \alpha(n) \le n \}$$

converge to a weak limit as $n \to \infty$ one can give necessary and sufficient conditions. This result can be used, for example, to describe the class of multiplicative functions which are uniformly summable (cf. Remark 11). We shall come back to this topic at a different place.

Put

$$\alpha(n) := \sum_{\substack{p \in \mathcal{P} \\ \partial(p) \le n}} \tilde{g}(p) q^{-\partial(p)} \tag{5.6}$$

and

$$\beta^2(n) := \sum_{\substack{p\in\mathcal{P}\\ \partial(p)\le n}} \tilde g^2(p) q^{-\partial(p)}. \tag{5.7}$$

We shall assume that the Feller-Lindeberg condition holds, i.e. for each fixed $\varepsilon > 0$, assume

$$\frac{1}{\beta^2(n)} \sum_{\substack{\partial(p)\le n\\ |\tilde g(p)|\ge\varepsilon\beta(n)}} \tilde g^2(p) q^{-\partial(p)} \to 0 \qquad \text{as} \quad n\to\infty. \tag{5.8}$$

To ease notational difficulties we shall confine ourselves to the case of completely additive functions, so that we can apply Theorem 1 with the condition $\lambda_f = \lambda_{f,1}$. The general case can easily be done following the proof of Theorem 1. We show

Theorem 5. *Let (G,∂) be an additive arithmetical semigroup satisfying the conditions of Theorem 4. Let $\tilde g$ be a real-valued completely additive function on G such that* (5.8) *holds. Then*

$$\frac{1}{G(n)}\#\left\{g\in G, \partial(g) = n : \frac{\tilde g(g)-\alpha(n)}{\beta(n)} \le x\right\} \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\, dt$$

as $n\to\infty$.

First we prove some properties of $\beta(n)$.

Lemma 7. *The following two assertions are equivalent.*

(i) *There exists $\varepsilon(n)\searrow 0$ such that, if $\varepsilon(n)n \le u\le n$,*

$$\frac{\beta(u)}{\beta(n)} \to 1 \qquad as \quad n\to\infty.$$

(ii) *For each fixed positive number c*

$$\frac{\beta(cn)}{\beta(n)} \to 1 \qquad as \quad n\to\infty.$$

Proof. Assume (i). Since $\beta(n)$ is non-decreasing and if $0 < c\le 1$ choose u such that $\varepsilon(n)n\le u < cn\le n$. Then

$$\frac{\beta(u)}{\beta(n)} \le \frac{\beta(cn)}{\beta(n)} \le 1,$$

and (ii) follows. If $1 < c$, then $0 < 1/c < 1$, and we have

$$\frac{\beta(cn)}{\beta(n)} = \left(\frac{\beta(\frac{1}{c}(cn))}{\beta(cn)}\right)^{-1} \to 1$$

as $n \to \infty$.

Next, assume (ii). Then for each $k \le n$ there exist x_k such that

$$1 - \frac{1}{k} < \frac{\beta(\frac{1}{k}x)}{\beta(x)} \le 1.$$

Let $1 \le x_1 < x_2 < \dots$ and $x_k \to \infty$ as $k \to \infty$. Put $n_0 = 1$, $n_k = x_k + x_{k+1}$ ($k \in \mathbb{N}$) and

$$\varepsilon(n) = \frac{1}{k} \qquad \text{if} \quad n_{k-1} \le n < n_k.$$

Then, if $n_{k-1} \le n < n_k$ and $\varepsilon(n)n \le u \le n$

$$1 - \frac{1}{k} \le \frac{\beta(\frac{1}{k}n_{k-1})}{\beta(n)} \le \frac{\beta(u)}{\beta(n)} \le 1$$

which proves (i). ■

Next we obtain

Lemma 8. *Assume that the Feller-Lindeberg condition* (5.8) *holds. Then assertion* (i) *of Lemma* 7 *is valid.*

Proof. It is enough to show that (ii) of Lemma 7 holds for $0 < c \le 1$. Let $\varepsilon > 0$ be fixed. Then

$$0 \le 1 - \frac{\beta^2(cn)}{\beta^2(n)} = \frac{1}{\beta^2(n)} \sum_{\substack{p \in \mathcal{P} \\ cn < \partial(p) \le n \\ |\tilde{g}(p)| \le \varepsilon\beta(n)}} \tilde{g}^2(p) q^{-\partial(p)} +$$

$$+ \frac{1}{\beta^2(n)} \sum_{\substack{p \in \mathcal{P} \\ cn < \partial(p) \le n \\ |\tilde{g}(p)| > \varepsilon\beta(n)}} \tilde{g}^2(p) q^{-\partial(p)} \le$$

$$\le \varepsilon^2 \sum_{cn < \partial(p) \le n} q^{-\partial(p)} + o_\varepsilon(1) =$$

$$= \varepsilon^2 \sum_{cn < m \le n} q^{-n} P(n) + o_\varepsilon(1).$$

Since $q^{-m}P(m) = O(1/m)$ the last sum can be estimated by

$$\ll \sum_{cn < m \le n} \frac{1}{m} = \log \frac{1}{c} + O((cn)^{-1}).$$

Therefore

$$\limsup_{n\to\infty}\left(1-\frac{\beta^2(cn)}{\beta^2(n)}\right) \ll \varepsilon^2 \log\frac{1}{c}.$$

Letting $\varepsilon \to 0$ shows

$$\lim_{n\to\infty}\left(1-\frac{\beta^2(cn)}{\beta^2(n)}\right) = 0$$

which ends the proof of Lemma 8. ■

Put, if $m \le n$,

$$f_n(m) = q^{-m} \sum_{\substack{g\in G\\ \partial(g)=m}} e^{it\tilde{g}(g)/\beta(n)}. \tag{5.9}$$

We shall show that

$$\frac{f_n(n)}{G(n)} = \exp\left(it\frac{\alpha(n)}{\beta(n)} - \frac{t^2}{2}\right)\{1+o(1)\} \qquad \text{as} \quad n\to\infty \tag{5.10}$$

which proves Theorem 5. ■

Since $\tilde{g}(g)$ is completely additive the function $e^{it\tilde{g}(g)/\beta(n)}$ is completely multiplicative and

$$F_n(y) := \sum_{m=1}^{\infty} f_n(m)y^m = \exp\left(\sum_{m=1}^{\infty}\frac{\lambda_{f,n}(m)}{m}y^m\right) \tag{5.11}$$

implies

$$\begin{aligned} A_n :&= \exp\left(\sum_{m=1}^{n}\frac{\lambda_{f,n}(m)-\lambda(m)}{m}\right) = \\ &= \exp\left(\sum_{\partial(p)\le n} q^{-\partial(p)}(e^{it\tilde{g}(p)/\beta(n)}-1) + \right. \\ &\left. + \sum_{m\le n} q^{-m}\sum_{\substack{d\mid m\\ d\le \frac{m}{2}}} d \sum_{\partial(p)=d}(e^{it\tilde{g}(p)/\beta(n)}-1)\right) = \\ &= \exp(\Sigma_1+\Sigma_2). \end{aligned} \tag{5.12}$$

Split the sum Σ_2 into two parts, where $m \le m_0$ and $m > m_0$, respectively. The sum over $m \ge m_0$ can be estimated by

$$\ll \sum_{m>m_0} q^{-m/2} m^{\varrho} \log m \le \varepsilon$$

if m_0 is big enough. Concerning $m \le m_0$ observe that

$$e^{it\tilde{g}(p)/\beta(n)} - 1 \to 0 \qquad \text{as} \quad n \to \infty$$

if $\partial(p) \le m_0$. Thus

$$\Sigma_2 = o(1).$$

By Lemma 7 and Lemma 8

$$\Sigma_1 = it\frac{\alpha(n)}{\beta(n)} - \frac{t^2}{2} + o(1) \qquad \text{as} \quad n \to \infty$$

uniformly in $|t| \le T$. Then (5.12) gives

$$A_n = \exp\left(\sum_{m\le n} \frac{\lambda_{f,n}(m) - \lambda(m)}{m}\right) = \tag{5.13}$$
$$= \exp\left(it\frac{\alpha(n)}{\beta(n)} - \frac{t^2}{2} + o(1)\right).$$

In the generating function (5.11) we may choose

$$\lambda_{f,n}(m) = \lambda(m) = 0 \qquad \text{if} \quad m > n$$

since these values do not influence $f_n(n)$.

Now by (5.13) condition (1.26) is satisfied for $a = 0$ where we choose $\lambda_{f,1} = \lambda_{f,n}$. Again by (5.13)

$$\sum_{m\le n} (\lambda(m) - \operatorname{Re}\lambda_{f,n}(m)) = o(n)$$

which implies (see (4.1))

$$\sum_{m\le n} |\lambda(m) - \lambda_{f,n}(m)| = o(n)$$

and this implies (1.27). Thus Theorem 1 gives (5.10) and Theorem 5 holds. ∎

Corollary 2. *Let (G, ∂) be an additive arithmetical semigroup satisfying the conditions of Theorem* 3*. Let $\tilde{g}$ be a completely additive function on G such that $|\tilde{g}(p)| = O(1)$ for all primes p, and such that $\beta(n) \to \infty$ as $n \to \infty$. Then*

$$\frac{1}{G(n)}\#\{g \in G, \partial(g) = n : \tilde{g}(g) - \alpha(n) \le x\beta(n)\} \to \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-t^2/2}\, dt$$

uniformly as $n \to \infty$.

Indeed, in the case of Corollary 2, $\varepsilon\beta(n) \to \infty$ as $n \to \infty$ for each $\varepsilon > 0$, and hence

$$\sum_{\substack{\partial(p) \le n \\ |\tilde{g}(p)| \ge \varepsilon\beta(n)}} \tilde{g}^2(p) q^{-\partial(p)} = 0$$

for n sufficiently large. As an interesting application of Corollary 2, we consider the function given by

$$\tilde{g}(g) = \bar{\Omega}(g) = \sum_{\substack{p \in \mathcal{P} \\ p^k || g}} k,$$

the total number of prime divisors of g. Then

$$\alpha(n) = \beta^2(n) = \sum_{\partial(p) \le n} q^{-\partial(p)} = \sum_{m \le n} q^{-m} P(m),$$

and by (1.8) and (1.20),

$$\alpha(n) = \beta^2(n) = \delta \log n + o(\log n).$$

Thus Corollary 2 for $\tilde{g}(g) = \bar{\Omega}(g)$ is just the analogue result of the Erdős-Kac theorem in probabilistic number theory.

Remark 9. If the power series

$$\log H(y) = \sum_{n=0}^{\infty} d_n y^n \qquad |y| < 1 \tag{5.14}$$

converges for $y = 1$ then, as $n \to \infty$,

$$\alpha(n) = \beta^2(n) = \sum_{m \le n} q^{-m} P(m) = \delta \log n + c + o(1)$$

and the following result holds true.

Corollary 3. *Assume that* (G, ∂) *satisfies the conditions of Theorem* 4 *such that the series* (5.14) *converges for* $y = 1$. *Then*

$$\frac{1}{G(n)} \# \left\{ g \in G, \partial(g) = n : \frac{\bar{\Omega}(g) - \log n}{\sqrt{\log n}} \leq x \right\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt.$$

Remark 10. The condition $\lambda_f(m) = O(1)$ can be omitted if $\gamma(n) \asymp n^\rho$ with $\rho \geq 0$. For, let (see (1.22))

$$F(y) = F_I(y) \exp\left(\sum_{\substack{m=1 \\ |\lambda_{f,2}(m)| \leq K}}^{\infty} \frac{\lambda_{f,2}(m)}{m} y^m \right) \exp\left(\sum_{\substack{m=1 \\ |\lambda_{f,2}(m)| > K}}^{\infty} \frac{\lambda_{f,2}(m)}{m} y^m \right) =$$

$$= \sum_{n=0}^{\infty} f_K(n) y^n \exp\left(\sum_{\substack{m=1 \\ |\lambda_{f,2}(m)| > K}}^{\infty} \frac{\lambda_{f,2}(m)}{m} y^m \right) =:$$

$$= \sum_{n=0}^{\infty} f_K(n) y^n \sum_{n=0}^{\infty} a_n y^n.$$

Then Theorem 1 and Theorem 2, respectively, may be applied to $f_K(n)$.

Let $\varepsilon > 0$ and choose $K > 0$ such that

$$\exp\left(\sum_{\substack{m=1 \\ |\lambda_{f,2}(m)| > K}}^{\infty} \frac{|\lambda_{f,2}(m)|}{m} \right) = 1 + \vartheta\varepsilon,$$

where $0 \leq \vartheta \leq 1$. Then, since $a_0 = 1$ and $a_m = 0$ for $1 < m \leq m_0(K)$ and $\sum\limits_{m > m_0} |a_m| \leq \varepsilon$,

$$|f(n) - f_K(n)| = \left| \sum_{m=m_0}^{n} a_m f_K(n-m) \right| \leq \varepsilon \max_{m \leq n - m_0} |f_K(m)| \ll \varepsilon \gamma(n).$$

Thus the following result holds.

Corollary 4. *Let* Z *be an element of the exp-log class* $\mathcal{F}$ *where*

$$\gamma(n) \asymp n^\varrho \qquad (\varrho \geq 0)$$

as $n \to \infty$. Let $F(y)$ in (1.4) satisfy (1.22), (1.23). Then the assertions of Theorem 1 and Theorem 2 are valid.

Remark 11. For a function $\tilde{f}: G \mapsto \mathbb{C}$ we introduce

$$M(n, \tilde{f}) := \frac{1}{G(n)} \sum_{\substack{g \in G \\ \partial(g)=n}} \tilde{f}(g)$$

and define, if $1 \le \alpha < \infty$ the seminorm

$$||\tilde{f}||_\alpha := \left(\limsup_{n \to \infty} M(n, |\tilde{f}|^\alpha) \right)^{1/\alpha}.$$

Let

$$\mathcal{L}^\alpha := \{\tilde{f}: G \mapsto \mathbb{C}; \quad ||\tilde{f}||_\alpha < \infty\}$$

denote the linear space of functions on G with bounded seminorm $||\tilde{f}||_\alpha$. If

$$\ell^\infty(G) := \left\{ \tilde{f}: G \mapsto \mathbb{C}; \quad \sup_{g \in G} |\tilde{f}(g)| < \infty \right\}$$

is the space of all bounded functions on G then we introduce the space $\mathcal{L}^*(G)$ of *uniformly summable functions* on G as the $||\cdot||_1$-closure of $\ell^\infty(G)$. Obviously $\tilde{f} \in \mathcal{L}^*$ if and only if

$$\lim_{K \to \infty} \sup_{n \ge 1} M(n, \ |\tilde{f}_K|) = 0,$$

where

$$\tilde{f}_K(g) = \begin{cases} \tilde{f}(g) & \text{if } |\tilde{f}(g)| \ge K, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to show that if $1 < \alpha < \infty$

$$\ell^\infty(G) \subsetneqq \mathcal{L}^\alpha \subsetneqq \mathcal{L}^* \subsetneqq \mathcal{L}^1.$$

It is worthwhile to give a characterization of multiplicative functions on G corresponding to that which was proved in the case of multiplicative functions on $\mathbb{N}$ (see Indlekofer [11], [12] and [13]). A description of multiplicative functions $\tilde{f} \in \mathcal{L}^*$ on arithmetical semigroups G satisfying $G(n) \sim Aq^n$ was done by Wehmeier in his thesis [24].

Here we show how our method can be used to deal with multiplicative functions which are bounded on prime powers.

For this let $\tilde{f} : G \mapsto \mathbb{C}$ be multiplicative such that, for some constant $c > 1$,

$$|\tilde{f}(p^k)| \leq c \qquad \text{for all prime powers } p^k. \tag{5.15}$$

Then we prove

Lemma 9. *Let (G, ∂) be an additive arithmetical semigroup satisfying $G(n) \ll n^\rho q^n$ where $q > 1$ and $\rho \in \mathbb{R}$. Assume that $\tilde{f}$ is multiplicative satisfying (5.15). Then there exists $m_0 \in \mathbb{N}$ such that*

$$\Pi(y) = \prod_{\substack{p \\ \partial(p) \geq m_0}} \left(1 + \sum_{k=1}^{\infty} \tilde{f}(p^k) q^{-k\partial(p)} y^{k\partial(p)} \right) =$$

$$= \exp\left(\sum_{m=m_0}^{\infty} \frac{\lambda_f(m)}{m} y^m \right) \qquad (|y| < 1),$$

where

$$\frac{\lambda_f(m)}{m} = \sum_{\substack{p \\ \partial(p)=m}} \tilde{f}(p) q^{-m} + O\left(q^{-\frac{m}{4}}\right)$$

as $m \to \infty$.

Proof. Put

$$\Pi(y) = \prod_{\substack{p \\ \partial(p) \geq m_0}} (1 - \tilde{f}(p) q^{-\partial(p)} y^{\partial(p)})^{-1} \times$$

$$\times \prod_{\substack{p \\ \partial(p) \geq m_0}} \left(1 + \sum_{k=2}^{\infty} (\tilde{f}(p^k) - \tilde{f}(p)\tilde{f}(p^{k-1})) q^{-k\partial(p)} y^{k\partial(p)} \right) =:$$

$$=: \Pi_1(y)\Pi_2(y).$$

Obviously

$$\log \Pi_1(y) = \sum_{\substack{p \\ \partial(p) \geq m_0}} \tilde{f}(p) q^{-\partial(p)} y^{\partial(p)} + \sum_{\substack{p \\ \partial(p) \geq m_0}} \sum_{k=2}^{\infty} \frac{(\tilde{f}(p))^k}{k} q^{-k\partial(p)} y^{k\partial(p)} =$$

$$= \sum_{m=m_0}^{\infty} \left\{ \sum_{\substack{p \\ \partial(p)=m}} \tilde{f}(p) + \sum_{\substack{k \mid m \\ 2 \leq k \leq \frac{m}{m_0}}} \frac{1}{k} \sum_{\substack{p \\ \partial(p)=\frac{m}{k}}} (\tilde{f}(p))^k \right\} q^{-m} y^m. \tag{5.16}$$

The last sum can be estimated by

$$\tag{5.17} \begin{aligned} &\ll \sum_{\substack{k\mid m\\ 2\le k\le \frac{m}{m_0}}} \frac{c^k}{k} m^{|\rho|} q^{\frac{m}{2}} \ll \\ &\ll q^{\frac{3}{4}m} \end{aligned}$$

if m_0 is large enough.

Now, we may choose m_0 such that $\Pi_2(y)$ is different from zero and holomorphic for $|y| < q^{1/2}$. Then $\log \Pi_2(y)$ is holomorphic for $|y| < q^{1/2}$, too, and obviously

$$\tag{5.18} \log \Pi_2(y) = \sum_{m\ge m_0} O\left(q^{-m(\frac{1}{2}-\varepsilon)}\right) y^m$$

for every $\varepsilon > 0$. Collecting (5.16), (5.17) and (5.18) gives the assertion of Lemma 9.

Motivated by the above mentioned results for multiplicative functions on $\mathbb{N}$ (see [11]) we shall assume that

$$\tag{5.19} \sum_p \frac{(|\tilde{f}(p)| - 1)^2}{q^{\partial(p)}} < \infty$$

and $\tilde{f} \in \mathcal{L}^1$, i.e.

$$\tag{5.20} M(n, |\tilde{f}|) \ll 1.$$

Using the notation of Lemma 9 we define the multiplicative function $\tilde{f}_1$ by

$$\tag{5.21} \tilde{f}_1(p^k) = \begin{cases} \tilde{f}(p^k) & \text{if } \partial(p) \ge m_0, \\ 0 & \text{if } \partial(p) < m_0. \end{cases}$$

Clearly $M(n, |\tilde{f}_1|) \le M(n, |\tilde{f}|)$, and (5.20) implies

$$\sum_{n=0}^{\infty} \sum_{\substack{g\in G\\ \partial(g)=n}} |\tilde{f}_1(g)| q^{-n} |y|^n \ll Z(|y|)$$

which shows, by Lemma 9 and (1.6),

$$\sum_{p} \frac{|\tilde{f}(p)|-1}{q^{\partial(p)}} r^{\partial(p)} \leq c_1 \qquad \text{with some } c_1>0 \tag{5.22}$$

uniformly as $r \to 1^-$.

Under these conditions we have

Theorem 6. *Let* (G,∂) *be as in Theorem* 4. *Let* $\tilde{f}$ *be multiplicative and assume* (5.19) *and* (5.20). *Let* $\tilde{f}_1$ *satisfy* (5.21) *and* (5.22). *Then, as* $n \to \infty$,

$$M(n,|\tilde{f}_1|) = c_2 \exp\left(\sum_{m_0\leq \partial(p)\leq n} \frac{|\tilde{f}(p)|-1}{q^{\partial(p)}}\right) + o(1)$$

with some positive constant c_2.

The *proof* is the same as that for Theorem 1. We observe that Lemma 1 holds because of (5.19), and that the estimate of Lemma 3 is valid, since

$$\begin{aligned}
&-2 \sum_{\substack{p\\ \partial(p)\geq m_0}} \frac{1-|\tilde{f}(p)|\operatorname{Re} e^{it\partial(p)}}{q^{\partial(p)}} r^{\partial(p)} + \\
&+ \sum_{\substack{p\\ \partial(p)\geq m_0}} \frac{1-\operatorname{Re} e^{it\partial(p)}}{q^{\partial(p)}} r^{\partial(p)} \leq \\
&\leq 2 \sum_{\substack{p\\ \partial(p)\geq m_0}} \frac{|\tilde{f}(p)|-1}{q^{\partial(p)}} r^{\partial(p)} \ll \\
&\ll 1.
\end{aligned}$$

Here we made use of the relations

$$\begin{aligned}
1-\operatorname{Re} e^{-it\partial(p)} &= \frac{1}{2}|1-e^{-it\partial(p)}|^2 \leq \\
&\leq |1-|\tilde{f}(p)||^2 + |1-|\tilde{f}(p)|e^{it\partial(p)}|^2
\end{aligned}$$

and

$$|1-|\tilde{f}(p)|e^{it\partial(p)}|^2 = |\tilde{f}(p)|^2 - 1 + 2(1-|\tilde{f}(p)|\operatorname{Re} e^{it\partial(p)}).$$

The further details are left to the reader.

Combining Theorem 2 and Theorem 6 will give

Theorem 7. *Let* (G,∂) *and* $\tilde{f}_1$ *as in Theorem 6. Then the following two assertions hold.*

(i) *Let*

$$\sum_{\substack{p \\ \partial(p)\geq m_0}} \frac{|\tilde{f}(p)| - \operatorname{Re}\tilde{f}(p)e^{i\partial(p)a}}{q^{\partial(p)}} \tag{5.23}$$

converge for some $a \in \mathbb{R}$. *Then*

$$M(n,\tilde{f}_1) = c(a)\exp\left(-ina + \sum_{\substack{p \\ m_0\leq\partial(p)\leq n}} \frac{\tilde{f}(p)e^{i\partial(p)a}-1}{q^{\partial(p)}}\right) + o(1)$$

with some $c(a)$ *which can be given explicitly.*

(ii) *Let* (5.23) *diverge for all* $a \in \mathbb{R}$. *Then*

$$M(n,\tilde{f}_1) = o(1) \qquad \text{as } n \to \infty.$$

The proof is left to the reader.

An easy consequence is the following

Corollary 5. *Let* (G,∂) *and* $\tilde{f}_1$ *as in Theorem 7. Assume in addition* $\gamma(n) - \gamma(n-1) = o(\gamma(n))$. *Then, if the mean value of* $\tilde{f}_1$ *exists the same holds for* $\tilde{f}$.

Since the (finite) product

$$\prod_{\substack{p \\ \partial(p)<m_0}} \left(1 + \sum_{k=1}^{\infty} \tilde{f}(p^k)q^{-k\partial(p)}y^{k\partial(p)}\right)$$

is holomorphic for $|y| < q$ and therefore absolutely convergent for $|y| \leq q^{\frac{1}{2}}$, say, then the assertion of Corollary 5 follows immediately.

6. Applications to decomposable combinatorial structures

In this chapter we use the notation of [2]. Let $\mathcal{P}$ be a class of combinatorial structures. A class $\mathcal{A}$ is said to be *decomposable* over $\mathcal{P}$ if each of its elements may be uniquely decomposed into components of elements of $\mathcal{P}$. Given an instance of size n, the most basic description reports only the number k of components. We are interested in the full component spectrum, specifying how many components there are of size one, two, three and so on. For a given combinatorial structure $\mathcal{A}$, the natural model assumes that size $n = |\alpha|, \alpha \in \mathcal{A}$ is given and that all $p(n)$ instances $\alpha \in \mathcal{A}$ of size n are equally likely. For such a random instance, we write C_i for the number of components of size i, so that the stochastic process

$$C^{(n)} := (C_1, C_2, \ldots, C_n) \tag{6.1}$$

specifies the entire component size counting process and the random variable

$$\Omega^{(n)} := C_1 + C_2 + \ldots + C_n \tag{6.2}$$

is the total number of components.

In the case of labelled structures, an element of $\mathcal{A}$ is formed by taking a multiset of (labelled) elements of $\mathcal{P}$ and performing all consistent relabellings. If there are m_i possible structures of size i in $\mathcal{P}$ then the combinatorial structure is determined by the sequence $m_1, m_2, \ldots$. In general, with

$$M(y) := \sum_{i=1}^{\infty} m_i y^i / i!, \qquad P(y) := \sum_{n=0}^{\infty} p(n) y^n / n! \tag{6.3}$$

labelled structures (or *assemblies*) are characterized by the exponential formula

$$P(y) = \exp(M(y)). \tag{6.4}$$

For *unlabelled* structures, we have the *multiset* construction, where elements of $\mathcal{A}$ are obtained by taking arbitrary sets (with repetitions allowed) of elements of $\mathcal{P}$. If $p(n)$ being the number of multisets of weight n,

$$P(y) := \sum_{n=0}^{\infty} p(n) y^n \qquad \text{and} \qquad M(y) := \sum_{i=1}^{\infty} m_i y^i \tag{6.5}$$

we have

$$P(y) = \prod_{i=1}^{\infty} (1 - y^i)^{-m_i} = \exp\left(\sum_{j=1}^{\infty} M(y^j)/j \right). \tag{6.6}$$

The class of *selections* is like multisets, as described above, except that all components must be distinct. Since multisets can be seen as arithmetical semi-groups, a natural description of selections would be “the subset of all squarefree elements in an additive arithmetical semigroup". With $p(n)$ being the number of sets of weight n

$$P(y) := \sum_{n=0}^{\infty} p(n)y^n \qquad \text{and} \qquad M(y) := \sum_{i=1}^{\infty} m_i y^i \tag{6.7}$$

we have

$$P(y) = \prod_{i=1}^{\infty} (1+y^i)^{m_i} = \exp\left(\sum_{j=1}^{\infty} M(-y^j)/j\right). \tag{6.8}$$

For more details see [2], Chapter 2.

We define a *completely multiplicative* function $\tilde{f} : \mathcal{A} \to \mathbb{C}$ as follows: Let $\{b_j\}$ be a sequence of complex numbers. For $\alpha \in \mathcal{A}$ of size n we put

$$\tilde{f}(\alpha) = \prod_{j=1}^{n} b_j^{C_j(\alpha)}. \tag{6.9}$$

Then $\tilde{f}$ is called *completely multiplicative*.

In a corresponding way we define *completely additive* functions $\tilde{g}$ by the linear combination

$$\tilde{g}(\alpha) := \sum_{j=1}^{n} a_j C_j(\alpha) \quad \text{if } \alpha \text{ is of size } n, \tag{6.10}$$

where $\{a_j\}$ is a sequence of complex numbers. The special choice $a_j = 1$, $j \in \mathbb{N}$, leads to $\tilde{g} = \Omega$ and

$$\Omega(\alpha) = \Omega^{(n)}(\alpha) = \sum_{j=1}^{n} C_j(\alpha) \quad \text{if } \alpha \text{ is of size } n.$$

Put

$$p(n;\tilde{f}) := \sum_{\substack{\alpha \in \mathcal{A} \\ |\alpha| = n}} \tilde{f}(\alpha).$$

(i) In the case of *assemblies* we put

$$M(y;\tilde{f}) := \sum_{i=1}^{\infty} m_i b_i y^i / i!$$

and obtain

$$P(y;\tilde{f}) = \sum_{n=0}^{\infty} p(n;\tilde{f}) y^n/n! = \exp(M(y;\tilde{f})).$$

(ii) For *multisets* we put

$$M(y;\tilde{f}) := \sum_{i=1}^{\infty} m_i b_i y^i$$

and conclude

$$\begin{aligned} P(y;\tilde{f}) &= \sum_{n=0}^{\infty} p(n;\tilde{f}) y^n = \\ &= \prod_{i=1}^{\infty} (1 - b_i y^i)^{-m_i} = \\ &= \exp\left(\sum_{j=1}^{\infty} M(y^j;\tilde{f})/j\right). \end{aligned}$$

(iii) Correspondingly for *selections* we have

$$M(y;\tilde{f}) := \sum_{i=1}^{\infty} m_i b_i y^i$$

and

$$\begin{aligned} P(y;\tilde{f}) &= \prod_{i=1}^{\infty} (1 + b_i y^i)^{m_i} = \\ &= \exp\left(\sum_{j=1}^{\infty} M(-y^j;\tilde{f})/j\right). \end{aligned}$$

Assume that P belongs to the exp-log class $\mathcal{F}$. Then the results of Theorems 1-5 may be applied to investigate the behaviour $p(n,f)/p(n)$ as $n \to \infty$. Especially, corresponding to Corollaries 1 and 2 we shall obtain generalizations of Proposition IX.14 et al. from [6]. As a typical result we formulate

Theorem 8. *Let P be a function of the exp-log class $\mathcal{F}$ and $\tilde{g}$ be a real-valued additive function. In order that the distribution functions*

$$\frac{1}{p(n)} \#\{\alpha \in \mathcal{A}, |\alpha| = n : \tilde{g}(\alpha) \le x\}$$

tend to a limit law as $n \to \infty$ it is necessary and sufficient that the series

$$\sum_{\substack{m=1\\|a_m|\le 1}}^{\infty} \frac{a_m\lambda(m)}{m}, \quad \sum_{\substack{m=1\\|a_m|\le 1}}^{\infty} \frac{{a_m}^2\lambda(m)}{m}, \quad \sum_{\substack{m=1\\|a_m|>1}}^{\infty} \frac{\lambda(m)}{m}$$

converge.

The remarks given in §5 may be transferred *cum grano salis* to investigations of exp-log schemas. We shall come back to this topic somewhere else.

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